Electrostatic

Coulomb's Law

- Nearly all electric field vary to some extent with time, but for many problems the time variation is slow and the field may be considered stationary in time.

- Coulomb stated that, the force between two very small objects separated in vacuum or in free space by a distance which is large compared to their size is proportional to charge on each and inversely proportional to the square of the distance between them.

$$F = k\frac{Q_1 Q_2}{R^2}$$

where, Q1 and Q2 are the charges, R is the separation between them and k is the proportionality constant.

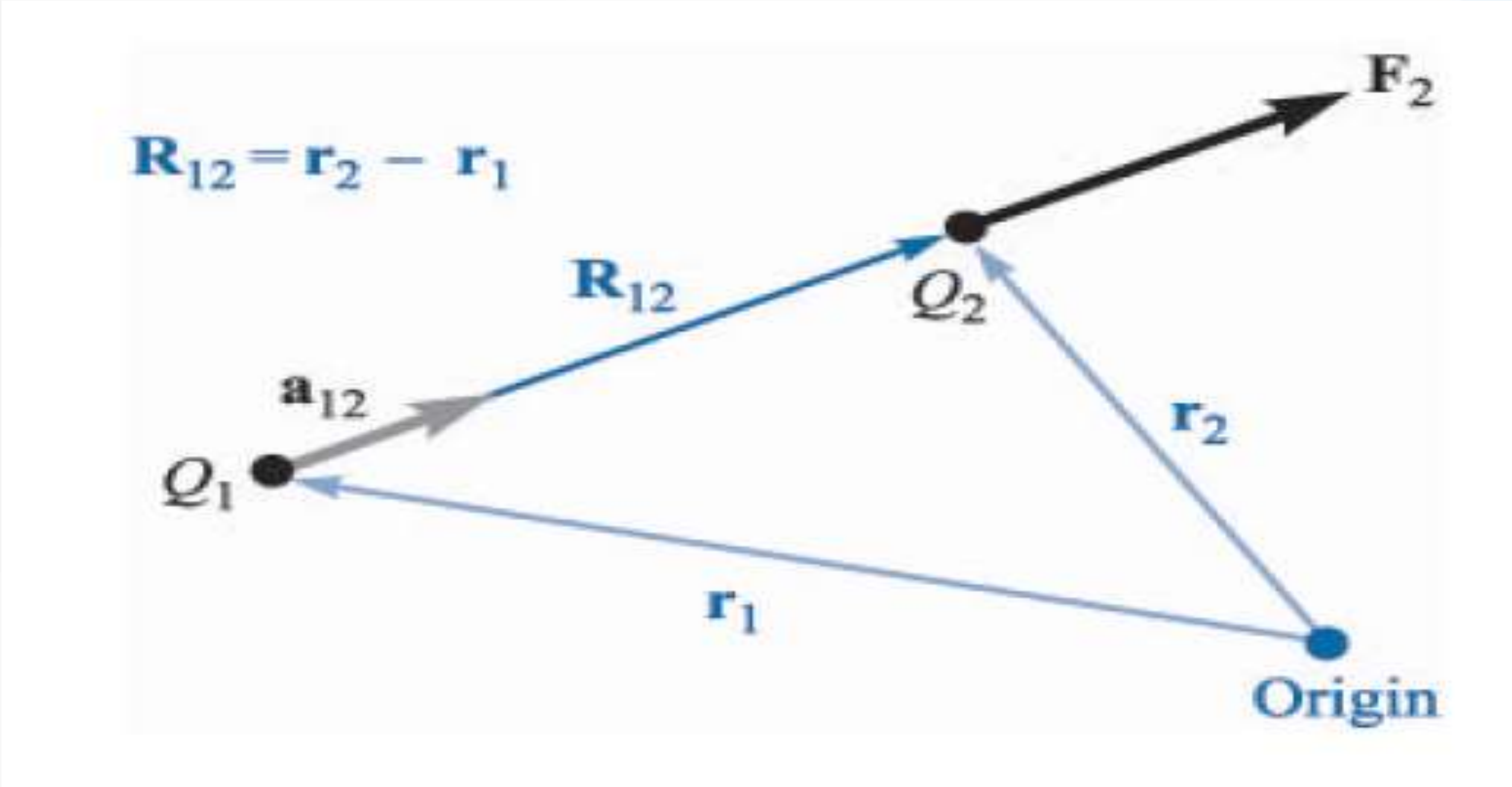
- The proportionality constant k is given by,

$$k = \frac{1}{4\pi\epsilon_0}$$

- So

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

- The force between two charges is repulsive if charges are alike and attractive if they are of opposite sign.



- Let vector r_1 located charge Q1 and vector r_2 located charge Q2, then the vector R12= r_2- r_1.
- The force F2 shown the force on the charge Q2, when the both charges of the same sign.

- The vector form of the coulomb's law is,

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$

where a12 is the unit vector in the direction R12,

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

- For coulomb's law is also write as,

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$

- The coulomb's law is linear, if we multiply Q1 by a factor n , the force on the Q2 is also multiply by the same factor n.

Electric field intensity:

- if one charge, say Q1 is fixed, and another charge is rotating around it, so there exist a force on the second charge everywhere , the second charge is call a test charge Qt.
- The force on the Qt due to the first is given by,
- The force per charge is,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

$$\frac{\mathbf{F}_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

- The equation is function of charge Q1, and the line segments are directed towards the test charge, this described the vector field, called electric field intensity.
- So finally electric field intensity E is,

$$\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}$$

- The above equation give, the electric field intensity due to single charge Q1 in vacuum.

