# 2D TRANSFORMATION & VIEWING

Outline

- Transformation

- Basic transformation

- Matrix representation and homogeneous coordinates

- Composite transformation

- Other transformation

- The viewing pipeline

- Viewing coordinate reference frame

- Window-to-viewport coordinate transformation

- Point clipping

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- Polygon clipping

Transformation

- Changing Position, shape, size, or orientation of an object is known as transformation.
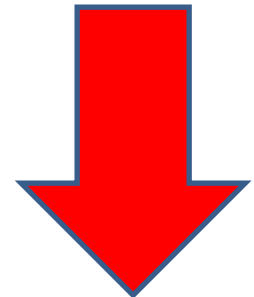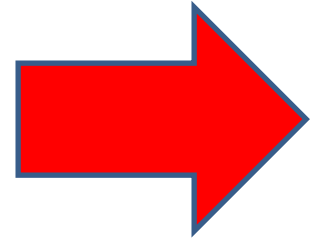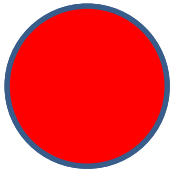
Basic Transformation

- includes three transformations
  - **Translation**,
  - **Rotation**
  - **Scaling**.
- These are known as basic transformation because with combination of them we can obtain any transformation.

Translation

- Transformation that used to reposition the object along the straight line path.

- Adding translation distance t_x and t_y to the original coordinate position (x, y) respectively we move point at new position (x', y').

- $x' = x + t_x$ & $y' = y + t_y$

- Translation distance pair (t_x, t_y) is called a **Translation Vector** or **Shift Vector**.

- It is rigid body transformation so we need to translate whole object.
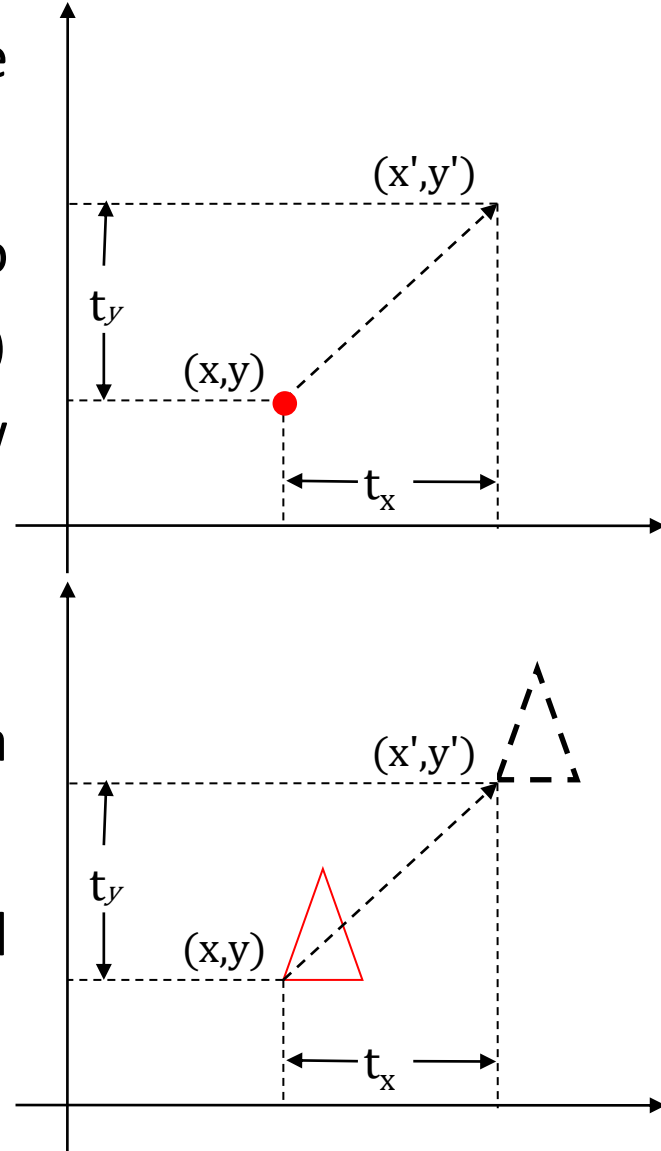
Contd.

- We can represent it into single matrix equation in column vector as;

$$P' = P + T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

- We can also represent it in row vector form as:

$$P' = P + T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} + \begin{bmatrix} t_x & t_y \end{bmatrix}$$

- Since column vector representation is standard mathematical notation and also many graphics package like **GKS** and **PHIGS** uses column vector we will also follow column vector representation.

Translation Example

- **Example**: - Translate the triangle [A (10, 10), B (15, 15), C(20, 10)] 2 unit in x direction and 1 unit in y direction.

We know that $P' = P + T = [P] + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

For point (10, 10) $A' = \begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$

For point (15, 15) $B' = \begin{bmatrix} 15 \\ 15 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \end{bmatrix}$

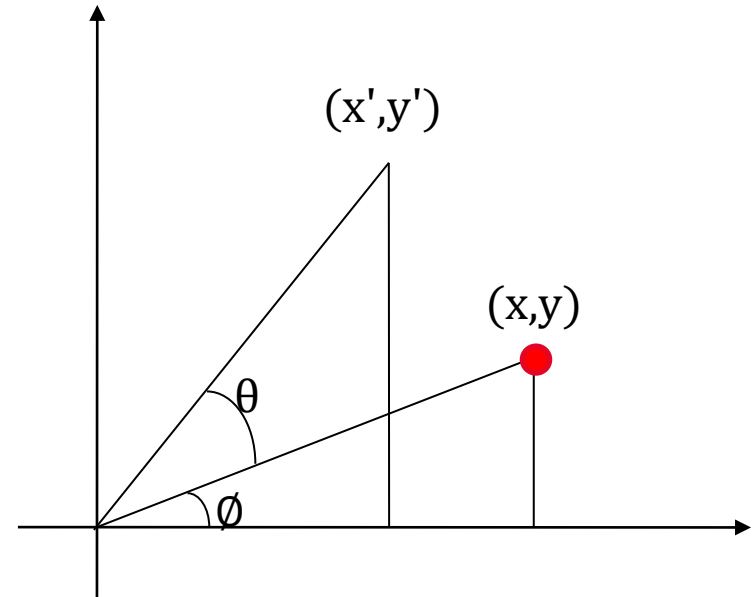For point (10, 10) $C' = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 11 \end{bmatrix}$

- Final coordinates after translation are:

[A$'$ (12, 11), B$'$ (17, 16), C$'$ (22, 11)]

Rotation

- It is a transformation that used to reposition the object along the circular path in the $xy - plane$.

