Engineering Electromagnetics

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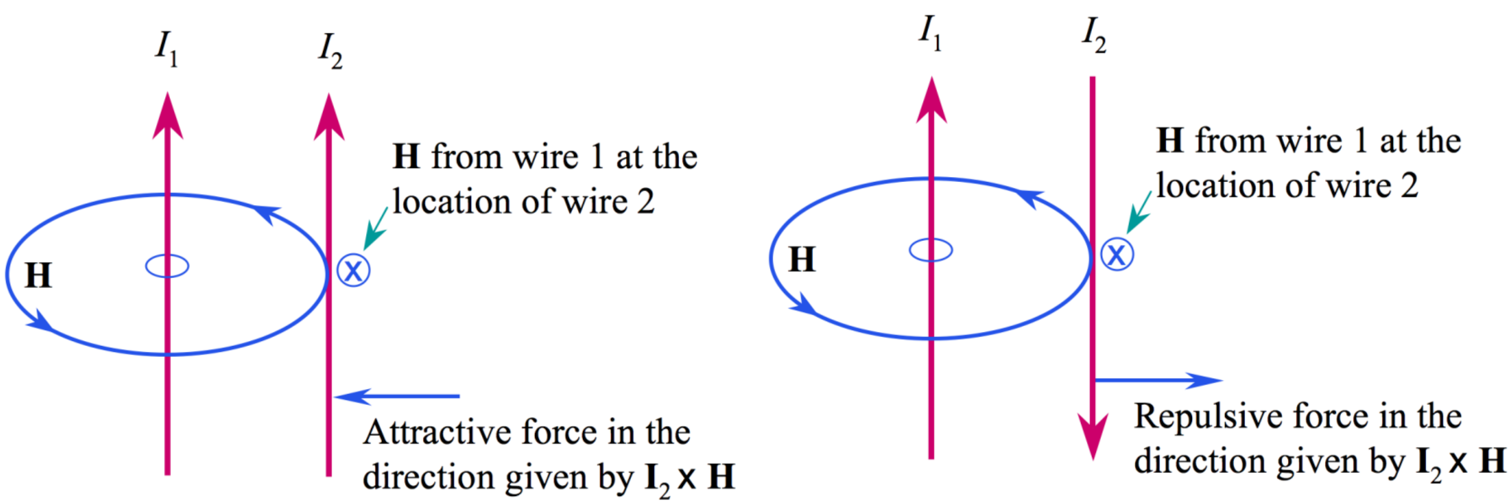
Chapter 7:

The Steady Magnetic Field

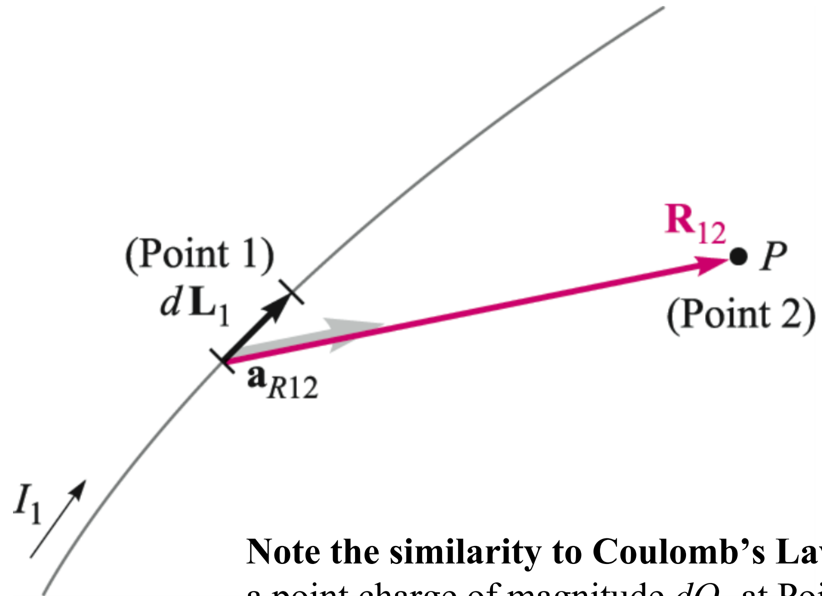
Magnetic Field

- A magntic field wih closed field lines is produced surrounding the current carrying wire.
- The magnetic field intensity, **H**, circulates *around* its source, I_1, in a direction determined by the *right-hand rule*: Right thumb in the direction of the current, fingers curl in the direction of **H.**



Biot-Savart Law

The Biot-Savart Law specifies the magnetic field intensity, **H**, arising from a "point source" current element of differential length $d\mathbf{L}$.