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

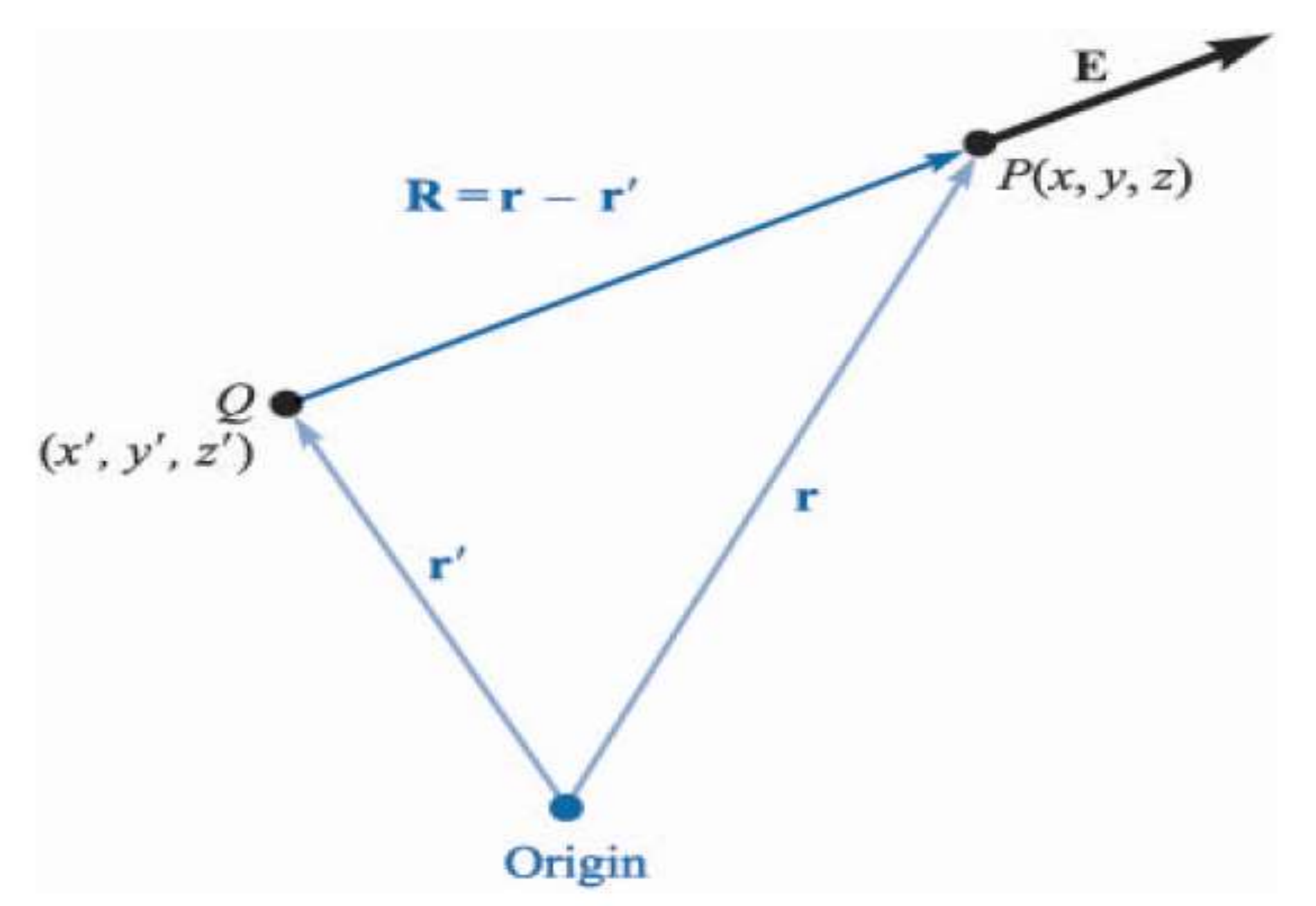
- In general,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R$$

- Here R is a vector, directed from point charge Q to the point at which E is desired, and a_R is the unit vector in the direction of R.
- In Cartesian coordinate, the electric field intensity is given by

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0(x^2+y^2+z^2)}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_x + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_y + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_z\right)$$

- For the charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z,$
- We find the field at $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$



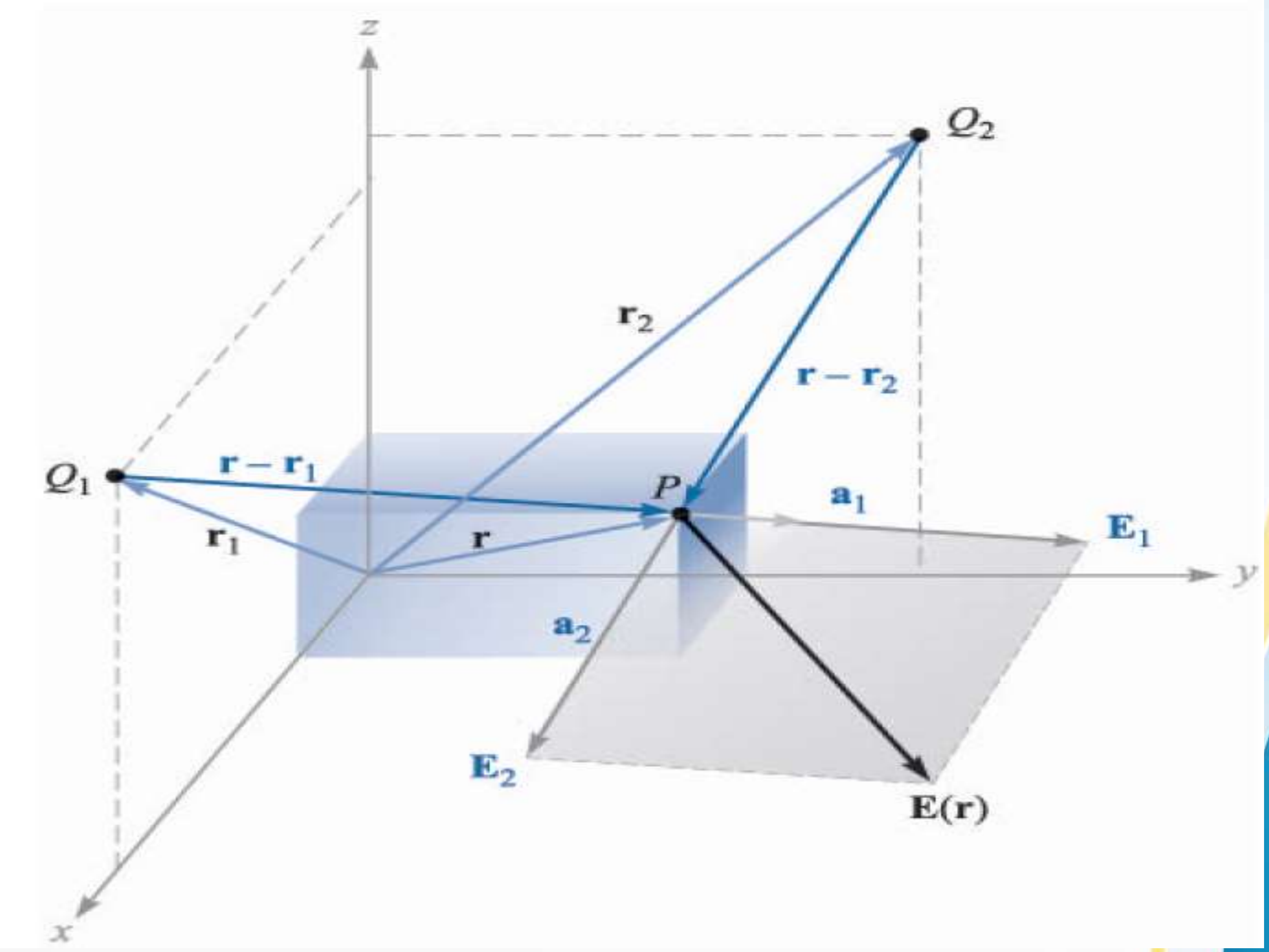
$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$$

- Suppose there are two charges Q1 and Q2, then the electric field intensity due to these two charges is,

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

- a_1 and a_2 are the unit vectors in the direction of $(r-r_1)$ and $(r-r_2)$ respectively.



z
Q_2
$\mathbf{r}_2$
$\mathbf{r} - \mathbf{r}_2$
Q_1
$\mathbf{r} - \mathbf{r}_1$
P
$\mathbf{a}_1$
$\mathbf{E}_1$
$\mathbf{r}_1$
$\mathbf{r}$
y
$\mathbf{a}_2$
$\mathbf{E}_2$
$\mathbf{E(r)}$
x