- To generate a rotation we specify a,

  - **Rotation angle θ.**

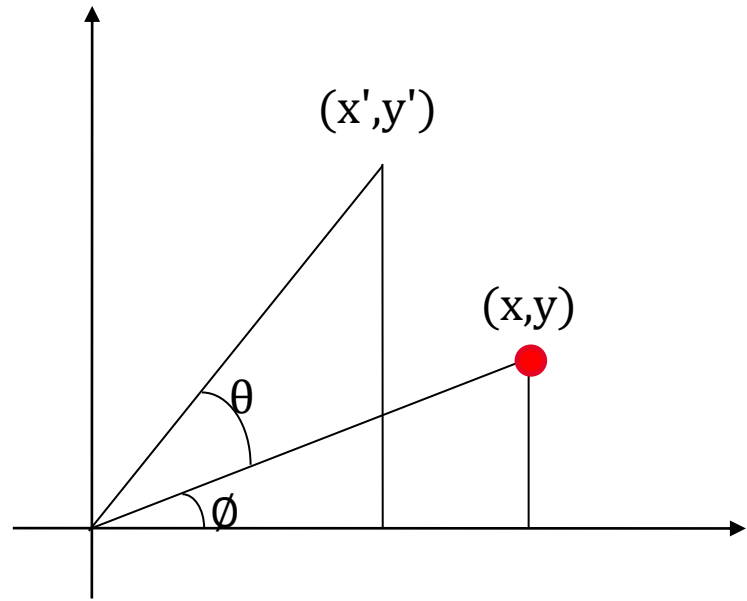 - **Rotation Point (Pivot Point)** (x_r, y_r).

- +ve value of rotation angle defines counter clockwise rotation.

- -ve value of rotation angle defines clockwise rotation.

Rotation Equation

- We first find the equation of rotation when pivot point is at coordinate origin(0,0).

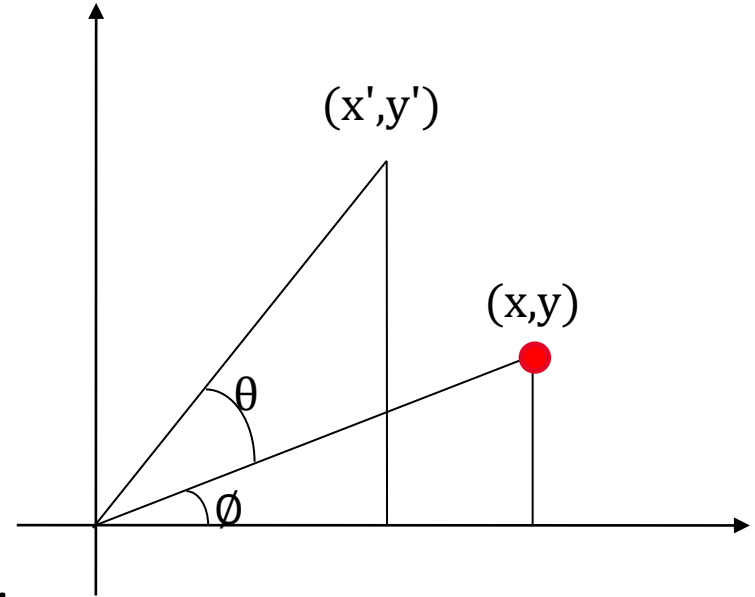- From figure we can write.

$$x = r\cos\emptyset$$
$$y = r\sin\emptyset$$

&

$$x' = r\cos(\theta + \emptyset) = r\cos\emptyset\cos\theta - r\sin\emptyset\sin\theta$$
$$y' = r\sin(\emptyset + \theta) = r\cos\emptyset\sin\theta + r\sin\emptyset\cos\theta$$

Contd.

- Now replace $r \cos \emptyset$ with x and $r \sin \emptyset$ with y in above equation.

- $x' = x \cos \theta - y \sin \theta$

- $y' = x \sin \theta + y \cos \theta$

- column vector matrix equation are,

- $P' = R \cdot P$

- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$

Rotation about arbitrary point

- Transformation equation for rotation of a point about pivot point (x_r, y_r) is:

- $x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$

- $y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$

- These equations are differing from rotation about origin and its matrix representation is also different.

- Its matrix equation can be obtained by simple method.

- Rotation is also rigid body transformation so we need to rotate each point of object.
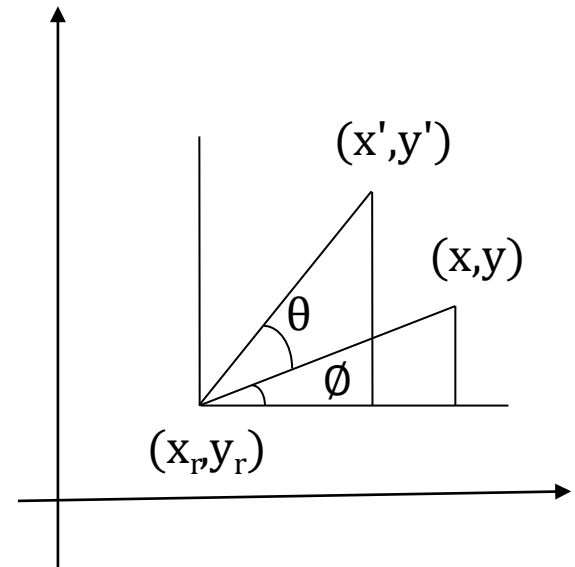
Rotation Example

- **Example**: - Locate the new position of the triangle

  [A (5, 4), B (8, 3), C (8, 8)] after its rotation by 90º clockwise about the origin.

- As rotation is clockwise we will take $\theta = -90^{\circ}$.

- $P' = R \cdot P$

- $P' = \begin{bmatrix} \cos(-90) & -\sin(-90) \\ \sin(-90) & \cos(-90) \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$

- $P' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$

- $P' = \begin{bmatrix} 4 & 3 & 8 \\ -5 & -8 & -8 \end{bmatrix}$

- Final coordinates after rotation are: [A' (4, -5), B' (3, -8), C' (8, -8)]

Scaling

- Transformation that used to alter the size of an object is known as scaling.

- This operation is carried out by multiplying coordinate value (x, y) with scale factors (s_x, s_y) respectively.

- Equation for scaling is given by,

$$x' = x \cdot s_x \ \& \ y' = y \cdot s_y$$

- These equation can be represented in column vector matrix equation as:

$$P' = S \cdot P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- Any positive value can be assigned to (s_x, s_y).

Contd.

- Normal scaling will scale as well as reposition the object.

- Scale factors less than 1 reduce the size and move object closer to origin.