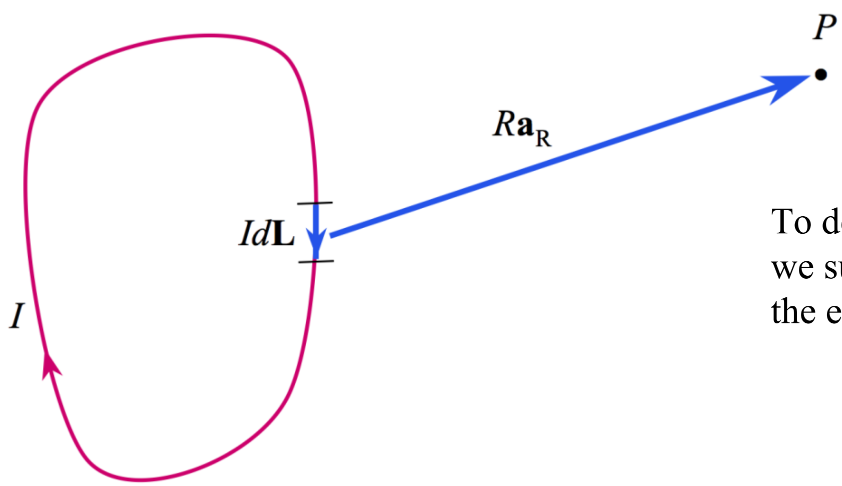
The units of **H** are [A/m]

Note in particular the inverse-square distance dependence, and the fact that the cross product will yield a field vector that points into the page. This is a formal statement of the right-hand rule

Note the similarity to Coulomb's Law, in which a point charge of magnitude dQ_1 at Point 1 would generate electric field at Point 2 given by:

Magnetic Field Arising From a Circulating Current

At point P, the magnetic field associated with the differential current element $Id\mathbf{L}$ is

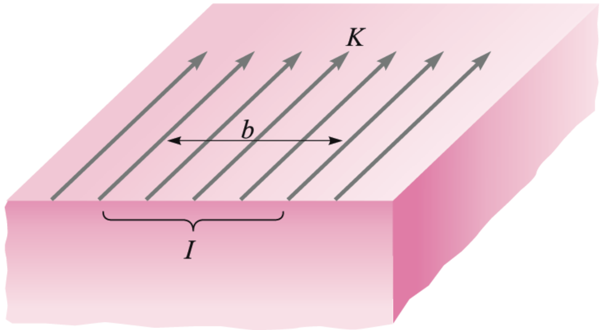


To determine the total field arising from the closed circuit path, we sum the contributions from the current elements that make up the entire loop, or

The contribution to the field at P from any portion of the current will be just the above integral evalated over just that portion.

Two- and Three-Dimensional Currents

On a surface that carries uniform surface current density **K** [A/m], the current within width *b* is



..and so the differential current quantity that appears in the Biot-Savart law becomes:

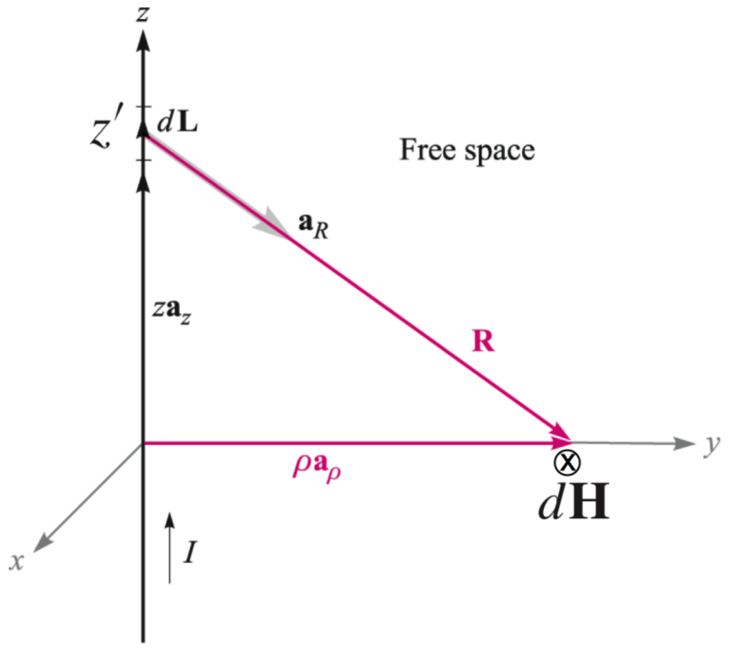
The magnetic field arising from a current sheet is thus found from the two-dimensional form of the Biot-Savart law:

In a similar way, a **volume current** will be made up of three-dimensional current elements, and so the Biot-Savart law for this case becomes:

Example of the Biot-Savart Law

In this example, we evaluate the magnetic field intensity on the y axis (equivalently in the xy plane) arising from a filament current of infinite length in on the z axis.