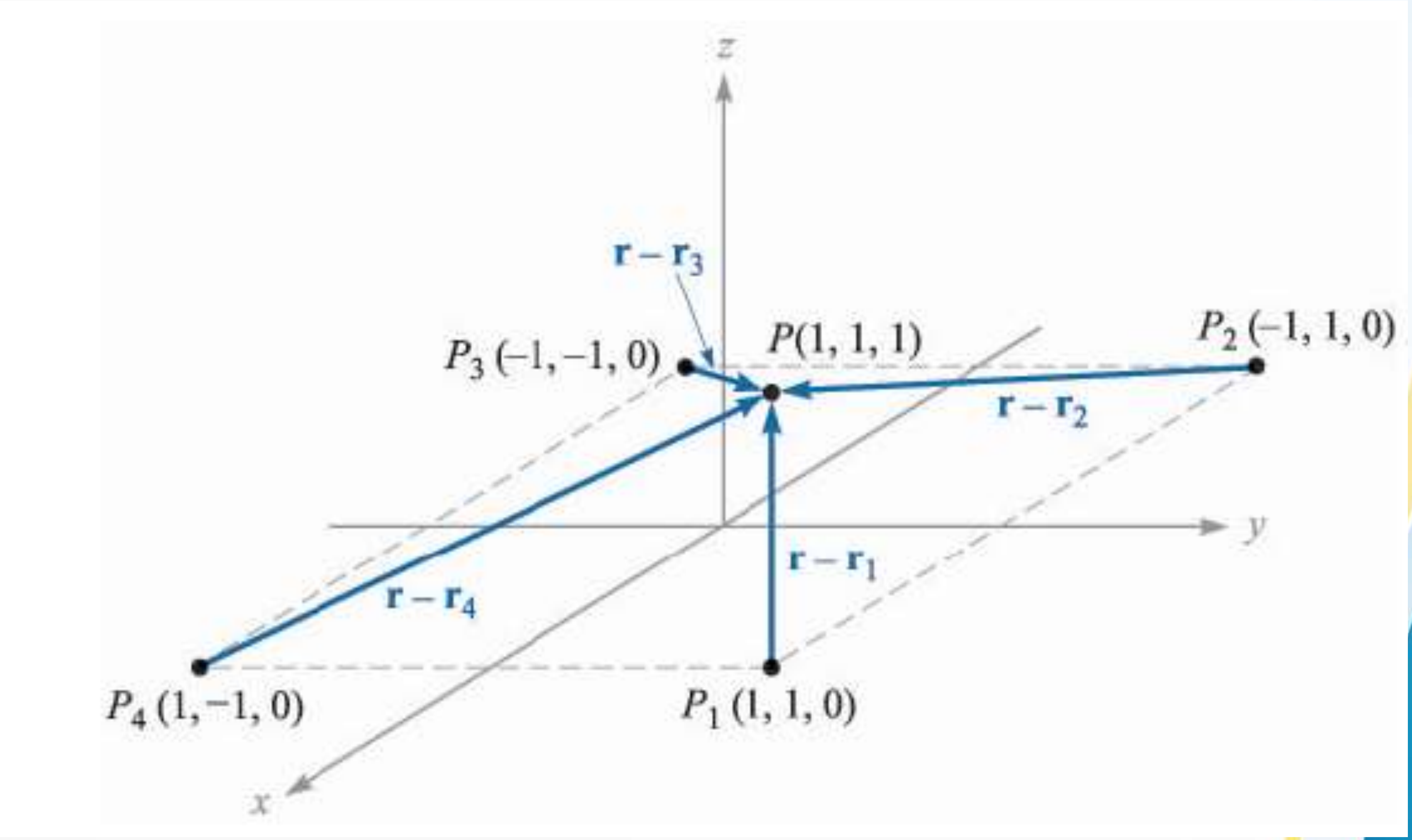
- If we add more charges at other positions, the field due to n points charges is

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2 + \ldots + \frac{Q_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|^2}\mathbf{a}_n$$

- This can be write as,

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

Ex 2.2



Volume Charge Distribution

- The volume charge density is ρ_v c/m^3
- The small amount of charge ΔQ in a small volume Δv is,

$$\Delta Q = \rho_v \Delta v$$

- We may defined ρ_v, mathematically by using limiting process,

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

- The total charge within the vo grating through out the volume,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

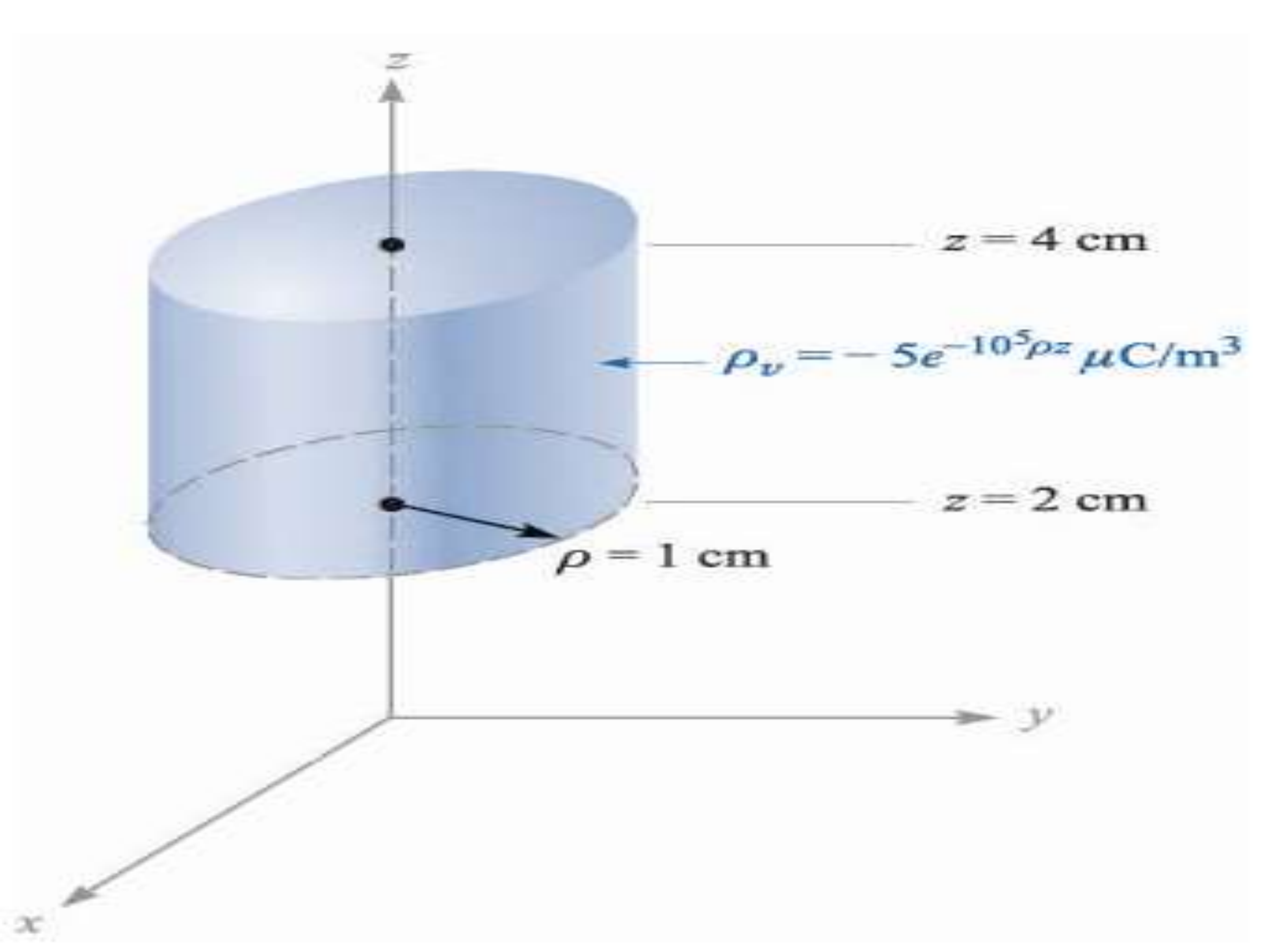
- The incremental electric field at distance 'r' produced by an incremental charge ΔQ at r' is,