- Scale factors greater than 1 enlarge the size of object and move object away from origin

- Example scale square with opposite corner coordinate point are (1, 1) and (2, 2) by scale factor (3, 3)

$$P' = S \cdot P$$

$$\begin{bmatrix} x_1' & x_2' \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix}$$
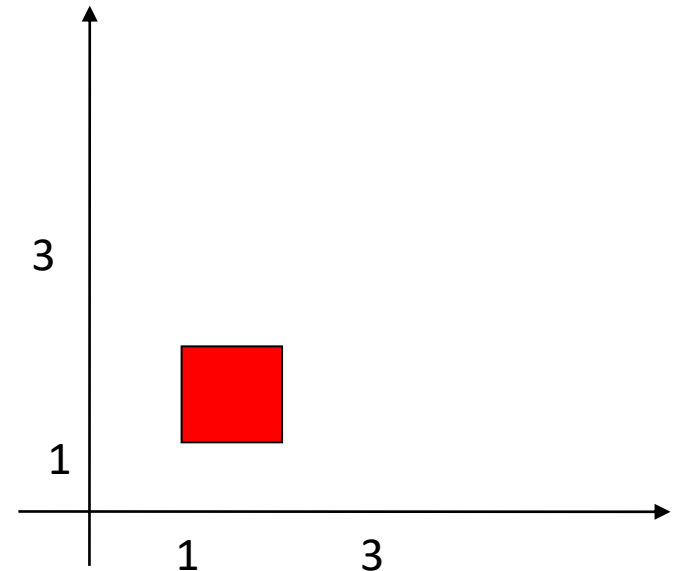
Contd.

- Same values of s_x and s_y will produce **Uniform Scaling**.

- Different values of s_x and s_y will produce **Differential (Non Uniform) Scaling**.

Uniform Scaling

Differential Scaling
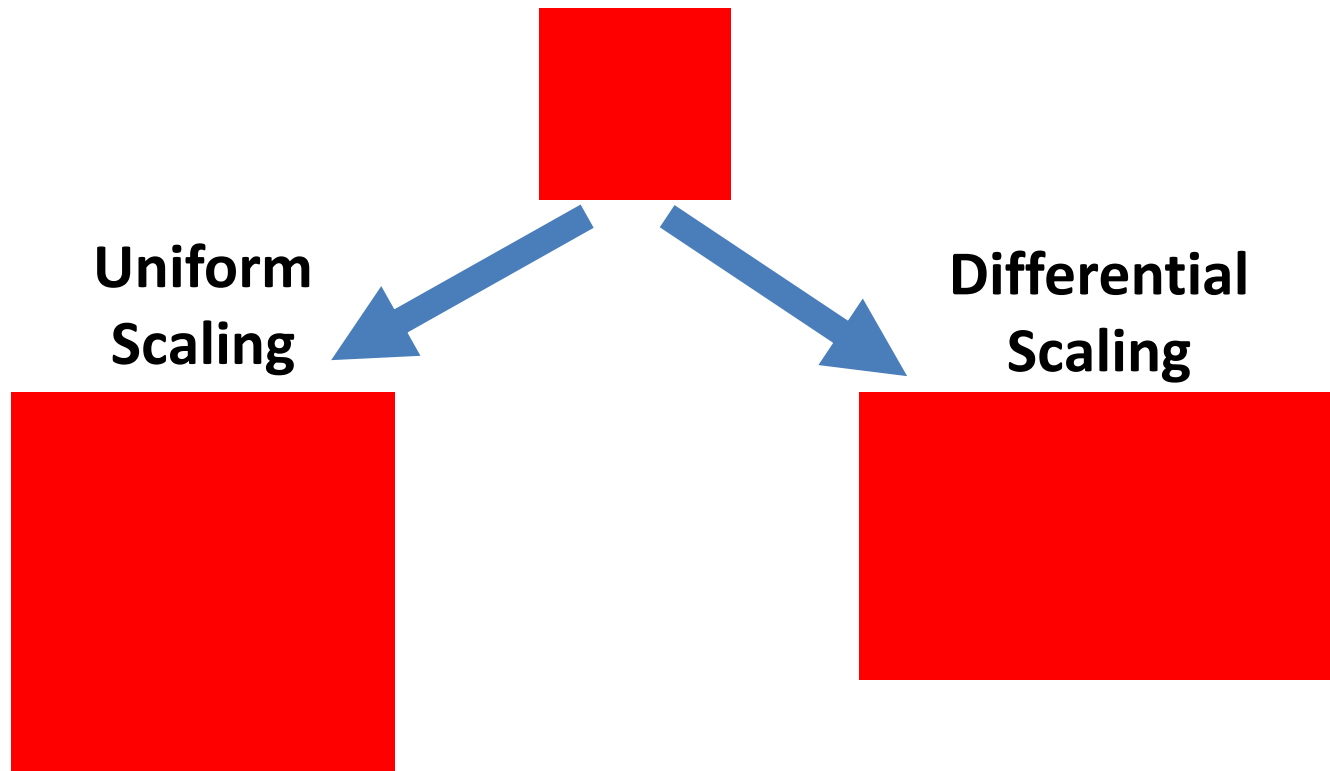
Fixed Point Scaling

- We can control the position of object after scaling by keeping one position fixed called **Fix point** (x_f, y_f).
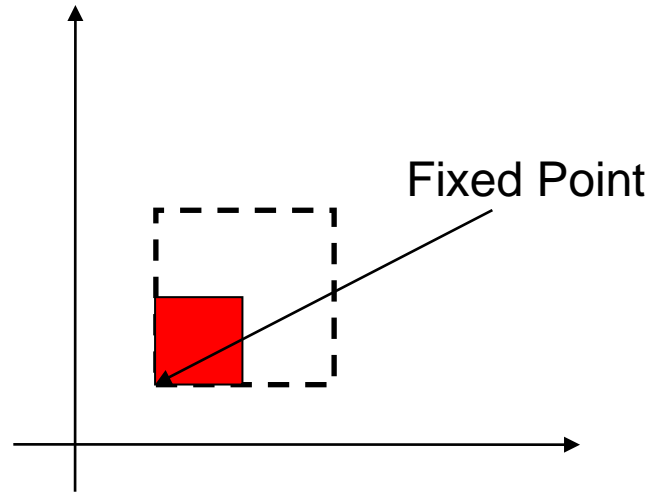
Fixed Point

Fixed Point Scaling Equation

- Equation for scaling with fixed point position as (x_f, y_f) is:

- $x' = x_f + (x - x_f)s_x$ $\qquad\qquad$ $y' = y_f + (y - y_f)s_y$

- $x' = x_f + xs_x - x_fs_x$ $\qquad\qquad$ $y' = y_f + ys_y - y_fs_y$

- $x' = xs_x + x_f(1 - s_x)$ $\qquad\qquad$ $y' = ys_y + y_f(1 - s_y)$

- Polygons are scaled by applying scaling at coordinates and redrawing.

- Other body like circle and ellipse will scale using its defining parameters.

- For example ellipse will scale using its semi major axis, semi minor axis and center point scaling and redrawing at that position.

Scaling Example

- **Example**: - Consider square with left-bottom corner at (2, 2) and right-top corner at (6, 6) apply the transformation which makes its size half.

- As we want size half so value of scale factor are $s_x = 0.5, s_y = 0.5$ and Coordinates of square are [A (2, 2), B (6, 2), C (6, 6), D (2, 6)].

