Example 4.37 *A rotor of mass 4 kg is mounted on 1 cm diameter shaft at a point 10 cm from one end. The 25 cm long shaft is supported by bearings. Calculate the critical speed. If the centre of gravity of the disc is 0.03 mm away from the geometric centre of rotor, find the deflection of the shaft when its speed of rotation is 5000 r.p.m. Take $E = 1.96 \times 10^{11}\ N/m^2$. Find the critical speed when the rotor is mounted midway on the shaft.*

(DCRU, Murthal, 2011)

Solution. The critical speed is given as

$$\omega_c = \sqrt{\frac{g}{\delta}} \quad \text{and} \quad \delta = \frac{Wbx}{6EIl}(l^2 - x^2 - b^2)$$

$$W = mg = 4 \times 9.81\ \text{kg} = 39.24\ \text{N}$$

$$l = 25\ \text{cm} = 25\ \text{m}, \qquad x = 0.10\ \text{m}$$

$$b = .25 - .10 = 0.15\ \text{m}, \qquad d = 1.0\ \text{cm}$$

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times (0.01)^4 = 4.9 \times 10^{-10}\ \text{m}^4$$

$$\delta = \frac{39.24 \times .15 \times .10\,(0.25^2 - 0.10^2 - 0.15^2)}{6 \times 1.96 \times 10^{11} \times 4.9 \times 10^{-10} \times 0.25} = 1.2257 \times 10^{-4}\ \text{m}$$

Critical speed

$$\omega_c = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.80}{1.2257 \times 10^{-4}}} = 282.7\ \text{rad/sec}$$

$$= \frac{282.7 \times 60}{2\pi} = 2701\ \text{rpm.}$$

We can find
$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 5000}{60} = 523.33 \text{ rad/sec}$$

Given,
$$e = 0.03 \text{ mm}$$
$$r = \omega/\omega_c = \frac{523.33}{282.7} = 1.85, \qquad r^2 = 3.42$$
$$x = \frac{er^2}{r^2 - 1} = \frac{.03 \times 3.42}{3.42 - 1} = 0.042 \text{ mm}$$

When the rotor is mounted midway on the shaft, the static deflection is given by
$$\delta = \frac{Wl^3}{48EI} = \frac{39.24 \times (.25)^3}{48 \times 1.96 \times 10^{11} \times 4.9 \times 10^{-10}} = 1.33 \times 10^{-4} \text{ m}$$
$$\omega_c = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.8}{1.33 \times 10^{-4}}} = 271.5 \text{ rad/sec}$$

Example 4.38 *A shaft of 2.5 cm diameter, freely supported by bearings 75 cm apart, carries a single concentrated load of 196.2 N midway between the bearings. Determine the first critical speed. Assume that shaft material has a density of 8×10^3 kg/m^3 and E is 2.1×10^{11} N/m^2.*

(Roorkee Uni., 96-97 ; KUK, 2012)

Solution. If a concentrated load is acting at the centre of beam whose weight is to be taken into account, the static deflection is given by

$$\delta = \frac{\left(W + \frac{17}{35}\rho l g\right) l^3}{48EI} \qquad \text{(From Strength of Materials)}$$

and critical speed
$$\omega_c = \sqrt{\frac{g}{\delta}}$$

Given
$$W = 196.2 \text{ N}, \quad l = 75 \text{ cm} = 0.75 \text{ m}$$
$$\rho = \text{mass per unit length}$$
$$= 8 \times 10^3 \times \frac{\pi}{4} d^2 = 8 \times 10^3 \frac{\pi}{4} (2.5 \times 10^{-2})^2 = 3.925 \text{ kg/m}$$
$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} \times (2.5 \times 10^{-2})^4 = 1.916 \times 10^{-8} \text{ m}^4$$

Using the values in above equation, one gets
$$\delta = \frac{\left(196.2 + \frac{17}{35} \times 3.925 \times (0.75) \times 9.81\right)(0.75)^3}{48 \times 2.1 \times 10^{11} \times 1.916 \times 10^{-8}} = 4.592 \times 10^{-4} \text{ m}$$
$$\sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{\ldots}} = 146.16 \text{ rad/sec}$$