$$\Delta \mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \frac{\rho_v \Delta v}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

- If we sum contributions of all the charges in a given volume, the total electric field is obtained by an integration,

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

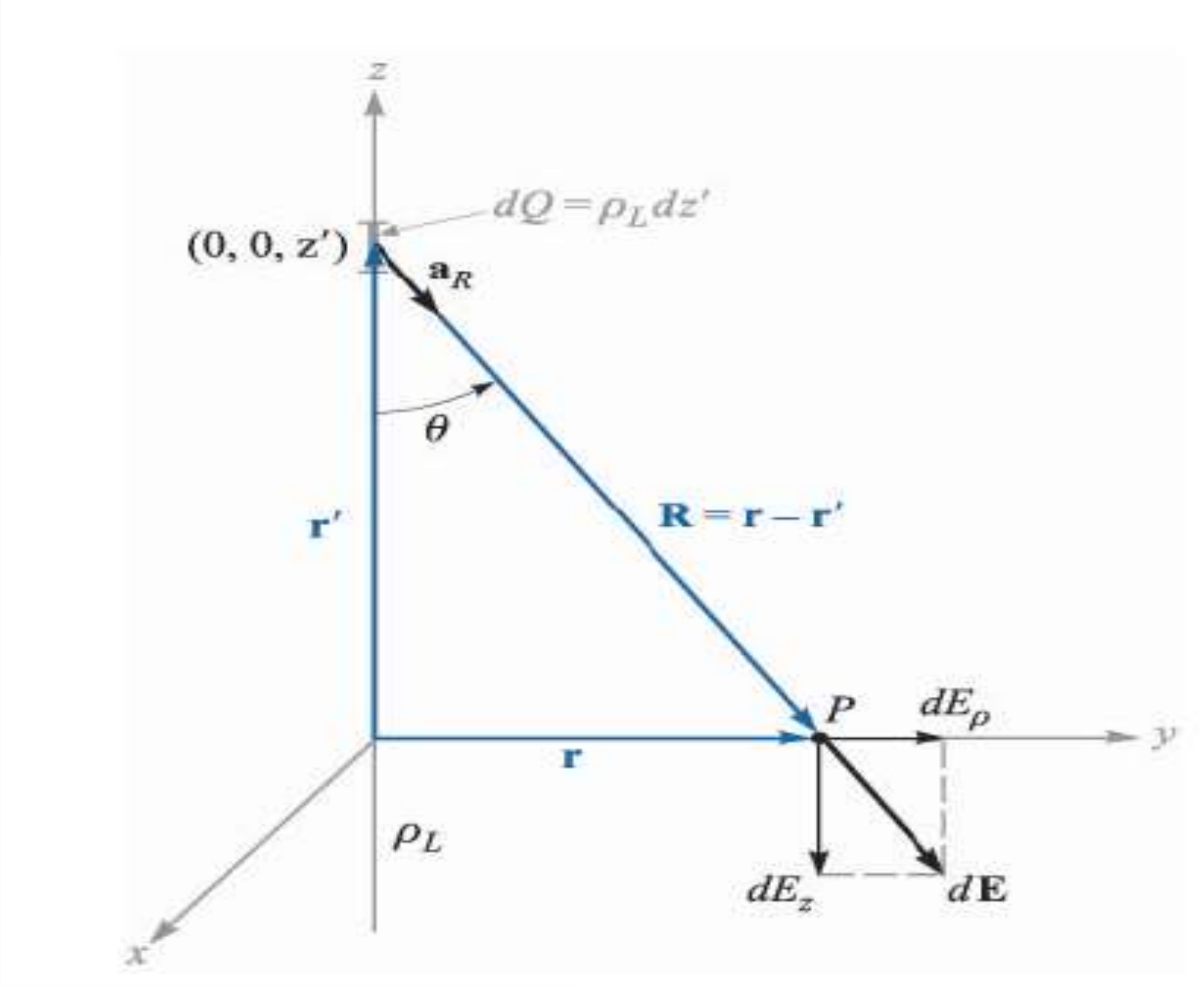
Ex 2.3



Field of Line Charge

- The charge of a very sharp beam in cathode ray tube or a charged conductor of very small radius, all these charge distribution can be treated as a line charge with charge density ρ_L c/m.
- Now let us assume that, a line charge extending along Z- axes, in a cylindrical co-ordinate system from $-\alpha$ to $+\alpha$.
- From the figure, as we move around the line charge, varying Φ while ρ and Z keeping constant, the line charge seems to be constant from every angle. In other words, the azimuthal symmetry is present, no field component will vary with Φ.
- As we move along the Z axes up and down while keeping ρ and Φ constant, the line charge appears to be unchanged.

- This is an axial symmetry, the field will not vary with Z.



- As we vary ρ while keeping z and Φ constant, according to coulomb's law, the field becomes weaker as ρ increases. Hence the field is vary only with ρ.

- Consider an incremental length of line charge which act as a point charge and produce an incremental electric field at particular point.
- This point charge does not contribute to produce Φ component of electric field, so E_Φ is zero.

- Each point cahrge does produce E_z and E_ρ components. However, contribution to E_z by elemnts of charge which are equal distances above & below the point at which we are determining the E field will cancel.

- E_ρ component is there and is vary with ρ only.

- Consider a point P(0,y,0) on the Y axes at which we want to determine the field.

