Mechanical Vibrations

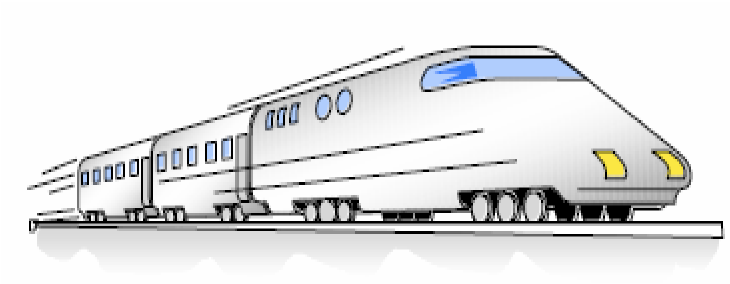
Basics & Beyond

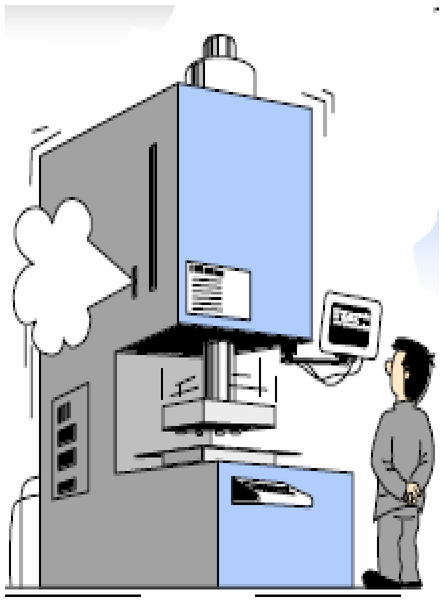
~Dr. Mitesh Mungla

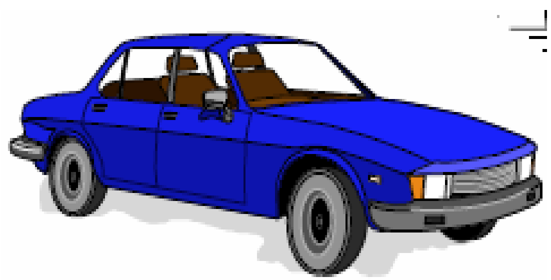
What is Vibration?

Vibration is the motion of a particle or body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable due to increased stresses and energy losses.

Vibration in Everyday Life

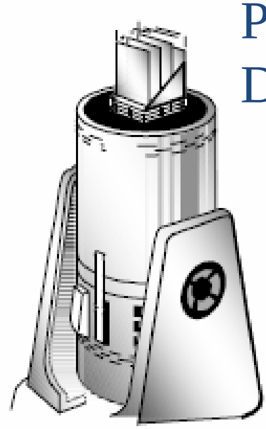




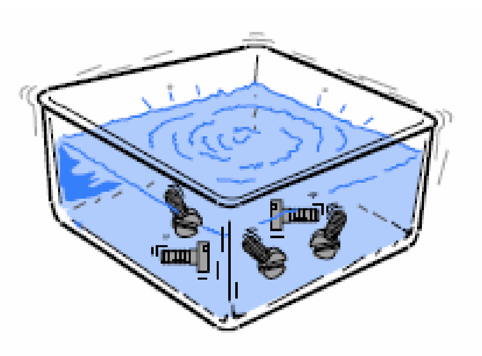


Useful Vibration

Concrete Compactor



Pile Drivers

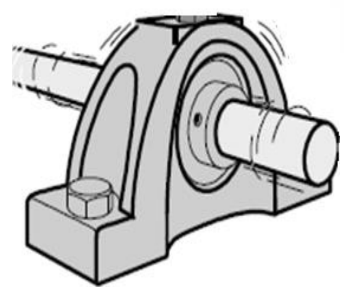


Ultrasonic Cleaning Baths

Harmful Vibration

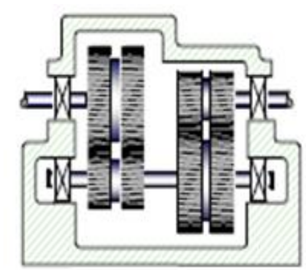
Noise

Destruction



Wear

Fatigue



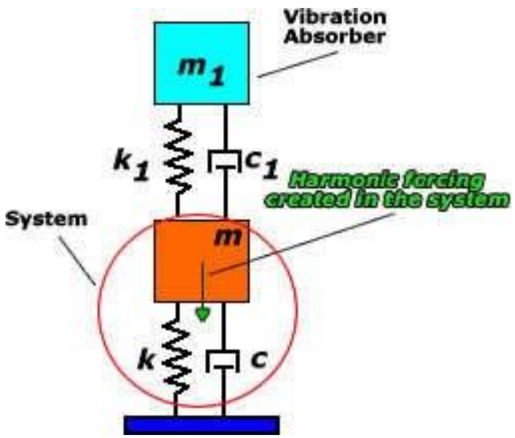
Vibration in our Lives

- Our heart beats
- Our lungs oscillate
- We hear because our ear drums vibrate
- Vibration makes us snore