Substituting the above values in the equations of motion, we get

$$(3k - m\omega^2)A_1 - kA_2 - kA_3 = 0$$
$$-kA_1 + (3k - m\omega^2)A_2 - kA_3 = 0$$
$$-kA_1 - kA_2 + (3k - m\omega^2)A_3 = 0$$

The frequency equation is obtained as

$$\begin{vmatrix} (3k - m\omega^2) & -k & -k \\ -k & (3k - m\omega^2) & -k \\ -k & -k & (3k - m\omega^2) \end{vmatrix} = 0$$

Expanding the above determinant, we get

$$\omega^6 - (9k/m)\omega^4 + (24k^2/m^2)\omega^2 - (16k^3/m^3) = 0$$

Solving it, we get three values of natural frequencies as

$$\omega_1 = \sqrt{\frac{k}{m}} \text{ rad/sec}, \qquad \omega_2 = \omega_3 = 2\sqrt{k/m} \text{ rad/sec.}$$

Example 6.9 *A shaft of negligible weight 6 cm diameter and 5 metres long is simply supported at the ends and carries four weights 50 kg each at equal distance over the length of the shaft. Find the frequency of vibration by Dunkerley's method. Take $E = 2 \times 10^6$ kg/cm².*

(K.U., 2008)

Solution. The system is shown in Fig. 6.24.

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64} \times 6^4 = 63.585 \text{ cm}^4$$

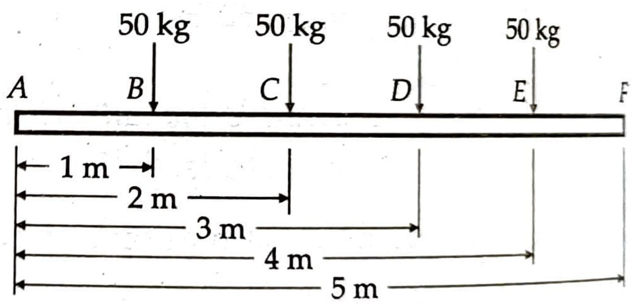

Fig. 6.24

The general expression for static deflection because of point load W is given by

$$y = \frac{Wl_1^2 l_2^2}{3EIl}$$

So static deflection at point B

$$y_B = \frac{50 \times 100^2 \times 400^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.419 \text{ cm}$$

$$y_C = \frac{50 \times 200^2 \times 300^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.943 \text{ cm}$$

$$y_D = \frac{50 \times 300 \times 300 \times 200 \times 200}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.943 \text{ cm}$$

$$y_E = \frac{50 \times 400^2 \times 100^2}{3 \times 2 \times 10^6 \times 63.585 \times 500} = 0.419 \text{ cm}$$

General expression for natural frequency is given by

$$\omega_n = \sqrt{\frac{g}{y}} \text{ rad/sec}$$

$$f_n = \omega/2\pi \text{ Hz}$$

$$f_B = \frac{1}{2\pi}\sqrt{\frac{9.81}{.419\times 10^{-2}}} = 7.7 \text{ Hz}, \qquad f_C = \frac{1}{2\pi}\sqrt{\frac{9.81}{.943\times 10^{-2}}} = 5.13 \text{ Hz}$$

$$f_D = f_C = 5.13 \text{ Hz}, \qquad f_E = 7.7 \text{ Hz}$$

According to Dunkerley's relation, we know

$$\frac{1}{f_n^2} = \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} + \ldots.$$

$$\frac{1}{f_n^2} = \frac{1}{(7.7)^2} + \frac{1}{(5.13)^2} + \frac{1}{(5.13)^2} + \frac{1}{(7.7)^2} = .03373 + .07599$$

$$f_n^2 = 9.113, \quad f = 3.01 \text{ Hz.}$$

Example 6.10 *Using matrix method, determine the natural frequencies of the system shown in Fig. 6.25.* **(M.D.U., 2010 ; PTU, 2012)**