- To find the incremental field at point P due to incremental charge $dQ = \rho_L \, dZ'$, we have

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$, $\mathbf{r}' = Z'\,\mathbf{a}_z$,

$$\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

- Therefore

$$d\mathbf{E} = \frac{\rho_L dz'(\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

- Since only E_ρ comp … bove equation as,

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

- The total electric field is given by,

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

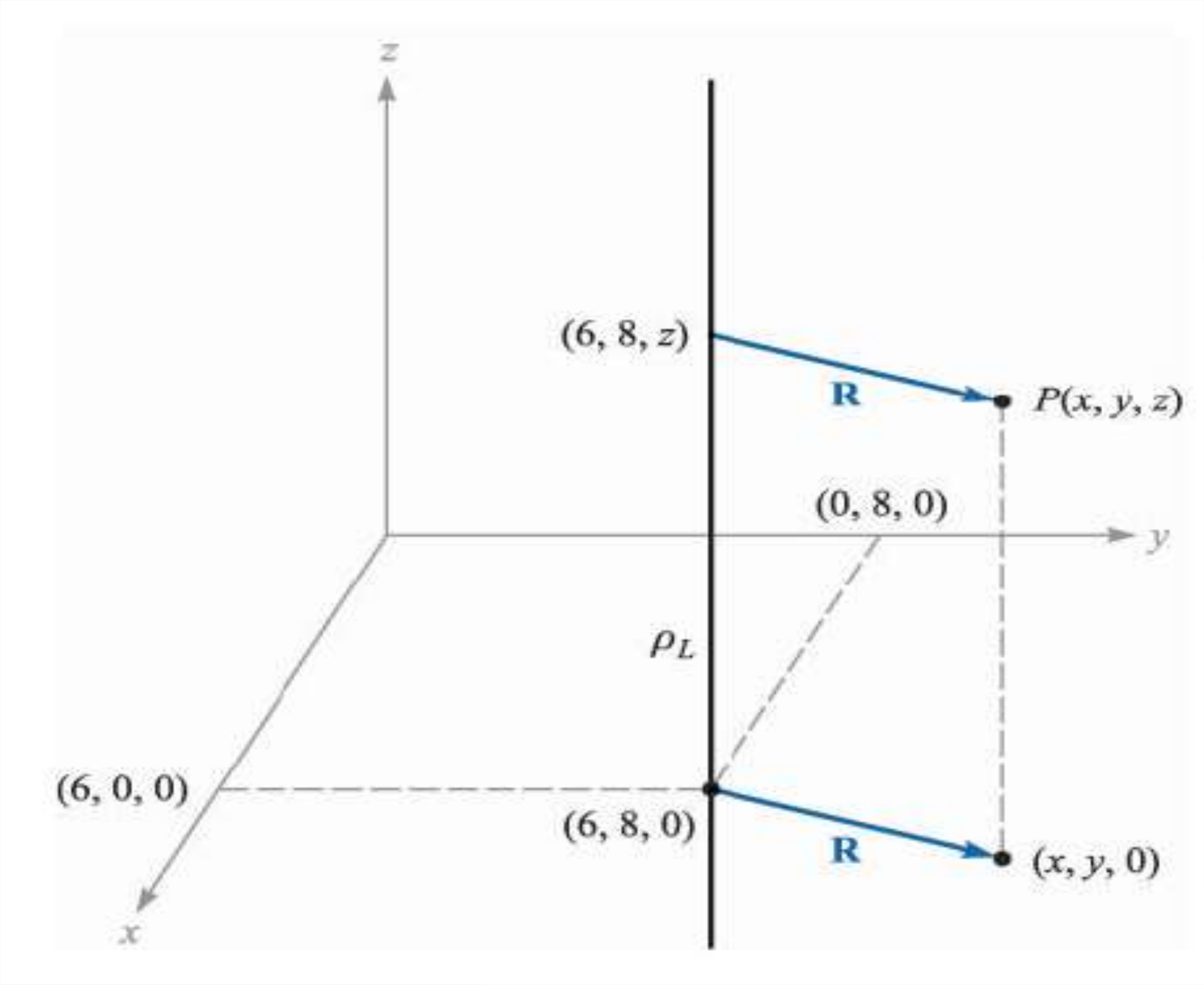
- For integration take $z' = \rho \cot\theta$ $R = \rho\csc\theta$

$$dE_\rho = \frac{\rho_L\, dz'}{4\pi\epsilon_0 R^2}\sin\theta = -\frac{\rho_L \sin\theta\, d\theta}{4\pi\epsilon_0\rho}$$

$$E_\rho = -\frac{\rho_L}{4\pi\epsilon_0\rho}\int_{\pi}^{0}\sin\theta\, d\theta = \frac{\rho_L}{4\pi\epsilon_0\rho}\cos\theta\Big]_{\pi}^{0}$$

$$= \frac{\rho_L}{2\pi\epsilon_0\rho}$$

- Let us consider infinite line charge distribution parallel to the Z-axes at X=6 and Y=8, as shown in figure.



- Find E at general point P(x,y,z).
- The radial distance between line charge and point P is,

$$R = \sqrt{(x-6)^2 + (y-8)^2}$$

- Thus,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\sqrt{(x-6)^2 + (y-8)^2}}\mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

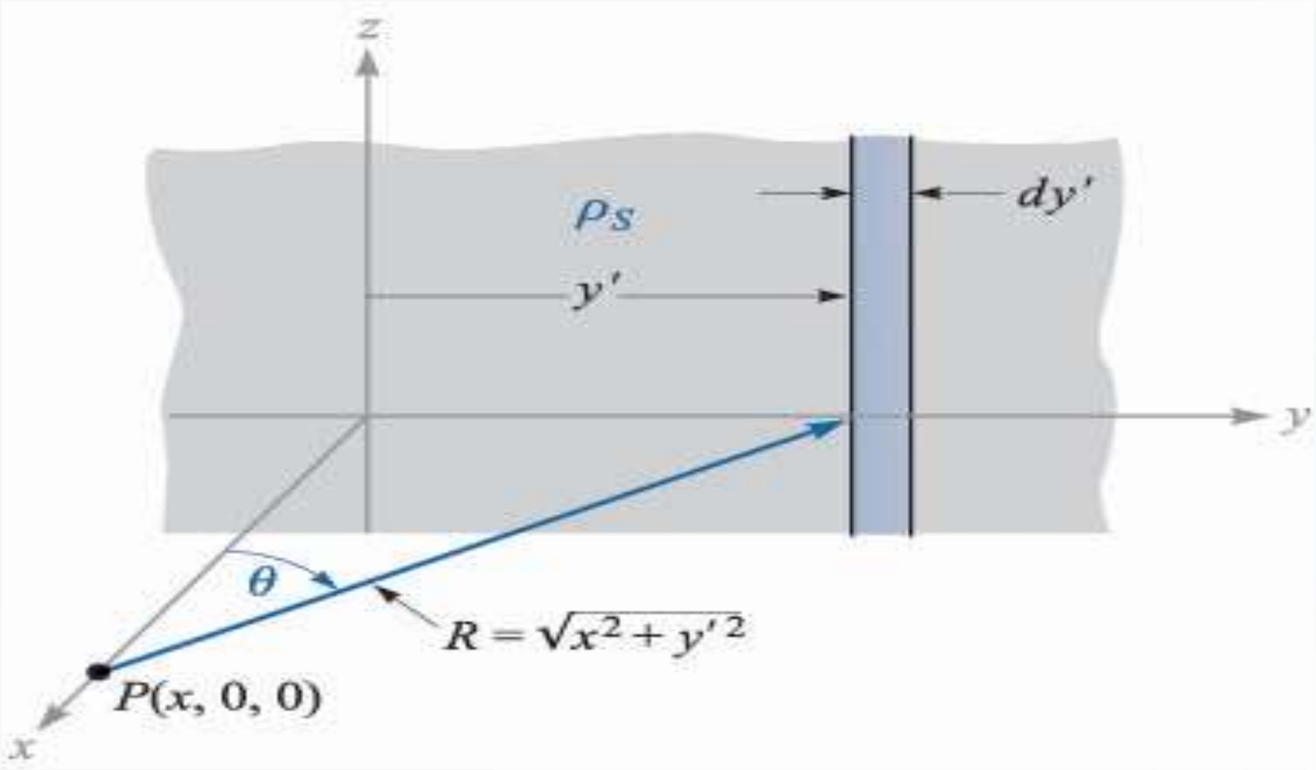
$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0}\frac{(x-6)\mathbf{a}_x + (y-8)\mathbf{a}_y}{(x-6)^2 + (y-8)^2}$$