Using the drawing, we identify:



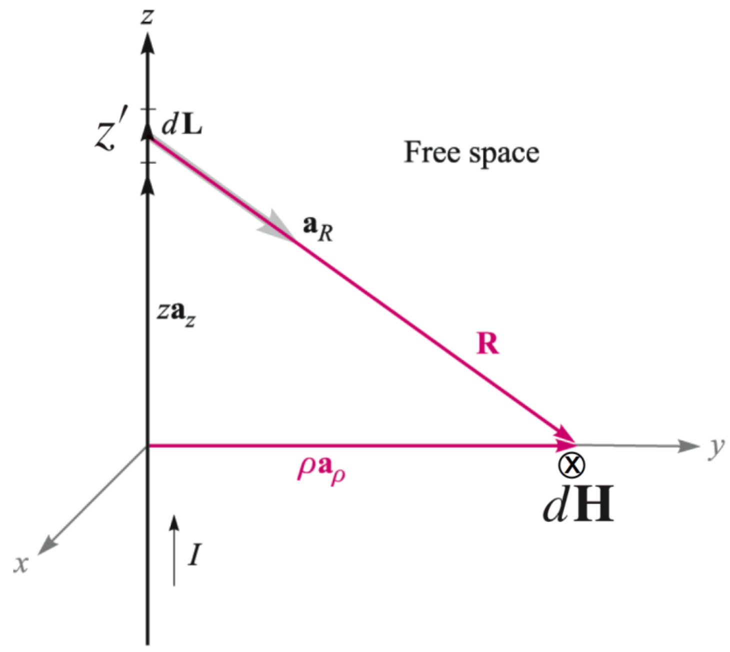
and so..

so that:

Example: continued

We now have:

Integrate this over the entire wire:

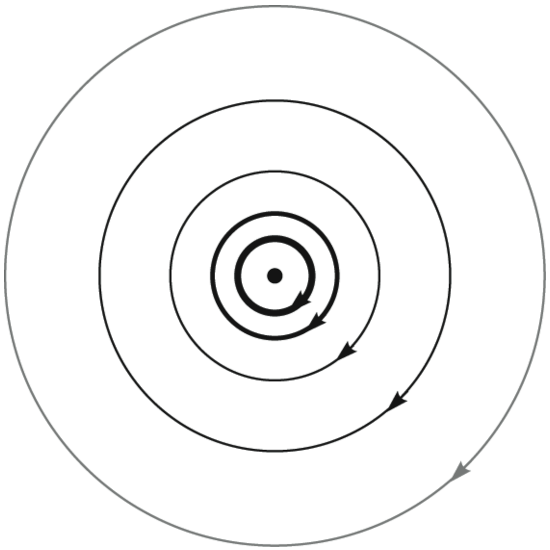


..after carrying out the cross product

Example: concluded

Evaluating the integral:

we have:

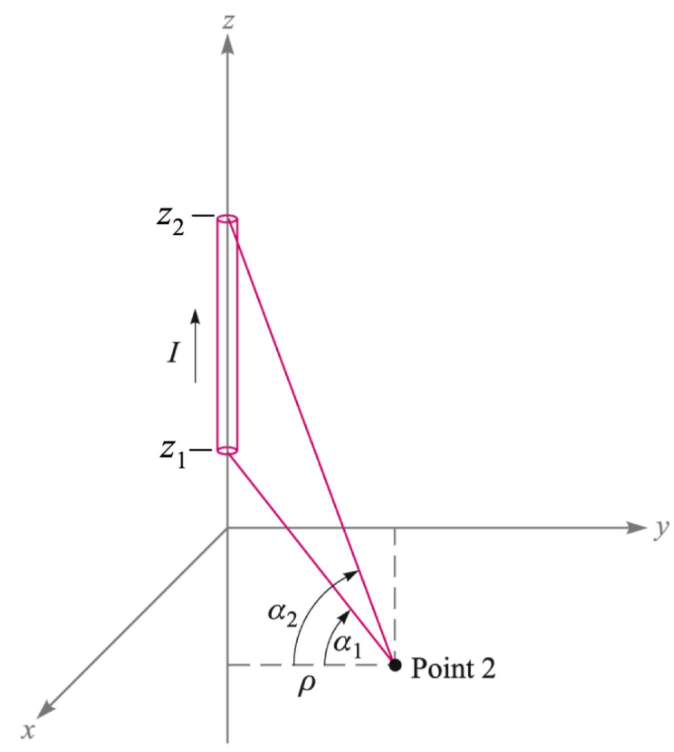


finally:

Current is into the page.
Magnetic field streamlines
are concentric circles, whose magnitudes
decrease as the inverse distance from the *z* axis

Field Arising from a Finite Current Segment

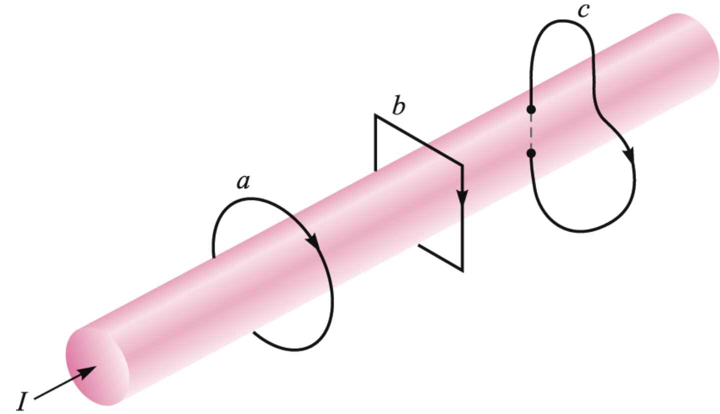
In this case, the field is to be found in the *xy* plane at Point 2.
The Biot-Savart integral is taken over the wire length:



..after a few additional steps (see Problem 7.8), we find:

Ampere's Circuital Law

Ampere's Circuital Law states that the line integral of H about *any closed path* is exactly equal to the direct current enclosed by that path.