- Light waves permit us to see
- Sound waves allow us to hear
- We move because of oscillation of legs
- We can not utter ‘vibration’ without the oscillation of larynges and vocal cords

Vibration in our Lives

- We limit our discussion to Mechanical Vibration
- Vibration of dynamic system of a structure
- It is the oscillations of a system that has mass and elasticity



Vibration – Friend or Foe

Friend

- Conveyors, Hoppers, Compactors, Pneumatic Drills, etc.
- Opening of a cork from a wine bottle
- Washing Machine
- Mechanical Shakers, Mixers, Sieves, Sorters, etc.
- Musical Instruments
- Clocks, Watches
- Medical Field – Massagers, etc.

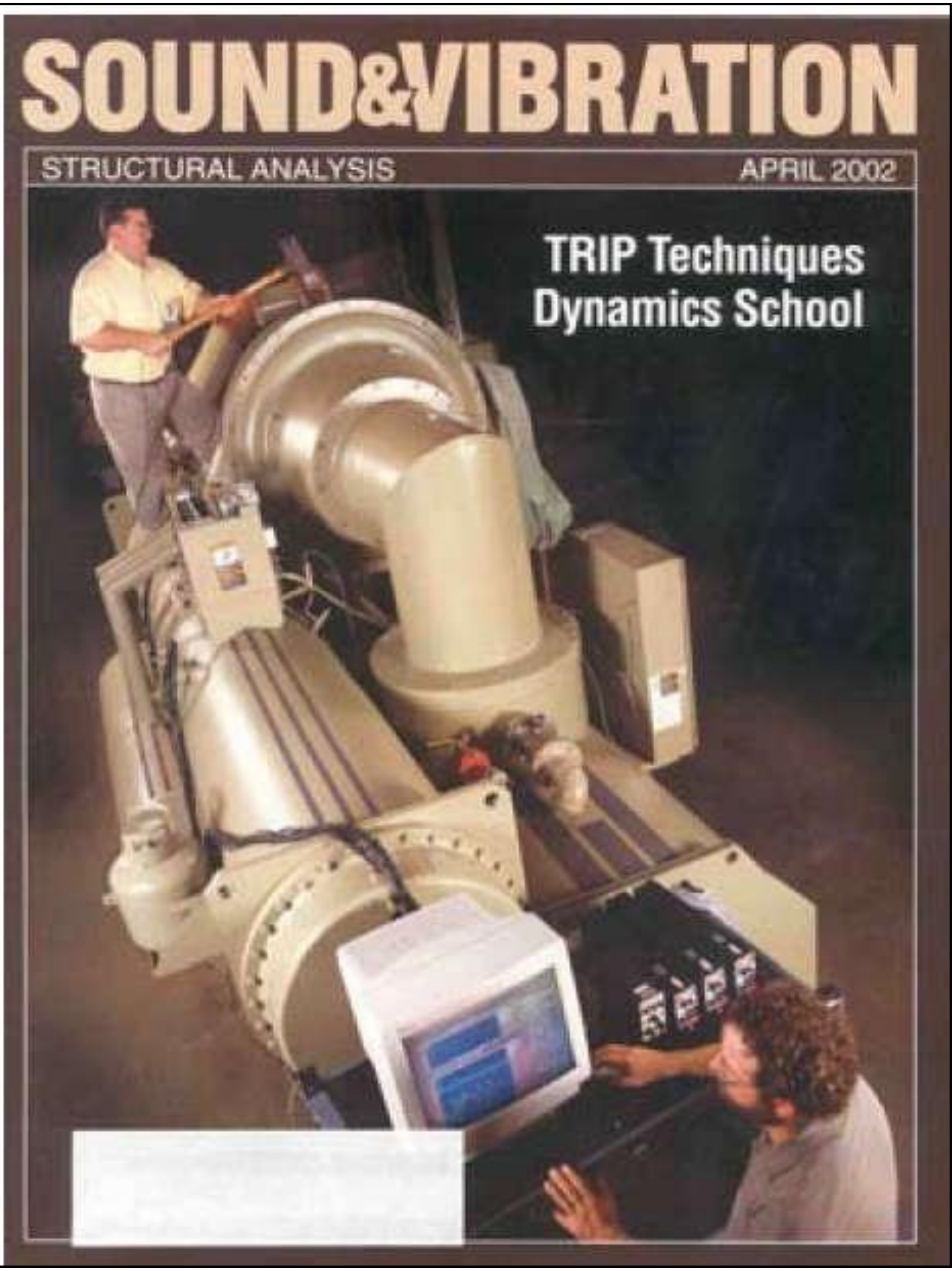
Foe

- RESONANCE – RESONANCE – RESONANCE

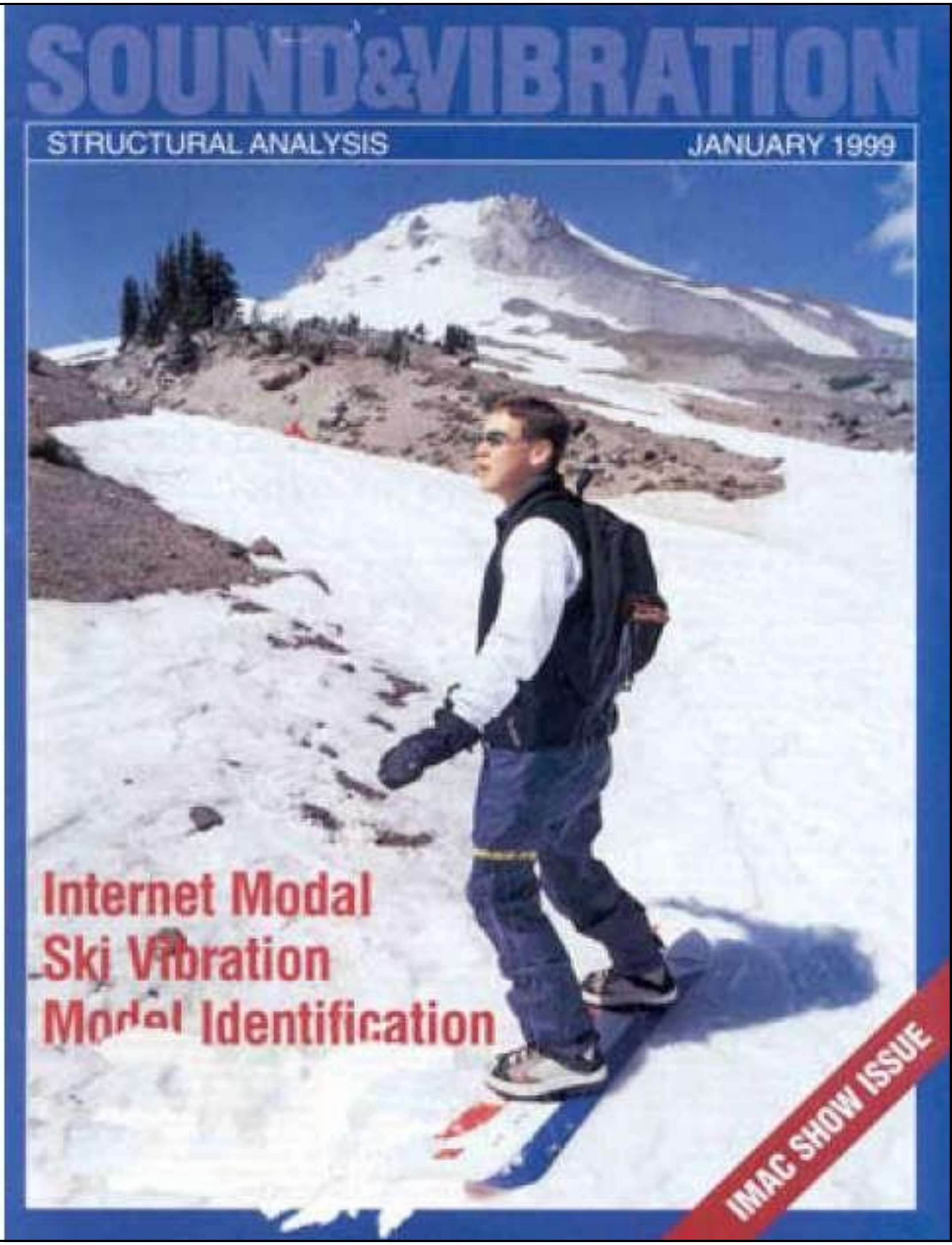
Tacoma Narrows Bridge

- Similar problems in machine tools, vehicles, turbines, pumps, compressors, buildings, aircraft & spacecraft systems
- Excessive vibration leads to loosening on parts, noise & eventual failure
- Effects of vibration on human body: Discomfort, Fatigue, Loss of Efficiency
- Sound quality in products