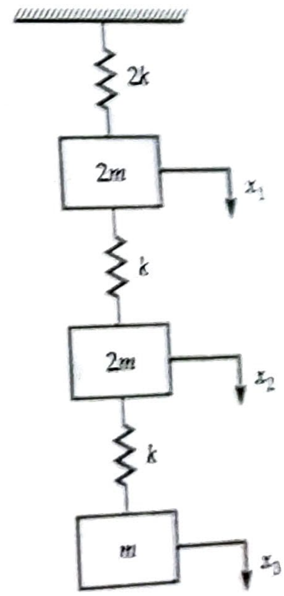

Fig. 6.25

Solution. The equations of motion are

$$2m\ddot{x}_1 + 2kx_1 + k(x_1 - x_2) = 0$$

$$2m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

$$\text{Mass matrix } [m] = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$m^{-1} = \frac{\text{adj } m}{|m|}$$

$$|m| = 2m \times 2m \times m = 4m^3$$

$$m^{-1} = \frac{\text{adj } m}{|m|} = \frac{1}{4m^3}\begin{bmatrix} 2m^2 & 0 & 0 \\ 0 & 2m^2 & 0 \\ 0 & 0 & 4m^2 \end{bmatrix}$$

We know that $[C] = [m]^{-1} . [k]$

$$[C] = \frac{1}{2m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\frac{k}{m} & -\frac{k}{2m} & 0 \\ -\frac{k}{2m} & \frac{k}{m} & -\frac{k}{2m} \\ 0 & -\frac{k}{m} & \frac{k}{m} \end{bmatrix}$$

Let us assume
$$\omega^2 = \lambda$$
$$|\lambda I - C| = 0$$

$$\begin{bmatrix} \lambda - \frac{3}{2}\frac{k}{m} & +\frac{k}{2m} & 0 \\ +\frac{k}{2m} & \lambda - \frac{k}{m} & +\frac{k}{2m} \\ 0 & +\frac{k}{m} & \lambda - \frac{k}{m} \end{bmatrix} = 0$$

Expanding it, we get
$$\lambda^3 - 3.5\lambda^2 \frac{k}{m} + 3.25\lambda \frac{k^2}{m^2} - .5\frac{k^3}{m^3} = 0$$

Solving it for λ, we have
$$\lambda_1 = .2\left(\frac{k}{m}\right), \qquad \omega_1 = \sqrt{.2\frac{k}{m}} = .44\sqrt{\frac{k}{m}} \text{ rad/sec}$$
$$\lambda_2 = 1.31\left(\frac{k}{m}\right), \qquad \omega_2 = 1.144\sqrt{\frac{k}{m}} \text{ rad/sec}$$
$$\lambda_3 = 2\left(\frac{k}{m}\right), \qquad \omega_3 = 1.414\sqrt{\frac{k}{m}} \text{ rad/sec}$$

Example 6.11 *Determine the natural frequencies and mode shapes of the system shown in Fig. 6.25, by matrix iteration method.* **(P.T.U., 2010)**

Solution. The influence coefficients are determined as
$$a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = \frac{1}{2k}, \qquad a_{22} = a_{23} = a_{32} = \frac{3}{2k}, \qquad a_{33} = \frac{5}{2k}$$

The equations for the above system in terms of influence coefficients can be written as
$$x_1 = 2ma_{11}\, x_1\omega^2 + 2ma_{12}\, x_2\omega^2 + ma_{13}x_3\omega^2$$
$$x_2 = 2ma_{21}\, x_1\omega^2 + 2ma_{22}\, x_2\omega^2 + ma_{23}x_3\omega^2$$
$$x_3 = 2ma_{31}\, x_1\omega^2 + 2ma_{32}\, x_2\omega^2 + ma_{33}x_3\omega^2$$

The equation can be written in matrix form as
$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = m\omega^2 \begin{bmatrix} 2a_{11} & 2a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = m\omega^2 \begin{bmatrix} \frac{1}{k} & \frac{1}{k} & \frac{1}{2k} \\ \frac{1}{k} & \frac{3}{k} & \frac{3}{2k} \\ \frac{1}{k} & \frac{3}{k} & \frac{5}{2k} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{m\omega^2}{k} \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 3 & 3/2 \\ 1 & 3 & 5/2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Example 6.13 *Three rail bogies are connected by two springs of stiffness 40×10^5 N/m each. The mass of each bogey is 20×10^3 kg. Determine the frequencies of vibration. Neglect friction between the wheels and rails.*

Solution. Refer Fig. 6.26.