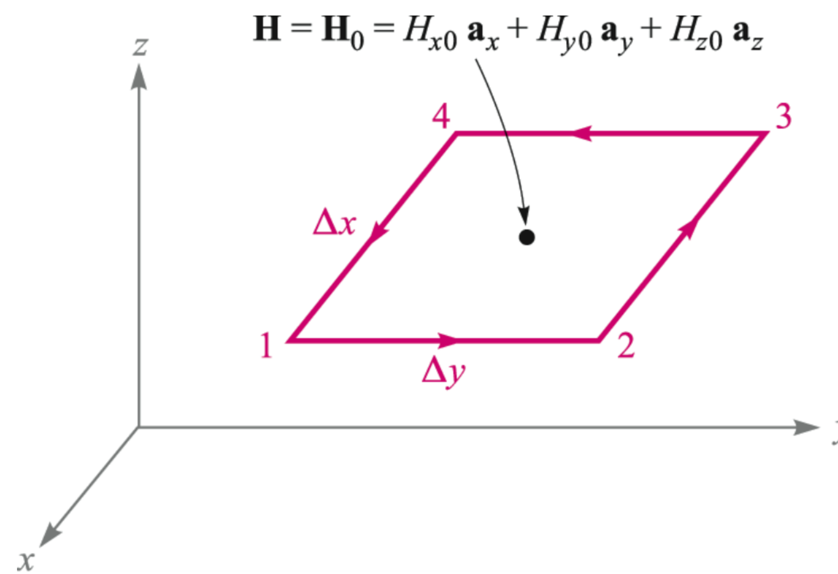


- In the figure at right, the integral of **H** about closed paths *a* and *b* gives the total current *I*, while the integral over path *c* gives only that portion of the current that lies within *c*.
- If H possess a circulation about a given path then current passes through this path.

Ampere's Law as Applied to a Small Closed Loop.

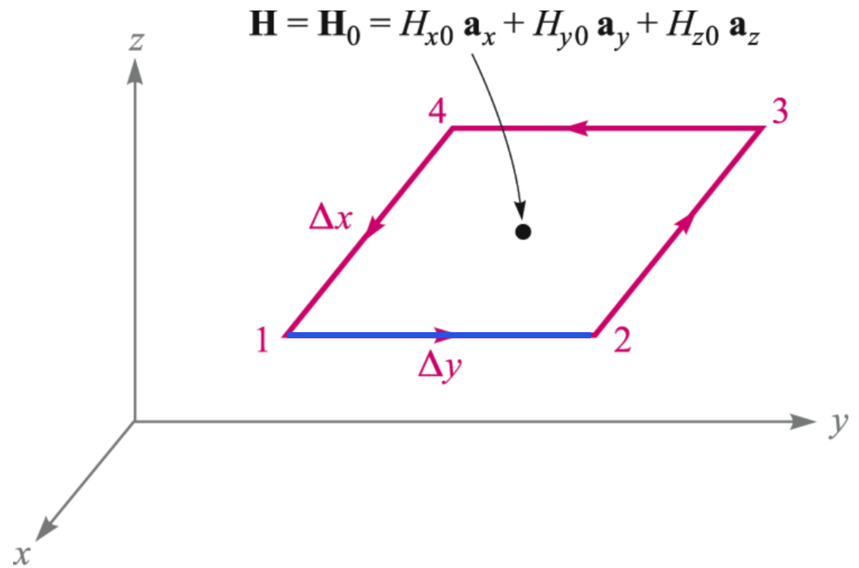
Consider magnetic field **H** evaluated at the point shown in the figure. We can approximate the field over the closed path 1234 by making appropriate adjustments in the value of **H** along each segment.

The objective is to take the closed path integral and ultimately obtain the point form of Ampere's Law.