- $P' = S \cdot P$

- $P' = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \end{bmatrix}$

- $P' = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$

- Final coordinate after scaling are:

  [A$'$ (1, 1), B$'$ (3, 1), C$'$ (3, 3), D$'$ (1, 3)]

Matrix Representation and Homogeneous Coordinates

- Many graphics application involves sequence of geometric transformations.

- For example in design and picture construction application we perform Translation, Rotation, and scaling to fit the picture components into their proper positions.

- For efficient processing we will reformulate transformation sequences.

Contd.

- We have matrix representation of basic transformation and we can express it in the general matrix form as:

$$P' = M_1 \cdot P + M_2$$

Where

P and P' are initial and final point position,

M_1 contains rotation and scaling terms and

M_2 contains translational terms associated with pivot point, fixed point and reposition.

Contd.

- For efficient utilization we must calculate all sequence of transformation in one step.

- For that reason we reformulate above equation to eliminate the matrix addition associated with translation terms in matrix M_2.

- We can combine that thing by expanding 2X2 matrix representation into 3X3 matrices.

- It will allows us to convert all transformation into matrix multiplication.

- We need to represent vertex position (x, y) with homogeneous coordinate triple (x_h, y_h, h).

- Where $x = \dfrac{x_h}{h}$, $y = \dfrac{y_h}{h}$ thus we can also write triple as $(h \cdot x, h \cdot y, h)$.

Contd.

- For two dimensional geometric transformation we can take value of h is any positive number.

- We can get infinite homogeneous representation for coordinate value (x, y).

- But convenient choice is set $h = 1$ as it is multiplicative identity, than (x, y) is represented as $(x, y, 1)$.

- Expressing coordinates in homogeneous coordinates form allows us to represent all geometric transformation equations as matrix multiplication.

Homogeneous Matrix for Translation

- Let's see each representation with $h = 1$

$$P' = T_{(t_x, t_y)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Inverse of translation matrix is obtain by putting $-t_x$ & $-t_y$ instead of t_x & t_y.

Homogeneous Matrix for Rotation

$$P' = R_{(\theta)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Inverse of rotation matrix is obtained by replacing θ by $-\theta$.

Homogeneous Matrix for Scaling

$$P' = S_{(s_x, s_y)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Inverse of scaling matrix is obtained by replacing s_x & s_y by $\frac{1}{s_x}$ & $\frac{1}{s_y}$ respectively.

Composite Transformation

- In practice we need to apply more than one transformations to get desired result.

- It is time consuming to apply transformation one by one on each point of object.

- So first we multiply all matrix of required transformations which is known as **composite transformation matrix.**

- Than we use composite transformation matrix to transform object.

- For column matrix representation, we form composite transformations by multiplying matrices from right to left.

Multiple Translations

- Two successive translations are performed as:

- $P' = T(t_{x2}, t_{y2}) \cdot \{T(t_{x1}, t_{y1}) \cdot P\}$

- $P' = \{T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})\} \cdot P$

- $P' = \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

Contd.

- $P' = \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = T(t_{x1} + t_{x2}, t_{y1} + t_{y2}) \cdot P$

- Here P' and P are column vector of final and initial point coordinate respectively.

- This concept can be extended for any number of successive translations.

Multiple Translations Example

- **Example:** Obtain the final coordinates after two translations on point $p(2,3)$ with translation vector $(4,3)$ and $(-1,2)$ respectively.

- $P' = T\left(t_{x1} + t_{x2}, t_{y1} + t_{y2}\right) \cdot P$

- $P' = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot P = \begin{bmatrix} 1 & 0 & 4 + (-1) \\ 0 & 1 & 3 + 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$

- Final Coordinates after translations are $p\prime(5,8)$.

Multiple Rotations

- Two successive Rotations are performed as:

- $P' = R(\theta_2) \cdot \{R(\theta_1) \cdot P\}$

- $P' = \{R(\theta_2) \cdot R(\theta_1)\} \cdot P$

- $P' = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 & \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

Contd.

- $P' = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = R(\theta_1+\theta_2) \cdot P$

- Here P' and P are column vector of final and initial point coordinate respectively.

- This concept can be extended for any number of successive rotations.

Multiple Rotations Example

- **Example:** Obtain the final coordinates after two rotations on point $p(6,9)$ with rotation angles are 30° and 60° respectively.

- $P' = R(\theta_1 + \theta_2) \cdot P$

- $P' = \begin{bmatrix} cos(\theta_1 + \theta_2) & -sin(\theta_1 + \theta_2) & 0 \\ sin(\theta_1 + \theta_2) & cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} cos(30 + 60) & -sin(30 + 60) & 0 \\ sin(30 + 60) & cos(30 + 60) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ 1 \end{bmatrix}$

- Final Coordinates after rotations are $p'(-9, 6)$.

Multiple Scaling

- Two successive scaling are performed as:

- $P' = S(s_{x2}, s_{y2}) \cdot \{S(s_{x1}, s_{y1}) \cdot P\}$

- $P' = \{S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1})\} \cdot P$

- $P' = \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

Contd.

- $P' = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2}) \cdot P$

- Here P' and P are column vector of final and initial point coordinate respectively.

- This concept can be extended for any number of successive scaling.

Multiple Scaling Example

- **Example:** Obtain the final coordinates after two scaling on line pq $[p(2,2),\ q(8,8)]$ with scaling factors are $(2,2)$ and $(3,3)$ respectively.

- $P' = S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2}) \cdot P$

- $P' = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P = \begin{bmatrix} 2 \cdot 3 & 0 & 0 \\ 0 & 2 \cdot 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 \\ 2 & 8 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 48 \\ 12 & 48 \\ 1 & 1 \end{bmatrix}$

- Final Coordinates after rotations are $p'(12,12)$ and $q'(48,48)$.

General Pivot-Point Rotation

- For rotating object about arbitrary point called pivot point we need to apply following sequence of transformation.

    1. Translate the object so that the pivot-point coincides with the coordinate origin.

    2. Rotate the object about the coordinate origin with specified angle.

    3. Translate the object so that the pivot-point is returned to its original position (i.e. Inverse of step-1).
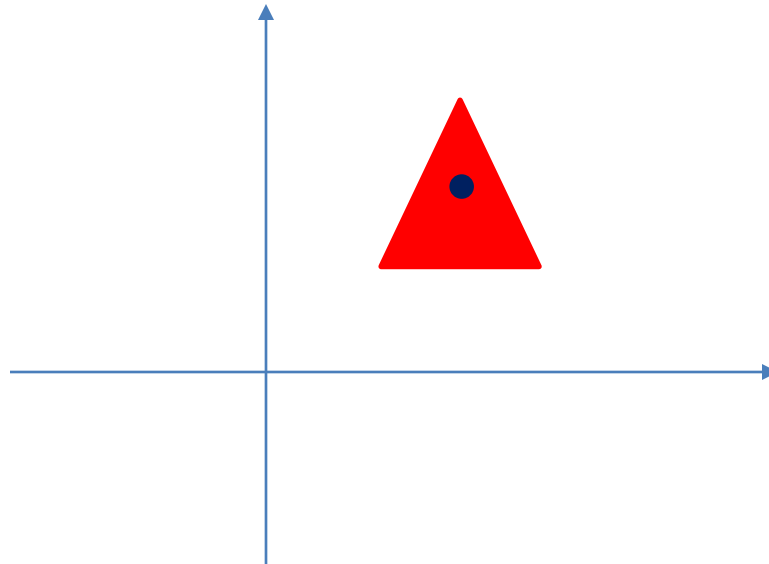
General Pivot-Point Rotation Equation

- Let's find matrix equation for this

- $P' = T(x_r, y_r) \cdot [R(\theta) \cdot \{T(-x_r, -y_r) \cdot P\}]$

- $P' = \{T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r)\} \cdot P$

- $P' = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

Contd.