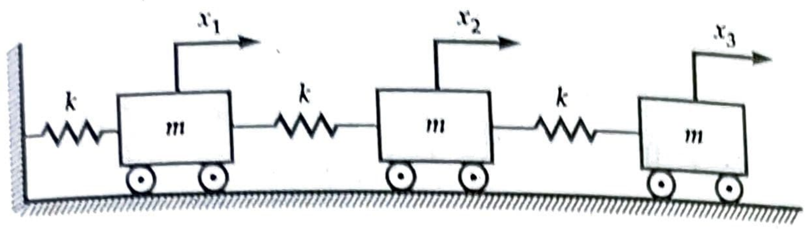

Fig. 6.26

The equations of motion can be written as

$$m\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Let us assume the oscillation of the form $x_i = A_i \sin \omega t$ and substituting this value in the above equations, we have

$$(k - m\omega^2) A_1 - kA_2 = 0$$

$$-kA_1 + (2k - m\omega^2) A_2 - kA_3 = 0$$

$$-kA_2 + (k - m\omega^2) A_3 = 0$$

The frequency equation is obtained by putting the determinant of coefficients of A's equal to zero,

$$\begin{bmatrix} k - m\omega^2 & -k & 0 \\ -k & (2k - m\omega^2) & -k \\ 0 & -k & k - m\omega^2 \end{bmatrix} = 0$$

$$(k - m\omega^2)(m^2\omega^4 - 3km\omega^2) = 0$$

which yields $\omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{m}}$, $\omega_3 = \sqrt{\frac{3k}{m}}$

Substituting the numerical values, we get

$$f_1 = 0$$

$$\omega_2 = \sqrt{\frac{400 \times 10^4}{20 \times 10^3}} = 14.14 \text{ rad/sec}, \qquad f_2 = 2.25 \text{ Hz}$$

$$\omega_3 = \sqrt{\frac{3 \times 40 \times 10^5}{20 \times 10^3}} = 24.49 \text{ rad/sec}, \qquad f_3 = \frac{\omega_3}{2\pi} = 3.9 \text{ Hz.}$$

Solution. The equations of motion can be written as

$$m_1\ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) - k_3(x_1 - x_3)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$
$$m_3\ddot{x}_3 = -k_3(x_3 - x_1)$$

Rearranging the above, we get

$$m_1\ddot{x}_1 + (k_1 + k_2 + k_3)x_1 - k_2x_2 - k_3x_3 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$
$$m_3\ddot{x}_3 - k_3x_1 + k_3x_3 = 0$$

Putting in the determinant form

$$\begin{vmatrix} (k_1 + k_2 + k_3 - m_1\omega^2) & -k_2 & -k_3 \\ -k_2 & k_2 - m_2\omega^2 & 0 \\ -k_3 & 0 & k_3 - m_3\omega^2 \end{vmatrix} = 0$$

Substituting $k_1 = 7k,\ k_2 = 5k,\ k_3 = 5k$ and $m_1 = 4m,\ m_2 = 3\text{m},\ m_3 = 2m$

$$\begin{vmatrix} (17k - 4m\omega^2) & -5k & -5k \\ -5k & 5k - 3m\omega^2 & 0 \\ -5k & 0 & 5k - 2m\omega^2 \end{vmatrix} = 0$$

Expanding the determinant, we get : $175k^3 - 400mk^2\omega^2 + 202km^2\omega^4 - 24m^3\omega^6 = 0$

Solving the equation for ω: $\omega = 0.78\sqrt{\dfrac{k}{m}}$ rad/sec

Example 6.16 *Determine the lowest natural frequency of the system shown in Fig. 6.28 by matrix method.*

Solution. The differential equations can be written as

$$m_1\ddot{x}_1 + (k_1 + k_2 + k_3)x_1 - k_2x_3 - k_3x_3 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$
$$m_3\ddot{x}_3 - k_3x_1 + k_3x_3 = 0$$