Approximation of $\mathbf{H}$ Along One Segment

Along path 1-2, we may write:

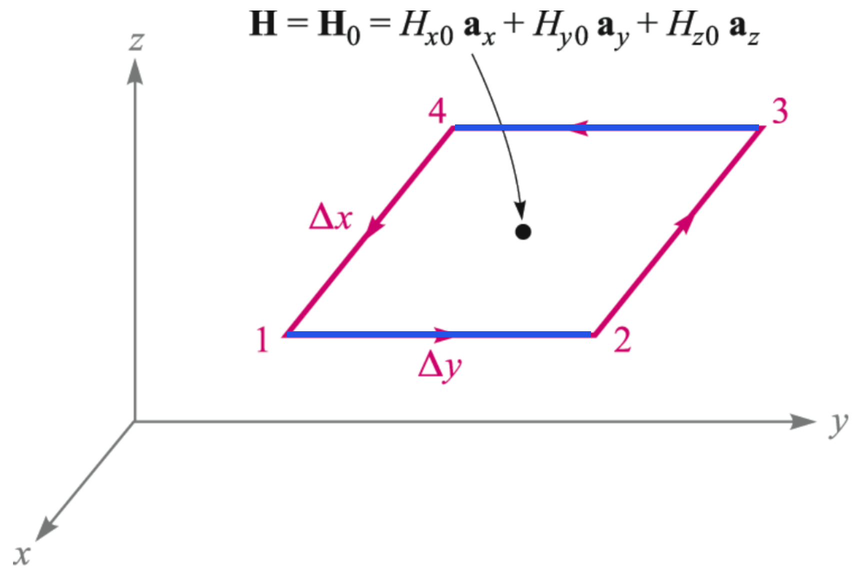


where:

And therefore:

Contributions of *y*-Directed Path Segments

The contributions from the front and back sides will be:



The contribution from the opposite side is:

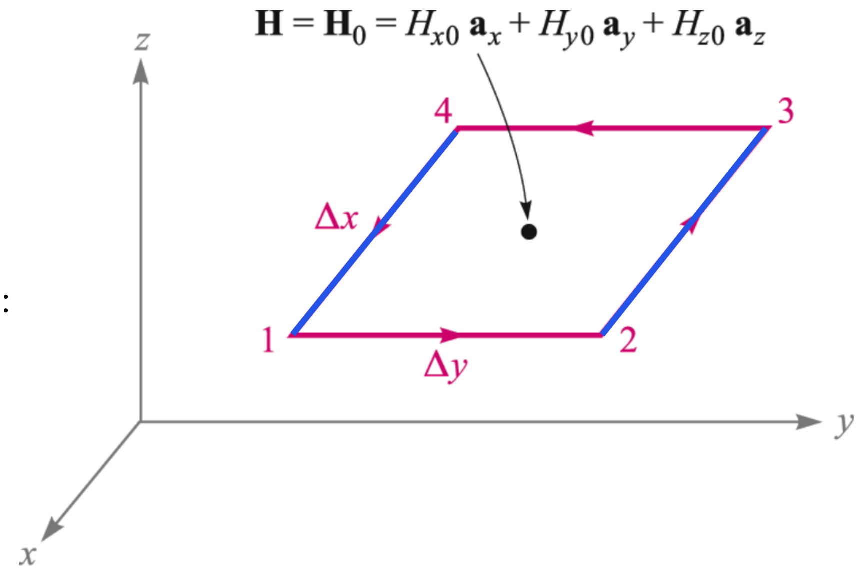
Note the path directions as specified in the figure, and how these determine the signs used .

This leaves the left and right sides…..

Contributions of *x*-Directed Path Segments

Along the right side (path 2-3):

…and the contribution from the left side (path 4-1) is:

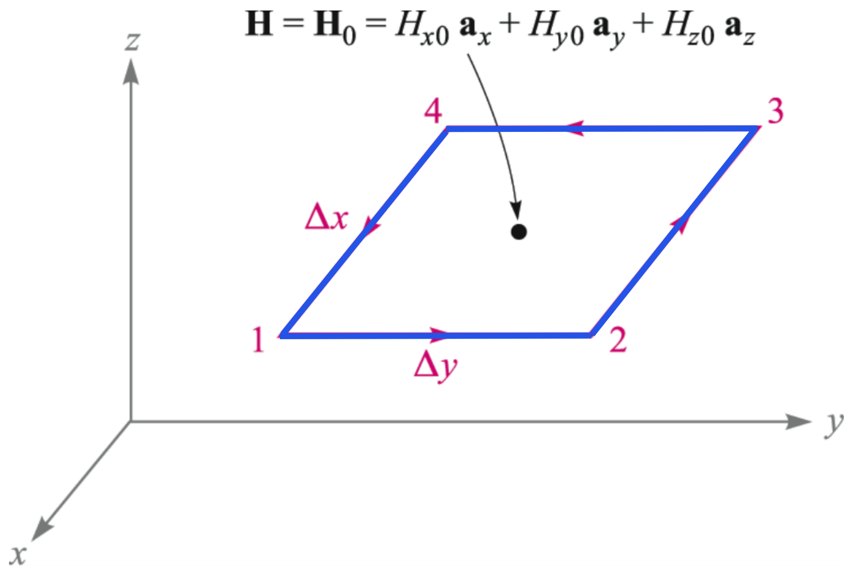


The next step is to add the contributions of all four sides to find the closed path integral:

Net Closed Path Integral

The total integral will now be the sum:

and using our previous results, the becomes:

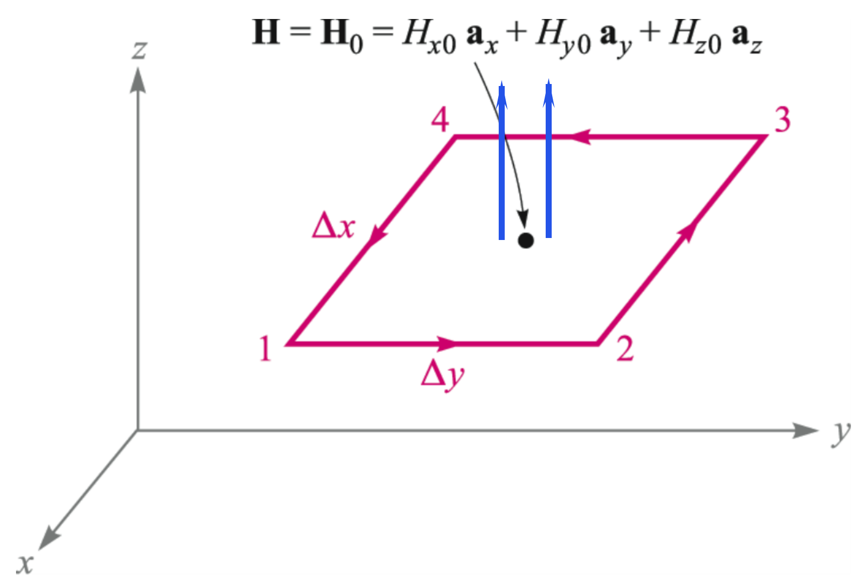


Relation to the Current Density

By Ampere's Law, the closed path integral of **H** is equal to the enclosed current, approximated in this case by the current density at the center, multiplied by the loop area:

Dividing by the loop area, we now have:

The expression becomes exact as the loop area approaches zero:



Other Loop Orientations

The same exercise can be carried with the rectangular loop in the other two orthogonal orientations. The results are:

Loop in *yz* plane:

Loop in *xz* plane:

Loop in *xy* plane:

This gives all three components of the current density field.

Curl of a Vector Field

The previous exercise resulted in the rectangular coordinate representation of the *Curl* of **H**.

In general, the curl of a vector field is another field that is normal to the original field.

The curl component in the direction *N*, normal to the plane of the integration loop is:

- **Any component of curl is given by the limit of quotient of closed line integral of a vector about a small closed path in plane normal to the component desired and of area enclosed as path shrinks to zero.**

Curl in Rectangular Coordinates

Assembling the results of the rectangular loop integration exercise, we find the vector field that comprises curl **H**:

An easy way to calculate this is to evaluate the following determinant:

Curl of any vector describes the infinitesimal rotation or circulation of a vector field in 3-D space.

Curl in Other Coordinate Systems

…a little more complicated!

Look these up as needed….

Another Maxwell Equation

It has just been demonstrated that:

…..which is in fact one of Maxwell's equations for static fields:

This is Ampere's Circuital Law in point form.

….and Another Maxwell Equation

We already know that for a *static* electric field:

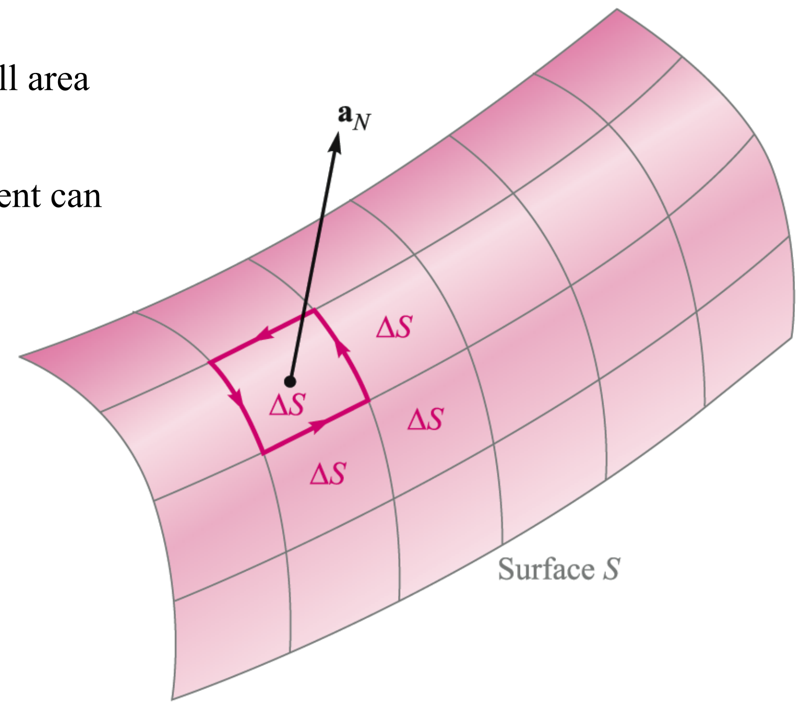
This means that: (applies to a static electric field)

Recall the condition for a conservative field: that is, its closed path integral is zero everywhere.