- $P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = R(x_r, y_r, \theta) \cdot P$

- Here P' and P are column vector of final and initial point coordinate respectively and (x_r, y_r) are the coordinates of pivot-point.

General Pivot-Point Rotation Example

- **Example**: - Locate the new position of the triangle ABC

  [A (5, 4), B (8, 3), C (8, 8)] after its rotation by 90º clockwise about the centroid.

- Pivot point is centroid of the triangle so:

- $x_r = \frac{5+8+8}{3} = 7$, $y_r = \frac{4+3+8}{3} = 5$

- As rotation is clockwise we will take $\theta = -90°$.

- $P' = R_{(x_r,\ y_r,\theta)} \cdot P$

- $P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$

Contd.

- $P' =$

$$\begin{bmatrix} \cos(-90) & -\sin(-90) & 7(1-\cos(-90))+5\sin(-90) \\ \sin(-90) & \cos(-90) & 5(1-\cos(-90))-7\sin(-90) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

- $P' = \begin{bmatrix} 0 & 1 & 7(1-0)-5(1) \\ -1 & 0 & 5(1-0)+7(1) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 12 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$

Contd.

- $P' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 12 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 6 & 5 & 10 \\ 7 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$

- Final coordinates after rotation are [A$'$ (6, 7), B$'$ (5, 4), C$'$ (10, 4)].

General Fixed-Point Scaling

- For scaling object with position of one point called fixed point will remains same, we need to apply following sequence of transformation.

  1. Translate the object so that the fixed-point coincides with the coordinate origin.

  2. Scale the object with respect to the coordinate origin with specified scale factors.

  3. Translate the object so that the fixed-point is returned to its original position (i.e. Inverse of step-1).
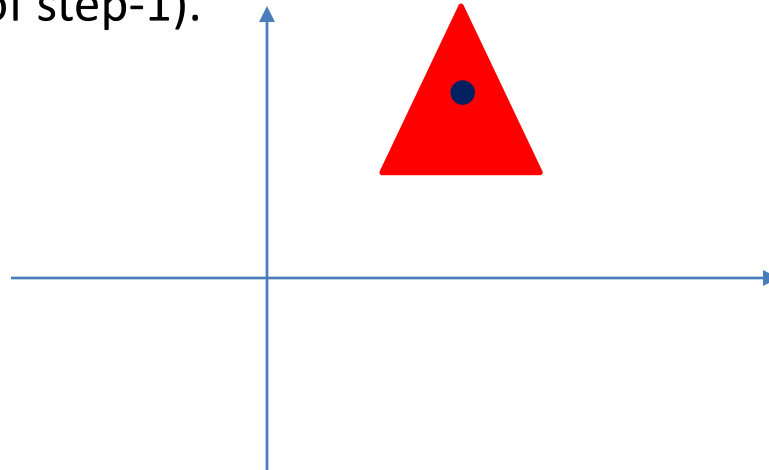
General Fixed-Point Scaling Equation

- Let's find matrix equation for this

- $P' = T(x_f, y_f) \cdot [S(s_x, s_y) \cdot \{T(-x_f, -y_f) \cdot P\}]$

- $P' = \{T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f)\} \cdot P$

- $P' = \begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

Contd.

- $P' = \begin{bmatrix} s_x & 0 & x_f(1 - s_x) \\ 0 & s_y & y_f(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = S(x_f, y_f, s_x, s_y) \cdot P$

- Here P' and P are column vector of final and initial point coordinate respectively and (x_f, y_f) are the coordinates of fixed-point.

General Fixed-Point Scaling Example

- **Example**: - Consider square with left-bottom corner at (2, 2) and right-top corner at (6, 6) apply the transformation which makes its size half such that its center remains same.

- Fixed point is center of square so:

- $x_f = \dfrac{6+2}{2}$, $\quad y_f = \dfrac{6+2}{2}$

- As we want size half so value of scale factor are $s_x = 0.5, s_y = 0.5$ and Coordinates of square are [A (2, 2), B (6, 2), C (6, 6), D (2, 6)].

- $P' = S(x_f, y_f, s_x, s_y) \cdot P$

Contd.

- $P' = S(x_f, y_f, s_x, s_y) \cdot P$

- $P' = \begin{bmatrix} s_x & 0 & x_f(1 - s_x) \\ 0 & s_y & y_f(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 0.5 & 0 & 4(1 - 0.5) \\ 0 & 0.5 & 4(1 - 0.5) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 0.5 & 0 & 2 \\ 0 & 0.5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Contd.

- $P' = \begin{bmatrix} 0.5 & 0 & 2 \\ 0 & 0.5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 3 & 5 & 5 & 3 \\ 3 & 3 & 5 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- Final coordinate after scaling are:

  [A' (3, 3), B' (5, 3), C' (5, 5), D' (3, 5)]

General Scaling Directions

- Parameter s_x and s_y scale the object along x and y directions.

- We can scale an object in other directions also.

- By rotating the object to align the desired scaling directions with the coordinate axes before applying the scaling transformation.

Contd.

- Suppose we apply scaling factor s_1 and s_2 in direction shown in figure than we will apply following transformations.

  1. Perform a rotation so that the direction for s_1 and s_2 coincide with x and y axes.

  2. Scale the object with specified scale factors.

  3. Perform opposite rotation to return points to their original orientations. (i.e. Inverse of step-1).
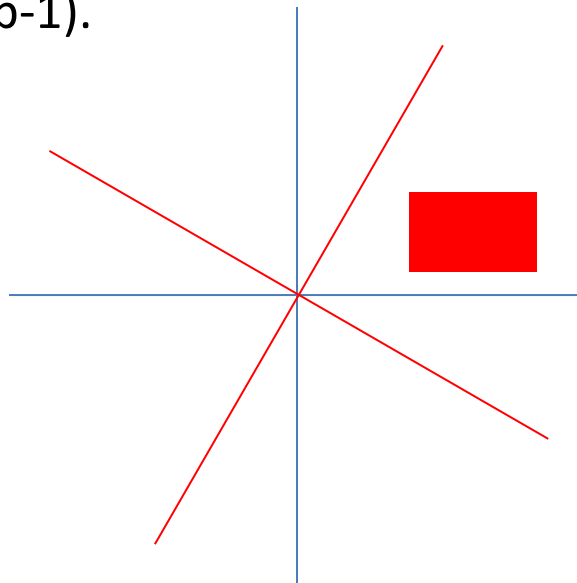
General Scaling Directions Equation

- Let's find matrix equation for this

- $P' = R^{-1}(\theta) \cdot [S(s_1, s_2) \cdot \{R(\theta) \cdot P\}]$

- $P' = \{R^{-1}(\theta) \cdot S(s_1, s_2) \cdot R(\theta)\} \cdot P$

- $P' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- $P' = \begin{bmatrix} s_1 \cos^2\theta + s_2 \sin^2\theta & (s_2 - s_1)\cos\theta\sin\theta & 0 \\ (s_2 - s_1)\cos\theta\sin\theta & s_1 \sin^2\theta + s_2 \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$

- Here P' and P are column vector of final and initial point coordinate respectively and θ is the angle between actual scaling direction and our standard coordinate axes.