Substituting the values of masses and stiffnesses, we get

$$4m\ddot{x}_1 + 17kx_1 - 5kx_2 - 5kx_3 = 0$$
$$3m\ddot{x}_2 - 5kx_1 + 5kx_2 = 0$$
$$2m\ddot{x}_3 - 5kx_1 + 5kx_3 = 0$$

Putting the above equations in matrix form, we get

$$\begin{bmatrix} 4m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 17k & -5k & -5k \\ -5k & 5k & 0 \\ -5k & 0 & 5k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

Dynamic matrix [C] can be written as

$$[C] = [M]^{-1}[k]$$

$$[M]^{-1} = \frac{\text{adj } M}{|M|} = \frac{1}{24m^3}\begin{bmatrix} 6m^2 & 0 & 0 \\ 0 & 8m^2 & 0 \\ 0 & 0 & 12m^2 \end{bmatrix}$$

$$[C] = \frac{1}{24m^3}\begin{bmatrix} 6m^2 & 0 & 0 \\ 0 & 8m^2 & 0 \\ 0 & 0 & 12m^2 \end{bmatrix}\begin{bmatrix} 17k & -5k & -5k \\ -5k & 5k & 0 \\ -5k & 0 & 5k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17}{4}\frac{k}{m} & \frac{-5}{4}\frac{k}{m} & \frac{-5}{4}\frac{k}{m} \\ \frac{-5}{3}\frac{k}{m} & \frac{5}{3}\frac{k}{m} & 0 \\ \frac{-5}{2}\frac{k}{m} & 0 & \frac{5}{2}\frac{k}{m} \end{bmatrix}$$

(∵ $\ddot{x} = -\omega^2 x$)

$$-\omega^2\{x\} + [C]\{x\} = 0 \quad \text{or} \quad |\lambda I - C| = 0$$

where $\omega^2 = \lambda$

$$\begin{vmatrix} \lambda - \frac{17k}{4m} & +\frac{5}{4}\frac{k}{m} & +\frac{5}{4}\frac{k}{m} \\ \frac{+5k}{3m} & \lambda - \frac{5}{3}\frac{k}{m} & 0 \\ \frac{+5}{2}\frac{k}{m} & 0 & \lambda - \frac{5}{2}\frac{k}{m} \end{vmatrix} = 0$$

$$\left(\lambda - \frac{17}{4}\frac{k}{m}\right)\left(\lambda - \frac{5}{3}\frac{k}{m}\right)\left(\lambda - \frac{5}{2}\frac{k}{m}\right) - \frac{5}{4}\frac{k}{m}\left(\frac{-5}{3}\frac{k}{m}\right)\left(\lambda - \frac{5}{2}\frac{k}{m}\right) \frac{+5}{4}\frac{k}{m}\frac{5}{2}\frac{k}{m}\left(\lambda - \frac{5}{3}\frac{k}{m}\right) = 0$$

$$\lambda^3 - \frac{101}{12}\lambda^2\frac{k}{m} + \frac{310}{24}\lambda\frac{k^2}{m^2} - \frac{175}{24}\frac{k^3}{m^3} = 0$$

$$\lambda = .6084\frac{k}{m}$$

So $$\omega_n = \sqrt{.6084\frac{k}{m}} = .78\sqrt{\frac{k}{m}} \text{ rad/sec.}$$

Example 6.17 *A steel shaft of diameter 10 cm is carrying three masses 2.5 kg, 3.75 kg and 7 kg respectively as shown in Fig. 6.29.*