Therefore, a field is conservative if it has *zero curl* at all points over which the field is defined.

Curl Applied to Partitions of a Large Surface

Surface S is paritioned into sub-regions, each of small area



The curl component that is normal to a surface element can be written using the definition of curl:

or:

We now apply this to every partition on the surface, and add the results….

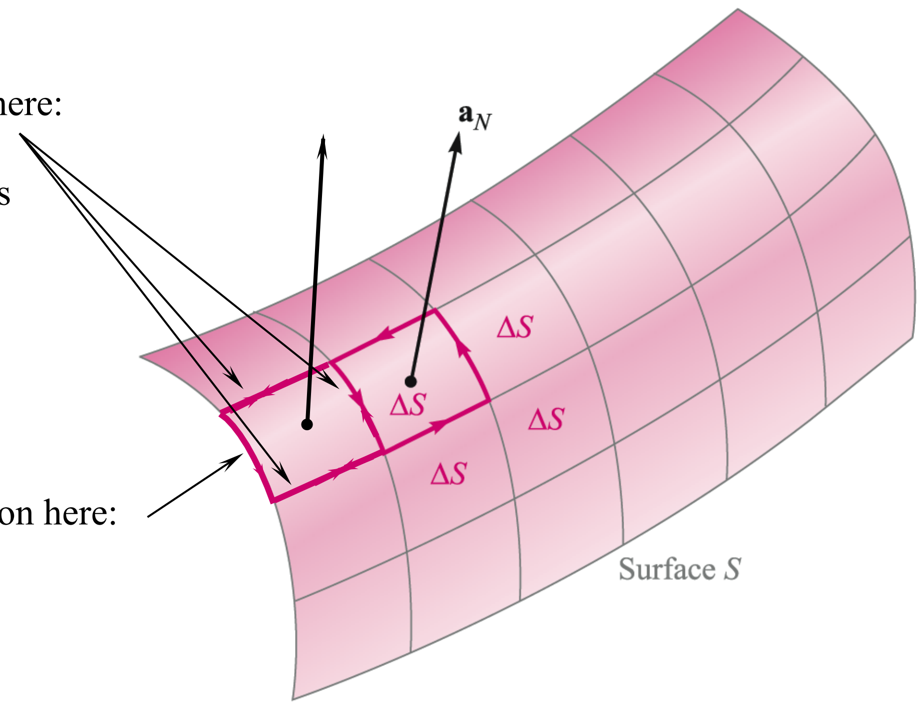
Adding the Contributions

Cancellation here:

We now evaluate and add the curl contributions from all surface elements, and note that adjacent path integrals will all cancel!

This means that the only contribution to the overall path integral will be around the outer periphery of surface *S*.

No cancellation here:



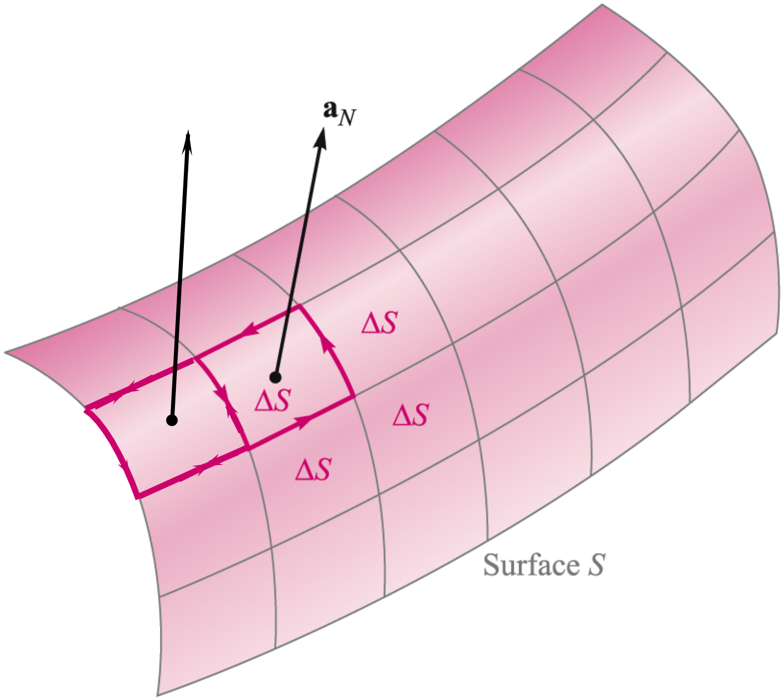
Using our previous result, we now write:

Stokes' Theorem

We now take our previous result, and take the limit as

In the limit, this side becomes the path integral of **H** over the outer perimeter because all interior paths cancel

In the limit, this side becomes the integral of the curl of **H** over surface *S*



The result is Stokes' Theorem

This is a valuable tool to have at our disposal, because it gives us two ways to evaluate the same thing!