Other Transformation

- Some package provides few additional transformations which are useful in certain applications.

- Two such transformations are:

  1. Reflection
  2. Shear.

1. Reflection

- A reflection is a transformation that produces a mirror image of an object.

- The mirror image for a two –dimensional reflection is generated relative to an **axis of reflection** by rotating the object 180° about the reflection axis.

- Reflection gives image based on position of axis of reflection. Transformation matrix for few positions are discussed here.

Reflection About X-Axis

- For reflection about the line

- $y = 0$, $the\ x\ axis$

- Transformation matrix:

- $Ref_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

y

1

Original
Position

2   3

x

2'   3'

Reflected
Position

1'
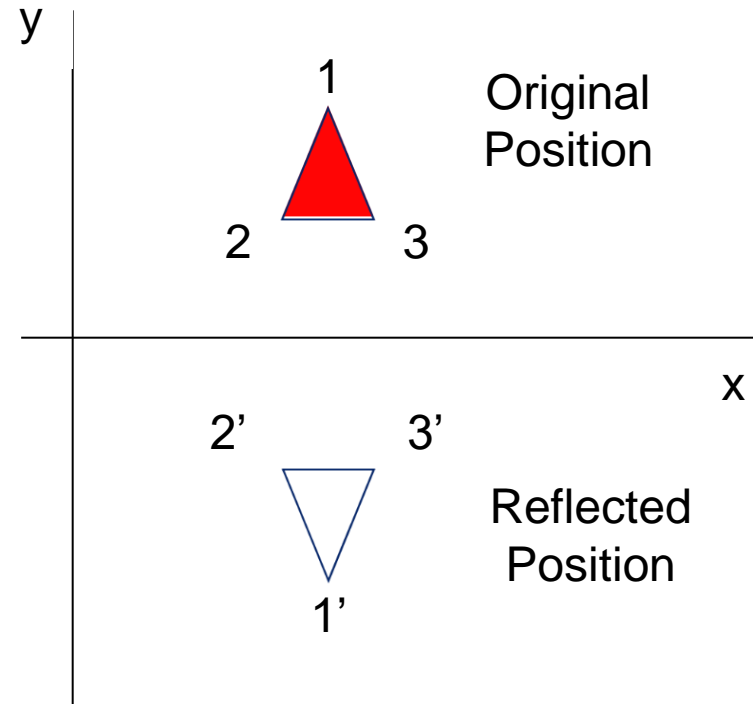
- This transformation keeps x values are same, but flips (Change the sign) y values of coordinate positions.

Reflection About Y-Axis

- For reflection about the line $x = 0$, *the y axis*

- Transformation matrix: $Ref_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- This transformation keeps y values are same, but flips (Change the sign) x values of coordinate positions.
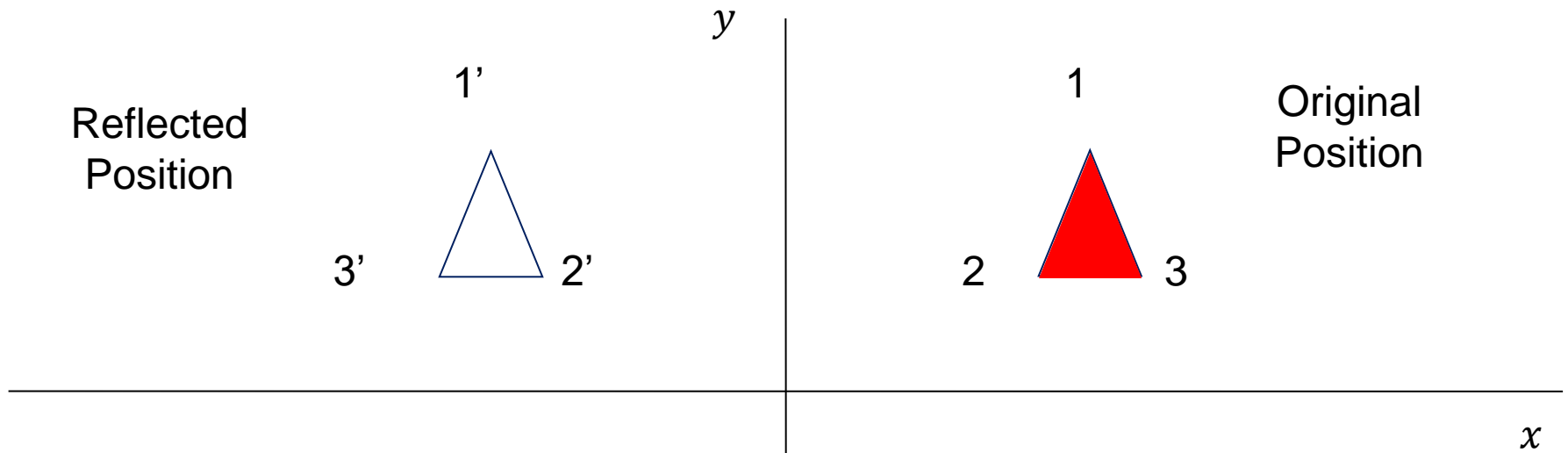
Reflection About Origin

- For reflection about the $\boldsymbol{Origin}$.