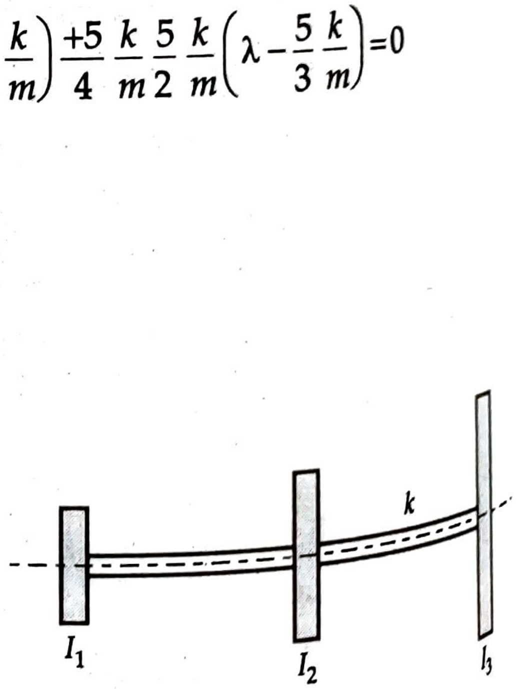

Fig. 6.29

The distances between the rotors are 0.70 m. Determine the natural frequencies of torsional vibrations. The radii of gyration of three rotors are 0.20, 0.30 and 0.40 m respectively. Take $G = 9 \times 10^8\ N/m^2$.

(M.D.U., 2010)

Solution. Let us assume that θ_1, θ_2 and θ_3 are the angular displacements of the three rotors having moment of inertias as I_1, I_2 and I_3.

The equations of motion can be written as

$$I_1\ddot{\theta}_1 = -k(\theta_1 - \theta_2)$$

$$I_2\ddot{\theta}_2 = k(\theta_1 - \theta_2) - k(\theta_2 - \theta_3)$$

$$I_3\ddot{\theta}_3 = +k(\theta_2 - \theta_3)$$

Assuming the motion of the form $\theta_i = A_i \cos \omega t$

So $\ddot{\theta}_1 = -\omega^2 A_1$ and so on.

Substituting these values in the above equations, we get

$$(k - I_1\omega^2)A_1 - kA_2 = 0$$

$$-kA_1 + (2k - I_2\omega^2)A_2 - kA_3 = 0$$

$$-kA_2 + (k - I_3\omega^2)A_3 = 0$$

Equating the coefficients of A_1, A_2 and A_3 in determinant form to zero.

$$\begin{vmatrix} k - I_1\omega^2 & -k & 0 \\ -k & 2k - I_2\omega^2 & -k \\ 0 & -k & k - I_3\omega^2 \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\omega^2\left[\omega^4 - \left(\frac{k}{I_3} + \frac{2k}{I_2} + \frac{k}{I_1}\right)\omega^2 + \frac{k^2}{I_1 I_2} + \frac{k^2}{I_1 I_3} + \frac{k^2}{I_2 I_3}\right] = 0$$

So $\omega_1 = 0$

According to strength of materials we know

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$k = \frac{T}{\theta} = \frac{GJ}{l} = \text{Stiffness of shaft}$$

$$k = \frac{9 \times 10^8}{.7} \cdot \frac{\pi}{32} \times .10^4 = 1.2616 \times 10^4\ \text{N-m/rad}$$

$$I_1 = m_1 k_1^2 = 2.5 \times .2^2 = .10\ \text{kg-m sec}^2$$

$$I_2 = m_2 k_2^2 = 3.75 \times .3^2 = 0.3375\ \text{kg-m sec}^2$$

$$I_3 = m_3 k_3^2 = 7 \times .4^2 = 1.12\ \text{kg-m sec}^2$$

$$\omega^4 - \omega^2\left(\frac{1.2616\times 10^4}{1.12} + \frac{2\times 1.2616\times 10^4}{0.3375} + \frac{1.2616}{.10}\right) + \frac{(1.2616\times 10^4)^2}{.1\times .3375} + \frac{(1.2616\times 10^4)^2}{.1\times 1.12} + \frac{(1.2616\times 10^4)^2}{.3375\times 1.12}$$

$$\omega^4 - \omega^2 \times 1.2616\times 10^4(.8928 + 5.9259 + 10) + 65.581121\times 10^8 = 0$$

$$\omega^4 - 21.2184\times 10^4\,\omega^2 + 65.581121\times 10^8 = 0$$

$$\omega = \frac{21.2184\times 10^4 \pm \sqrt{(21.2184\times 10^4)^2 - 4\times 65.5811\times 10^8}}{2} = \frac{21.2184\times 10^4 \pm 13.7075\times 10^4}{2}$$

$$\omega_1 = 0, \quad \omega_3 = 417.88 \text{ rad/sec}, \quad \omega_2 = 193.78 \text{ rad/sec}$$