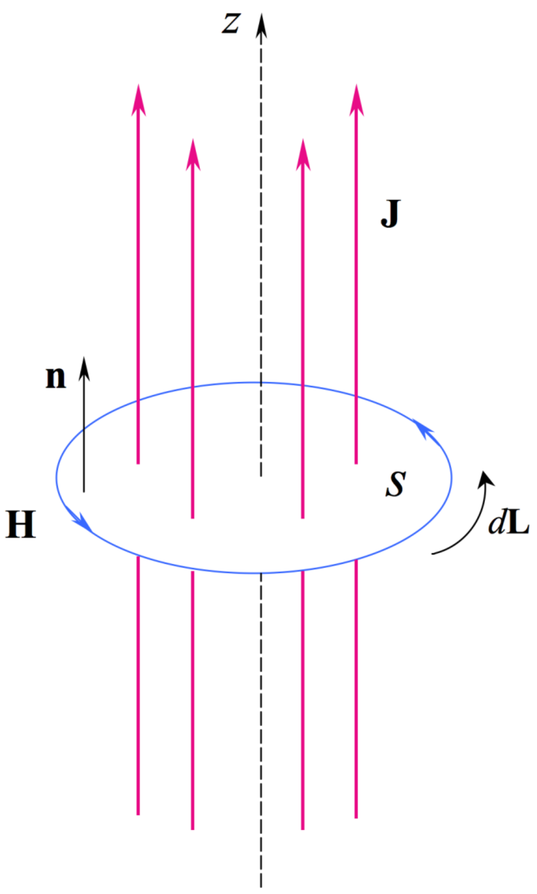
Obtaining Ampere's Circuital Law in Integral Form, using Stokes' Theorem

Begin with the point form of Ampere's Law for static fields:

Integrate both sides over surface *S*:

..in which the far right hand side is found from the left hand side using Stokes' Theorem. The closed path integral is taken around the perimeter of *S*. Again, note that we use the right-hand convention in choosing the direction of the path integral.

The center expression is just the net current through surface *S*, so we are left with the integral form of Ampere's Law:



Magnetic Flux and Flux Density

We are already familiar with the concept of electric flux:

Coulombs

in which the electric flux density in free space is:

and where the free space permittivity is

In a similar way, we can define the *magnetic flux* in units of Webers (Wb):

Webers

in which the *magnetic flux density* (or *magnetic* induction) in free space is:

and where the free space *permeability* is

A Key Property of B

If the flux is evaluated through a closed surface, we have in the case of electric flux, Gauss' Law:

If the same were to be done with magnetic flux density, we would find:

The implication is that (for our purposes) there are no magnetic charges
-- specifically, *no point sources of magnetic field exist*. A hint of this has already been observed, in that magnetic field lines always close on themselves.

Another Maxwell Equation

We may rewrite the closed surface integral of **B** using the divergence theorem, in which the right hand integral is taken over the volume surrounded by the closed surface:

Because the result is zero, it follows that

This result is known as Gauss' Law for the magnetic field in point form.

Maxwell's Equations for Static Fields

We have now completed the derivation of Maxwell's equations for no time variation. In point form, these are:

Gauss' Law for the electric field

Conservative property of the static electric field

Ampere's Circuital Law

Gauss' Law for the Magnetic Field

where, in free space:

Significant changes in the above four equations will occur when the fields are allowed to vary with time, as we'll see later.

Maxwell's Equations in Large Scale Form

The divergence theorem and Stokes' theorem can be applied to the previous four point form equations to yield the integral form of Maxwell's equations for static fields:

Gauss' Law for the electric field

Conservative property of the static electric field

Ampere's Circuital Law

Gauss' Law for the magnetic field

Scalar Magnetic Potential

We are already familiar with the relation between the scalar electric potential and electric field:

So it is tempting to define a scalar magnetic potential such that:

This rule must be consistent with Maxwell's equations, so therefore:

But the curl of the gradient of any function is identically zero! Therefore, the scalar magnetic potential is valid only in regions where the current density is zero (such as in free space).

So we define scalar magnetic potential with a condition:

Further Requirements on the Scalar Magnetic Potential

The other Maxwell equation involving magnetic field must also be satisfied. This is:

in free space

Therefore:

..and so the scalar magnetic potential satisfies Laplace's equation (again with the restriction that current density must be zero:

Vector Magnetic Potential

We make use of the Maxwell equation:

.. and the fact that the divergence of the curl of any vector field is identically zero (show this!)

This leads to the definition of the *magnetic vector potential,* **A**:

Thus:

and Ampere's Law becomes

Equation for the Vector Potential

We start with:

Then, introduce a vector identity that defines the *vector Laplacian*:

Using a (lengthy) procedure (see Sec. 7.7) it can be proven that

_x0014_We are therefore left with