- Transformation matrix:

- $Ref_{origin} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

y

Original
Position

x

Reflected
Position
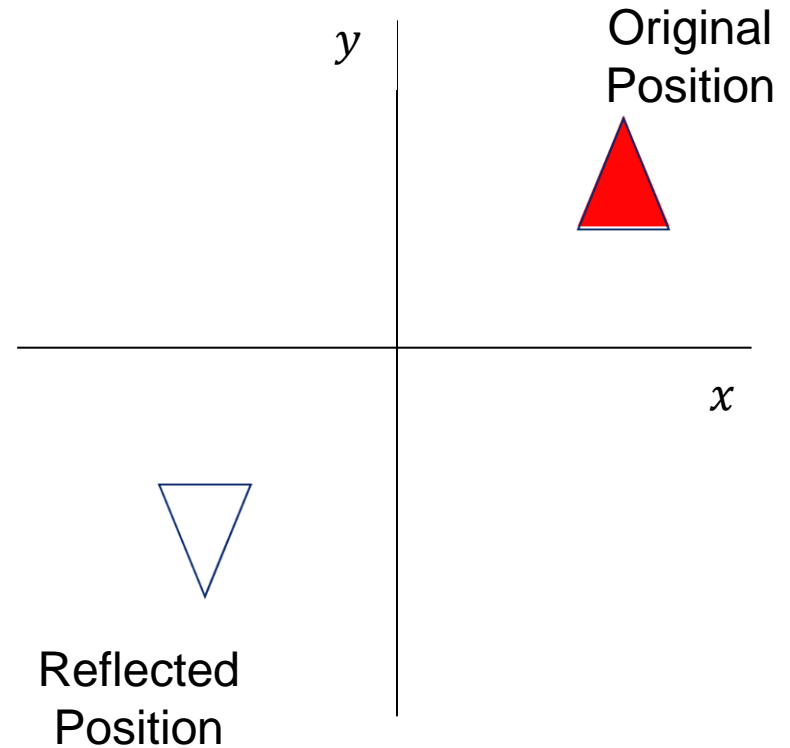
- This transformation flips (Change the sign) x and y both values of coordinate positions.

Reflection About X = Y Line

- For reflection about the line $x = y$.

- Transformation matrix:

- $Ref_{x=y\ line} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

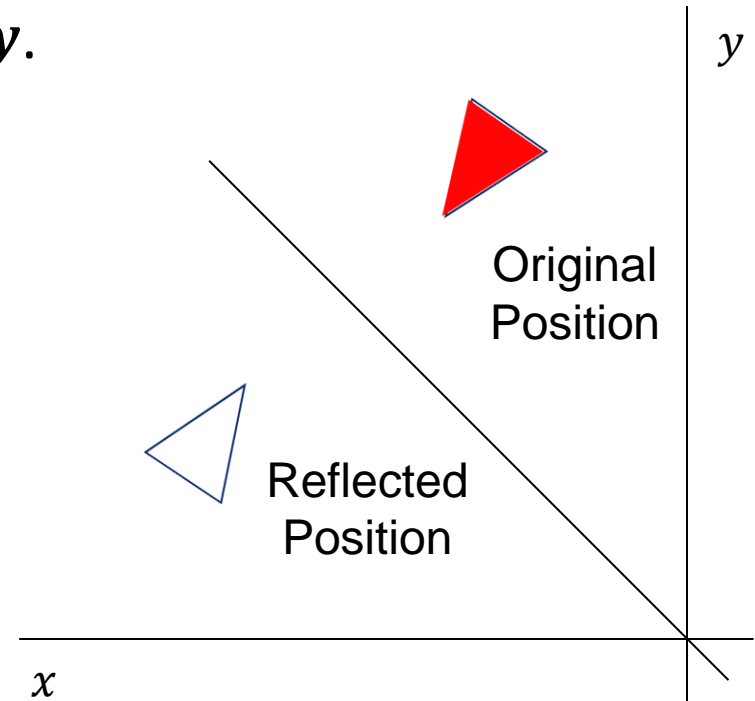- This transformation interchange x and y values of coordinate positions.

Reflection About X = - Y Line

- For reflection about the line $x = -y$.

- Transformation matrix:

- $Ref_{x=-y\ line} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Original Position
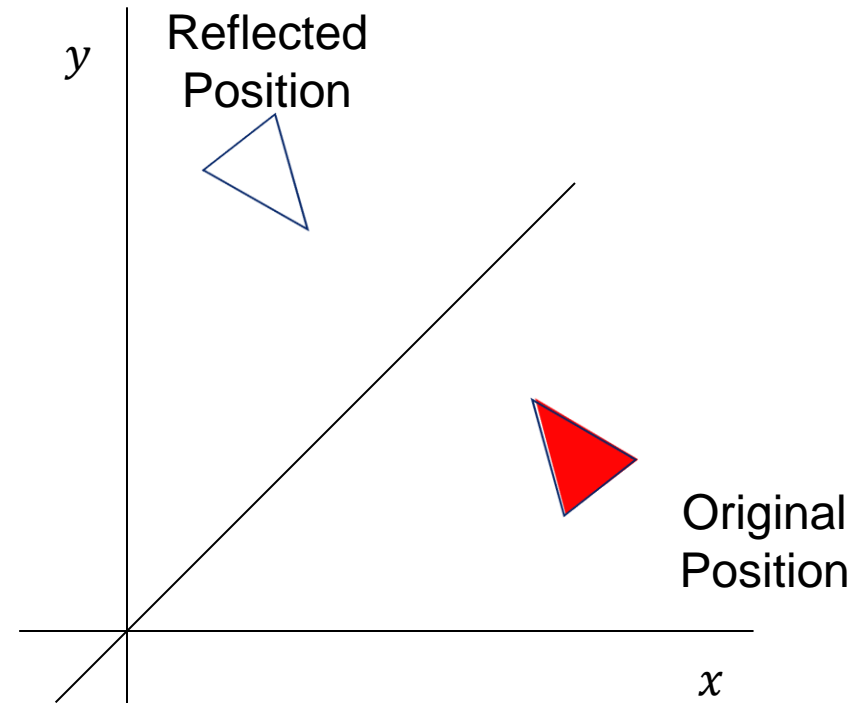
Reflected Position

- This transformation interchange x and y values of coordinate positions.

Reflection Example

- **Example**: - Find the coordinates after reflection of the triangle [A (10, 10), B (15, 15), C (20, 10)] about x axis.

- $P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 15 & 20 \\ 10 & 15 & 10 \\ 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 10 & 15 & 20 \\ -10 & -15 & -10 \\ 1 & 1 & 1 \end{bmatrix}$

- Final coordinate after reflection are:
  [A' (10, -10), B' (15, -15), C' (20, -10)]

2. Shear

- A transformation that distorts the shape of an object such that the transformed shape appears as if the object were composed of internal layers that had been caused to slide over each other is called **shear**.