Sheet Charge Distribution

- Another basic charge distribution is infinite sheet charge having a uniform density of ρ_s C/m^2.
- Such a charge distribution is found on the conductors of the strip transmission line, or on the parallel plate capacitor.
- Let as place a sheet of charge in the yz plane and again consider symmetry.
- Due to symmetry, charge along the 'y' and along the 'z', located symmetrically, so there are no Ey and Ez component of electric field. The electric field has only Ex component, and this component is a function of 'x' alone.

- Let us use the field of the infinite line charge, by dividing the infinite sheet into differential-width strips.
- The line charge density, or charge per unit length, is $\rho_L = \rho_s dy'$ and the distance from this line charge to our general point P on the x axis is

$$R = \sqrt{x^2 + y'^2}$$



- The Ex, due to this differential-width strip is then,

$$dE_x = \frac{\rho_S \, dy'}{2\pi\epsilon_0\sqrt{x^2 + y'^2}}\cos\theta = \frac{\rho_S}{2\pi\epsilon_0}\frac{x dy'}{x^2 + y'^2}$$

- Adding the effects of all the strips,

$$E_x = \frac{\rho_S}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\,dy'}{x^2 + y'^2} = \frac{\rho_S}{2\pi\epsilon_0}\tan^{-1}\frac{y'}{x}\Big]_{-\infty}^{\infty} = \frac{\rho_S}{2\epsilon_0}$$

- So electric field due to sheet charge is,

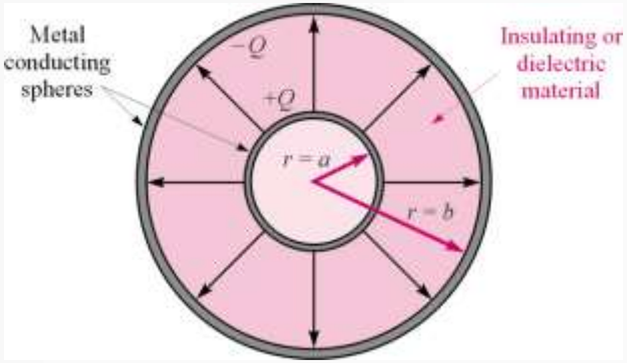
$$\mathbf{E} = \frac{\rho_S}{2\epsilon_0}\mathbf{a}_N$$

where a_N is unit vector, which is normal to the sheet and directed outward.

Faraday's Experiment

Faraday had of pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material, or simply dielectric) which would occupy the entire volume between the concentric spheres.

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

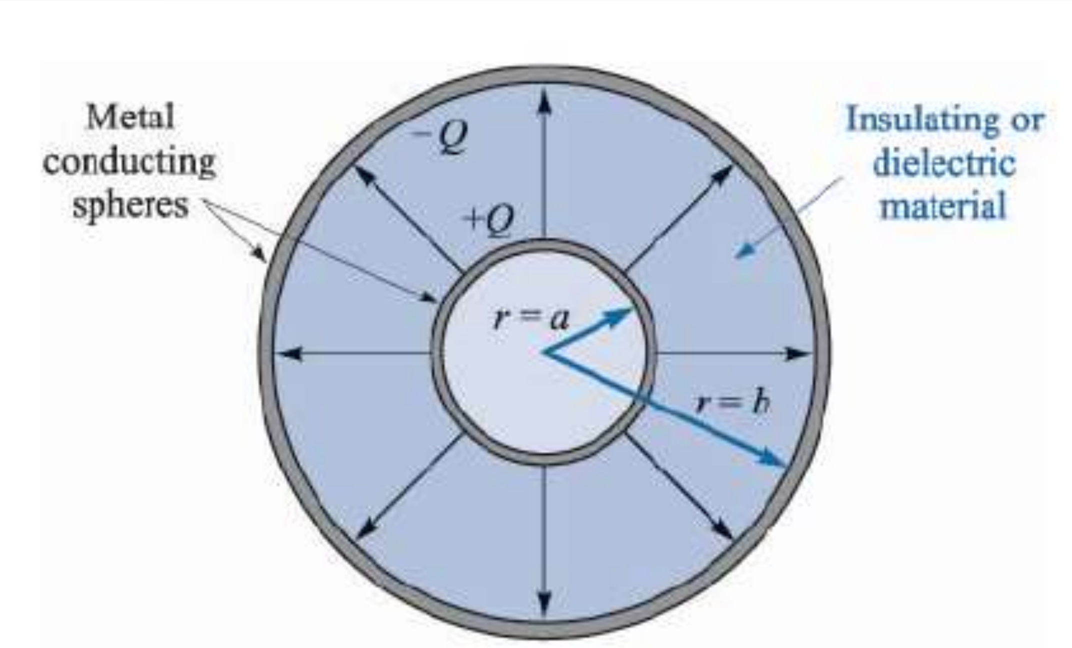


Electric Flux / displacement flux

- Faraday found that total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere.
- There was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we refer to this as displacement flux or electric flux.
- Faraday's also showed that, the electric flux is directly proportional to the charge on the sphere.
- If electric flux is denoted by ψ and total charge by Q, then from faraday's ,

$$\Psi = Q$$

- Now consider two sphere of radius ‘a’ and radius ‘b’ as shown in figure.



- On the inner sphere, the charge is Q, while on the outer sphere the charge is –Q.
- The paths electric flux ψ is extending from inner sphere to the outer sphere.

- At the surface of the inner sphere, ψ coulombs of electric flux are produce by the charge Q coulombs distributed uniformly over the surface having an area of $4\pi a^2\ \mathrm{m}^2$.

- The density of the flux at this surface is, $\Psi/4\pi a^2$ $\mathrm{C/m}^2$, this density of flux is ric flux density, denoted by D.

- The direction of 'D' at a point is the direction of the flux line at that point, and the magnitude is given by the number of flux line crossing the surface normal to the line divided by the surface area.