Example 6.18 *Find the natural frequencies and mode shapes of the system shown in Fig. 6.4 for $k_1 = k_2 = k_3$ and $m_1 = m_2 = m_3$ using the matrix iteration method.* (P.U., 93, ME)

Solution. The differential equations in terms of influence coefficients can be written as

$$x_1 = a_{11}m_1x_1\,\omega^2 + a_{12}m_2x_2\,\omega^2 + a_{13}m_3x_3\omega^2$$
$$x_2 = a_{21}m_1x_1\omega^2 + a_{22}m_2x_2\,\omega^2 + a_{23}m_3x_3\omega^2$$
$$x_3 = a_{31}m_1x_1\omega^2 + a_{32}m_2x_2\omega^2 + a_{33}m_3x_3\omega^2$$

The influence coefficients are

$$a_{11} = \frac{1}{k} = a_{12} = a_{13} = a_{21} = a_{31}$$

$$a_{22} = \frac{1}{k} + \frac{1}{k} = \frac{2}{k} = a_{23} = a_{32}$$

$$a_{33} = \frac{1}{k} + \frac{1}{k} + \frac{1}{k} = \frac{3}{k}$$

Substituting the values of influence coefficients in the above equations, we get

$$x_1 = \frac{m}{k}x_1\omega^2 + \frac{m}{k}x_2\omega^2 + \frac{m}{k}x_3\omega^2$$

$$x_2 = \frac{m}{k}x_1\omega^2 + \frac{2m}{k}x_2\omega^2 + \frac{2m}{k}x_3\omega^2$$

$$x_3 = \frac{m}{k}x_1\omega^3 + \frac{2m}{k}x_2\omega^2 + \frac{3m}{k}x_3\omega^2$$

This can be written in matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ \ldots \end{Bmatrix} = \frac{\omega^2 m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ \ldots \end{bmatrix}\begin{Bmatrix} x_1 \\ \ldots \end{Bmatrix}$$

The ratios of calculated deflections are

$$\frac{x_2'}{x_1'} = \frac{1069.85}{495.9} = \frac{2.15}{1.0}$$

This ratio is much different from the assumed ratio.

◆ **Second trial**

$$F_1' = m_1\omega^2 x_1' = 100\,\omega^2, \quad F_2' = m_2\,\omega^2 x_2' = 50\,\omega^2 \times 2.15 = 107.5\,\omega^2$$

$$x_1'' = F_1' a_{11} + F_2' a_{12}$$

$$= 100\omega^2 \times 2.4795 \times 10^{-8} + 107.5\,\omega^2 \times 4.959 \times 10^{-8} = 781 \times 10^{-8}\,\omega^2$$

$$x_2'' = F_1' a_{21} + F_2' a_{22}$$

$$= 100\,\omega^2 \times 4.959 \times 10^{-8} + 107.5\,\omega^2 \times 11.479 \times 10^{-8} = 1729.89 \times 10^{-8}\,\omega^2$$

So $x_2'' : x_1'' = 2.21 : 1$

The ratio is quite different from the assumed ratio in the start of this trial.

◆ **Third trial**

We get $x_1''' = 795.9 \times 10^{-8}\,\omega^2$ and $x_2''' = 1764.32 \times 10^{-8}\,\omega^2$

The ratio of deflections is

$$\frac{x_2'''}{x_1'''} = 2.21 : 1$$

This ratio is equal to the starting value for this trial. Thus the assumed and calculated values of ratio are equal.

So $x_1''' = 1$ (assumed)

$x_1''' = 795.9 \times 10^{-8}\,\omega^2$ or $1 = 795.9 \times 10^{-8}\,\omega^2$ (calculated)

$$\omega = 354.5 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = 56.45 \text{ Hz.}$$

Example 6.21 *Using Holzer method find the natural frequencies of the system shown in Fig. 6.4. Assume $m_1 = m_2 = m_3 = 1\,kg$ and $k_1 = k_2 = k_3 = 1\,N/m$.* **(KUK, 2012)**

Solution. Assuming the initial displacement $x_1 = 1$ and natural frequency $\omega = 30$ rad/sec.