- Two common shearing transformations are those that shift coordinate x values and those that shift y values.
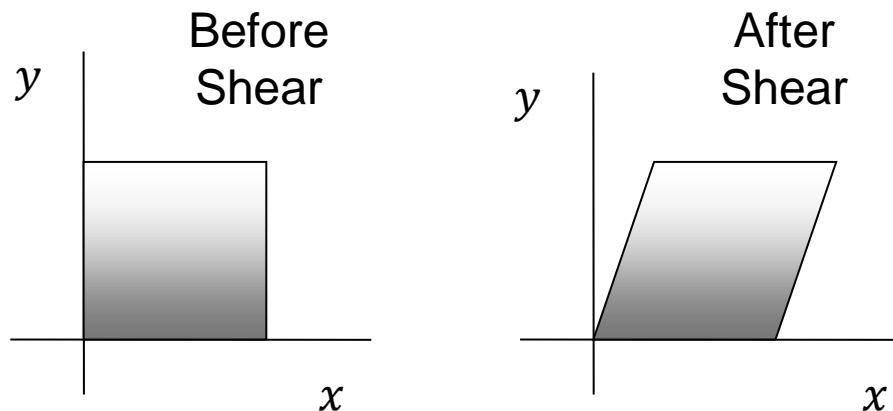
Shear in x − Direction

- Shear relative to $x - axis$ that is $y = 0$ line can be produced by following equation:

$$x' = x + sh_x \cdot y, \qquad y' = y$$

- Transformation matrix for that is:

$$Shear_{x-direction} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

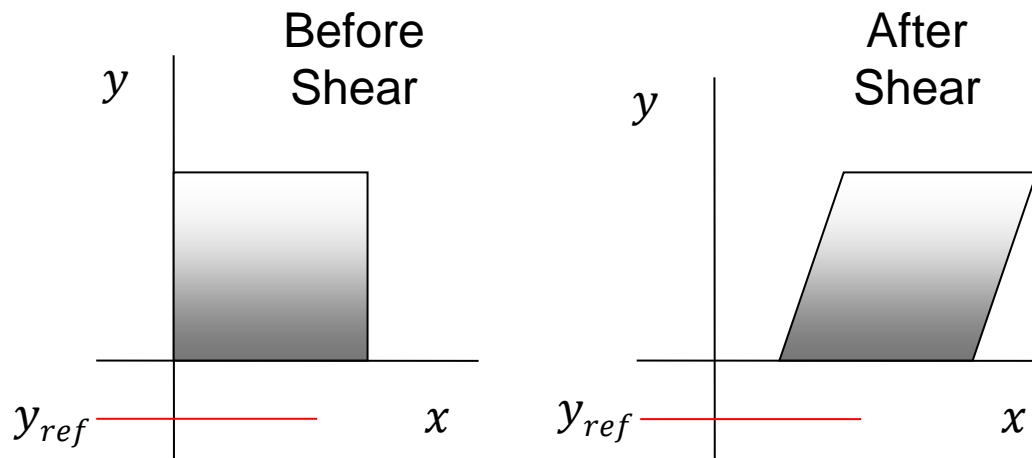- Here sh_x is shear parameter.

- We can assign any real value to sh_x.

Before Shear

After Shear

Contd.

- We can generate $x - direction$ shear relative to other reference line $y = y_{ref}$ with following equation:

$$x' = x + sh_x \cdot (y - y_{ref)}, \qquad y' = y$$

- Transformation matrix for that is:

$$Shear_{x-direction} = \begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shear in x − Direction Example

- **Example**: - Shear the unit square in x direction with shear parameter ½ relative to line $y = -1$.

- Here $y_{ref} = -1$ and $sh_x = 0.5$

- Coordinates of unit square are: [A (0, 0), B (1, 0), C (1, 1), D (0, 1)].

- $P' = \begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 1 & 0.5 & -0.5 \cdot (-1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Contd.

- $P' = \begin{bmatrix} 1 & 0.5 & -0.5 \cdot (-1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 0.5 & 1.5 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- Final coordinate after shear are:

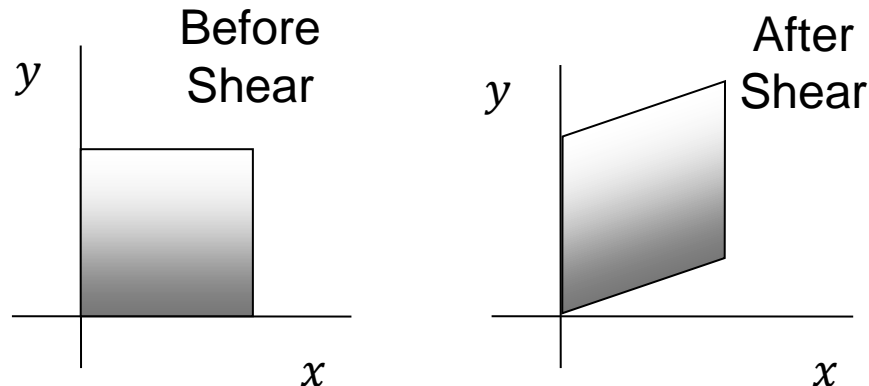 [A' (0.5, 0), B' (1.5, 0), C' (2, 1), D' (1, 1)]

Shear in y − Direction

- Shear relative to $y − axis$ that is $x = 0$ line can be produced by following equation:

$$x' = x , \qquad y' = y + sh_y \cdot x$$

- Transformation matrix for that is:

$$Shear_{y-direction} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Here sh_y is shear parameter.
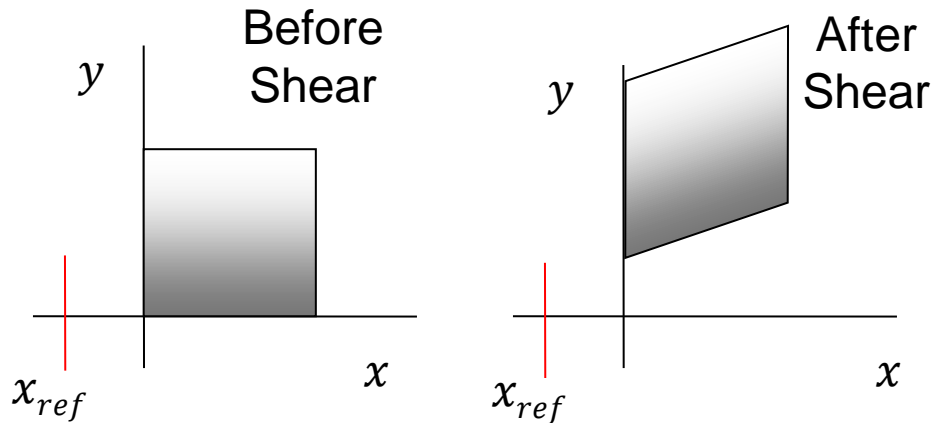
- We can assign any real value to sh_y.

Contd.

- We can generate $y - direction$ shear relative to other reference line $x = x_{ref}$ with following equation:

$$x' = x, \qquad y' = y + sh_y \cdot (x - x_{ref})$$

- Transformation matrix for that is:

$$Shear_{y-direction} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

Shear in y − Direction Example

- **Example**: - Shear the unit square in y direction with shear parameter ½ relative to line $x = -1$.

- Here $x_{ref} = -1$ and $sh_y = 0.5$

- Coordinates of unit square are [A (0, 0), B (1, 0), C (1, 1), D (0, 1)].

- $P' = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & -0.5 \cdot (-1) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Contd.

- $P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & -0.5 \cdot (-1) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- $P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0.5 & 1 & 2 & 1.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- Final coordinate after shear are:

  [A$'$ (0, 0.5), B$'$ (1, 1), C$'$ (1, 2), D$'$ (0, 1.5)]