- Form the figure, the electric flux density is in radial direction and has value of,

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

- The electric flux density at a radial distance ‘r’.

$$a \le r \le b,$$

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

- If the radius of the inner sphere is reduced to approximately zero, it become a point charge but still charge on it is Q coulombs and the electric flux density at a radial distance ‘r’ is still remain same,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

- The electric field intensity due to point charge is,

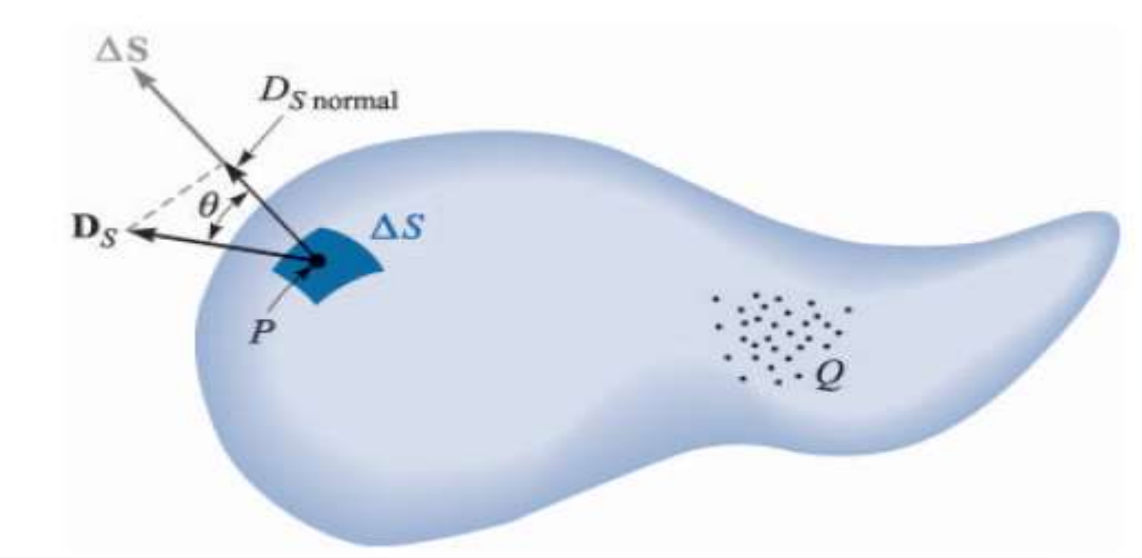
$$\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r$$

- So in free space,

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

Gauss Law

- This law is a generalization of faraday's experiments.
- Statement: ***" the electric flux passing through any closed surface is equal to total charge enclosed by that surface".***
- Consider a distribution of charge, shown as a cloud of point charges in the figure.



- If total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface.
- At every point, electric flux density vector have some value, D_s and it is vary in magnitude and direction from one point to another.
- At point P, consider an incremental element of the surface and let D_s make an angle of θ with .
- The flux crossing incremental surface is then the product of the normal component of D_s and ΔS ,

ΔS

ΔS

- Total flux crossing the close $\Delta \Psi$:e $= \mathbf{D}_S \cdot \Delta \mathbf{S}$ ing over the surface,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

- The mathematical formula of the gauss law is,

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

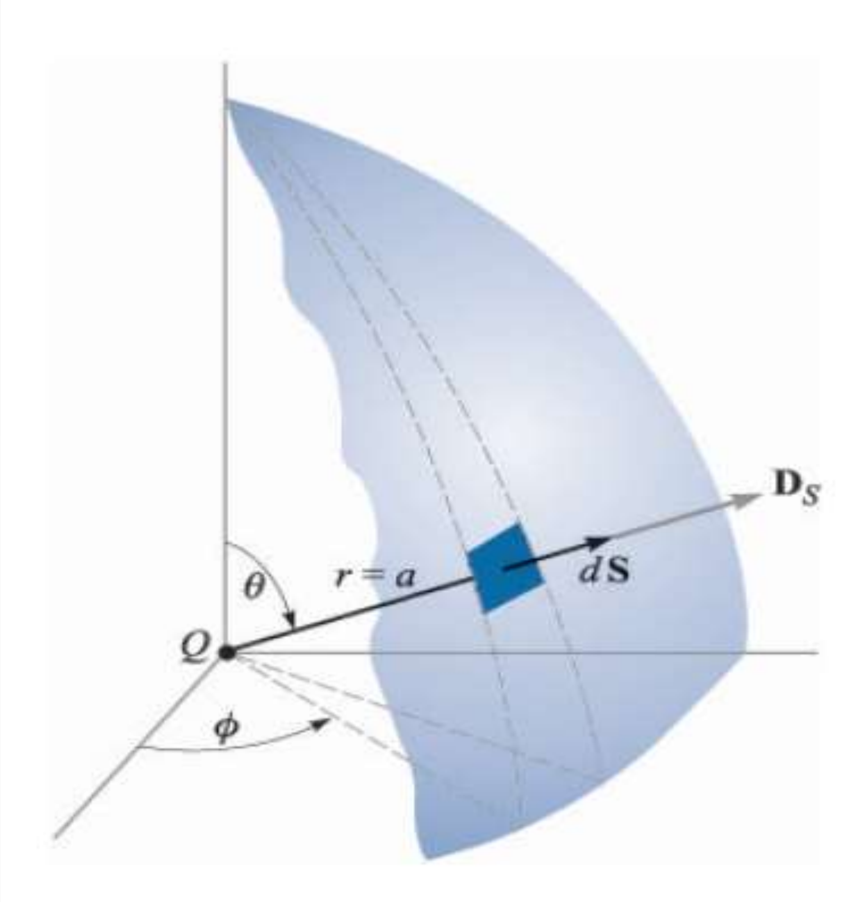
- If we consider a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