$$\omega^2 = .3 \times .3 = .09$$

$$x_2 = x_1 - \frac{m_1 x_1\,\omega^2}{k_1} = 1 - \frac{1}{1} \times .09 = .91$$

$$x_3 = x_2 - \omega^2 \frac{(m_1 x_1 + m_2 x_2)}{k_2} = .91 - .09(1 + .91) = 0.74$$

$$x_4 = x_3 - \omega^2 \frac{(m_1 x_1 + m_2 x_2 + m_3 x_3)}{k_3} = 0.74 - .09\,(1 + .91 + .74) = 0.51$$

Similarly, other deflections can be calculated and are directly put in the table 1 for different assumed frequency. The results for frequency are obtained by drawing a curve between ω and displacement x as shown in Fig. 6.31. The natural frequencies are

$\omega_1 = 0.44$ rad / sec

$\omega_2 = 1.24$ rad / sec and $\omega_3 = 1.80$ rad / sec.

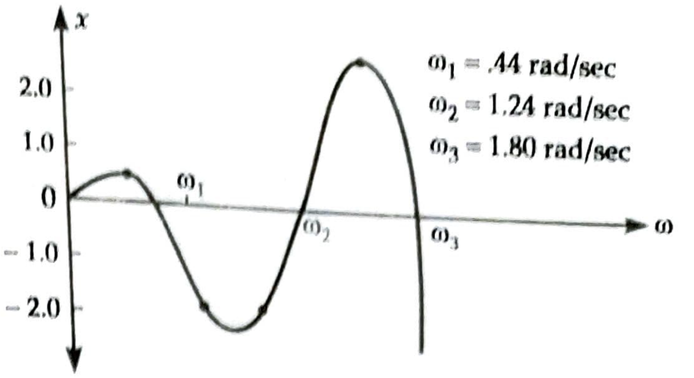

Fig. 6.31

Table 1

Assumed Frequency	Row	m	$m\omega^2$	x	$mx\omega^2$	$\Sigma\, mx\omega^2$	k	$\frac{1}{k}\Sigma\, mx\omega^2$
$\omega = 0.30$	1	1	.09	1.0	.09	0.09	1	.09
$\omega^2 = .09$	2	1	.09	0.91	.0819	0.17	1	.17
	3	1	.09	0.74	.0666	0.23	1	0.23
				0.51				
$\omega = 0.50$	1	1	.25	1.0	0.25	0.25	1	0.25
$\omega^2 = 0.25$	2	1	.25	0.75	0.19	0.44	1	0.44
	3	1	.25	0.31	0.07	0.51	1	0.51
				–0.20				
$\omega = 0.75$	1	1	0.56	1.0	0.56	0.56	1	0.56
$\omega^2 = 0.56$	2	1	0.56	0.44	0.24	0.80	1	0.80
	3	1	0.56	– 0.36	– 0.20	0.60	1	0.60
				– 0.96				
$\omega = 1.0$	1	1	1	1	1	1	1	1
$\omega^2 = 1.0$	2	1	1	0	0	1	1	1
	3	1	1	–1	–1	0	1	0
				–1				
$\omega = 1.25$	1	1	1.56	1.0	1.56	1.56	1	1.56
$\omega^2 = 1.56$	2	1	1.56	–.56	–.87	0.69	1	0.69
	3	1	1.56	–1.25	–1.95	–1.26	1	–1.26
				.01				
$\omega = 1.50$	1	1	2.25	1.0	2.25	2.25	1	2.25
$\omega^2 = 2.25$	2	1	2.25	–1.25	–2.82	–0.57	1	–0.57
	3	1	2.25	–0.68	–1.53	–2.10	1	–2.10
				1.40				
$\omega = 1.75$	1	1	3.06	1.0	3.06	3.06	1	3.06
$\omega^2 = 3.06$	2	1	3.06	–2.06	–6.30	–3.24	1	–3.24
	3	1	3.06	1.18	3.60	0.36	1	0.36
				.81				
$\omega = 2.0$	1	1	4.0	1	4	4	1	4
$\omega^2 = 4.0$	2	1	4.0	–3	–12	–8	1	–8
	3	1	4.0	5	20	12	1	12
				–7				