- The gauss law,

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv$$

Application of Gauss Law



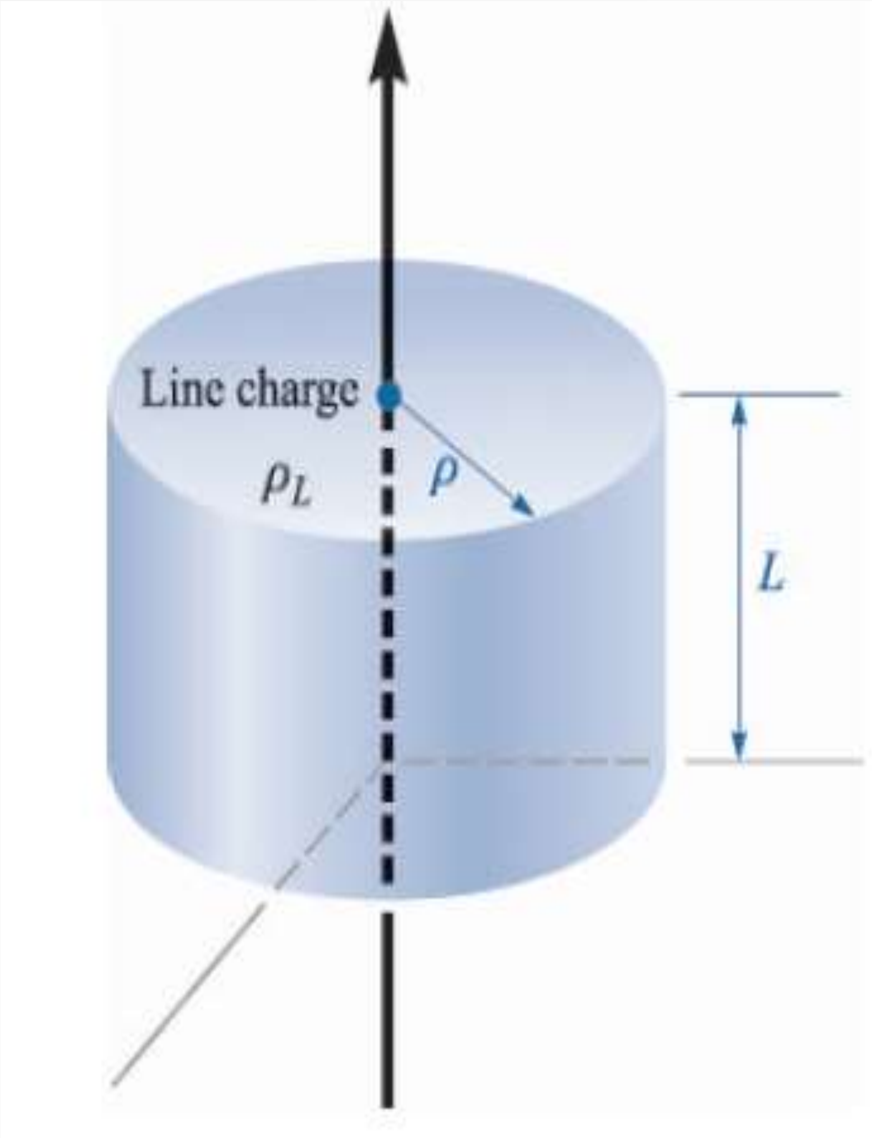
- Application of gauss law to the field of a point charge Q on a spherical closed surface of radius 'a'.

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta \, d\phi$$

$$\int_0^{2\pi} \frac{Q}{4\pi}(-\cos\theta)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$



- The Gaussian surface for an infinite uniform line charge, the length is 'L' and radius is ρ

$$Q = \oint_{cyl} \mathbf{D}_S \cdot d\mathbf{S} = D_S \int_{sides} dS + 0 \int_{top} dS + 0 \int_{bottom} dS$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S 2\pi\rho L$$

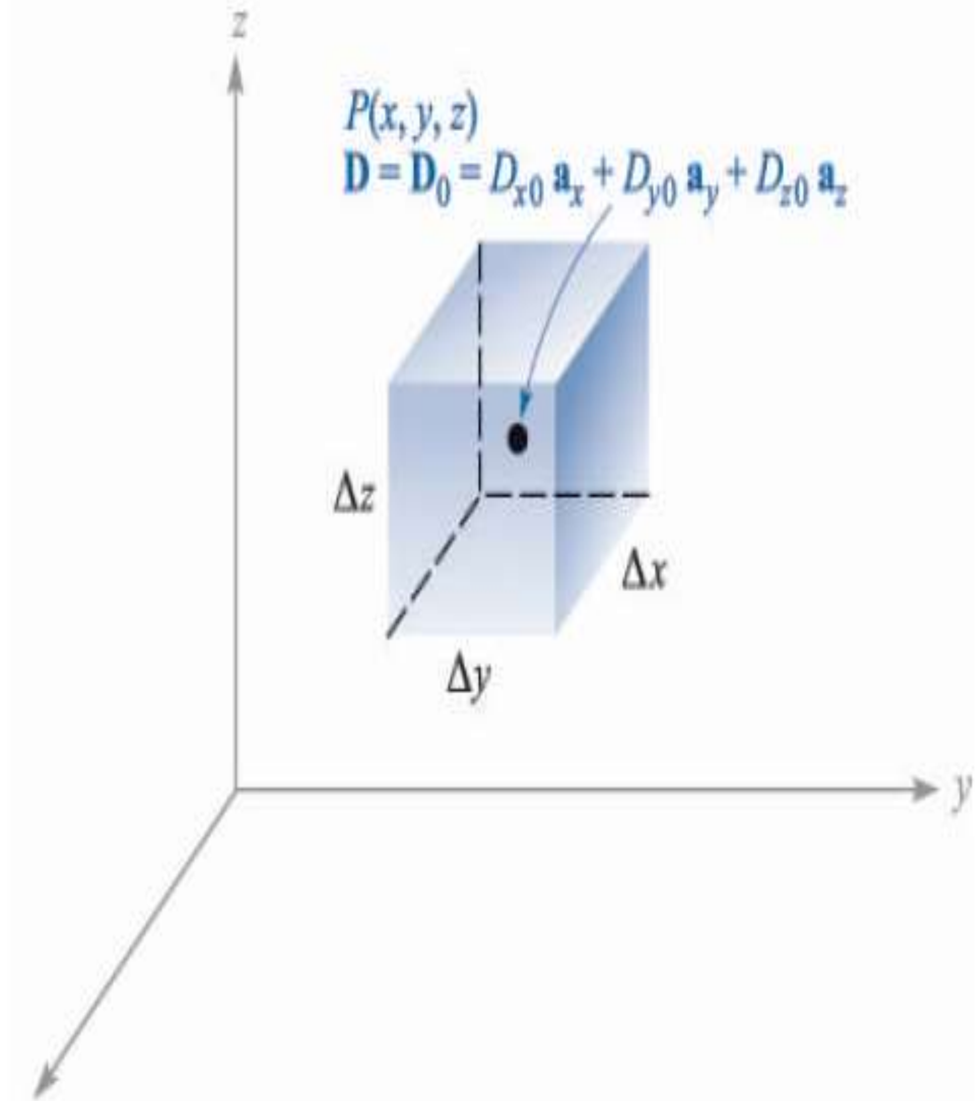
$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$

$$Q = \rho_L L$$

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}$$

Point form (differntial equtaion) of Gauss law



- Consider a differential sized Gaussia surface about the point 'P'.
- As shown in figure, the closed surface, small rectangular box, centered at 'P having a sides of length Δx, Δy, Δz.
- The value of D at point 'P' is
- As per the gauss law,

$$\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z.$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

- In order to integrate over the closed surface, the integral must break up into six integrals,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

- Since the surface is very small, D is essentially constant, so

$$\begin{aligned}\int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y \, \Delta z \, \mathbf{a}_x \\ &\doteq D_{x,\text{front}} \Delta y \, \Delta z\end{aligned}$$

- The front face is at distance of $\Delta x/2$ from the P, and hence

$$\begin{aligned}D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{ rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\end{aligned}$$

- So we have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

- Consider the integral over the back surface,

$$\begin{aligned} \int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y \, \Delta z \end{aligned}$$

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

- If we combined these two integral,

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

- By the same way,

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

- So

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta v$$

Divergence

- Divergence is the outflow of flux from a small closed surface area (per unit volume) as volume shrinks to zero.

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right)$$

$$\text{div } \mathbf{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, D_\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi}$$

Thank You