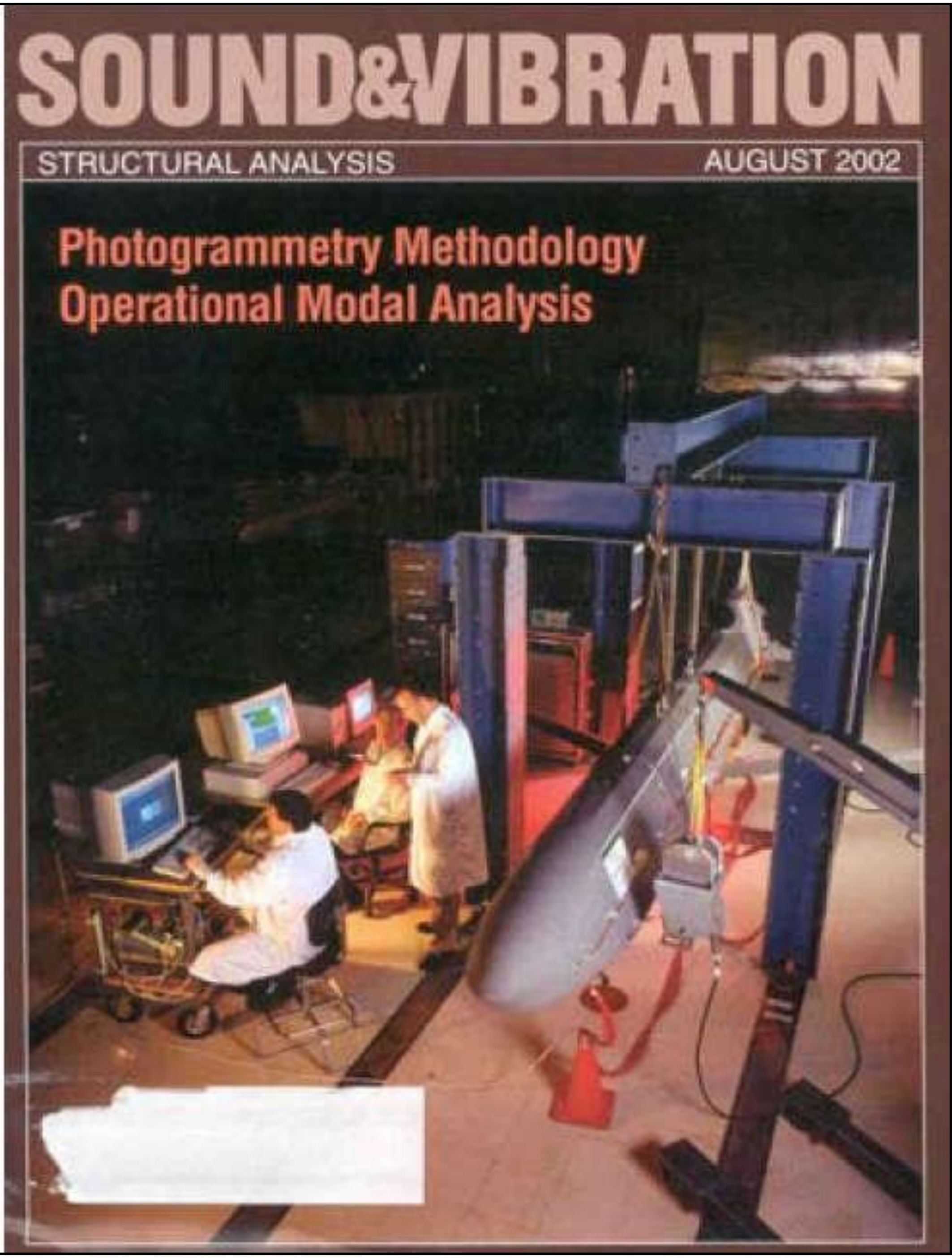
Vibration in Machinery



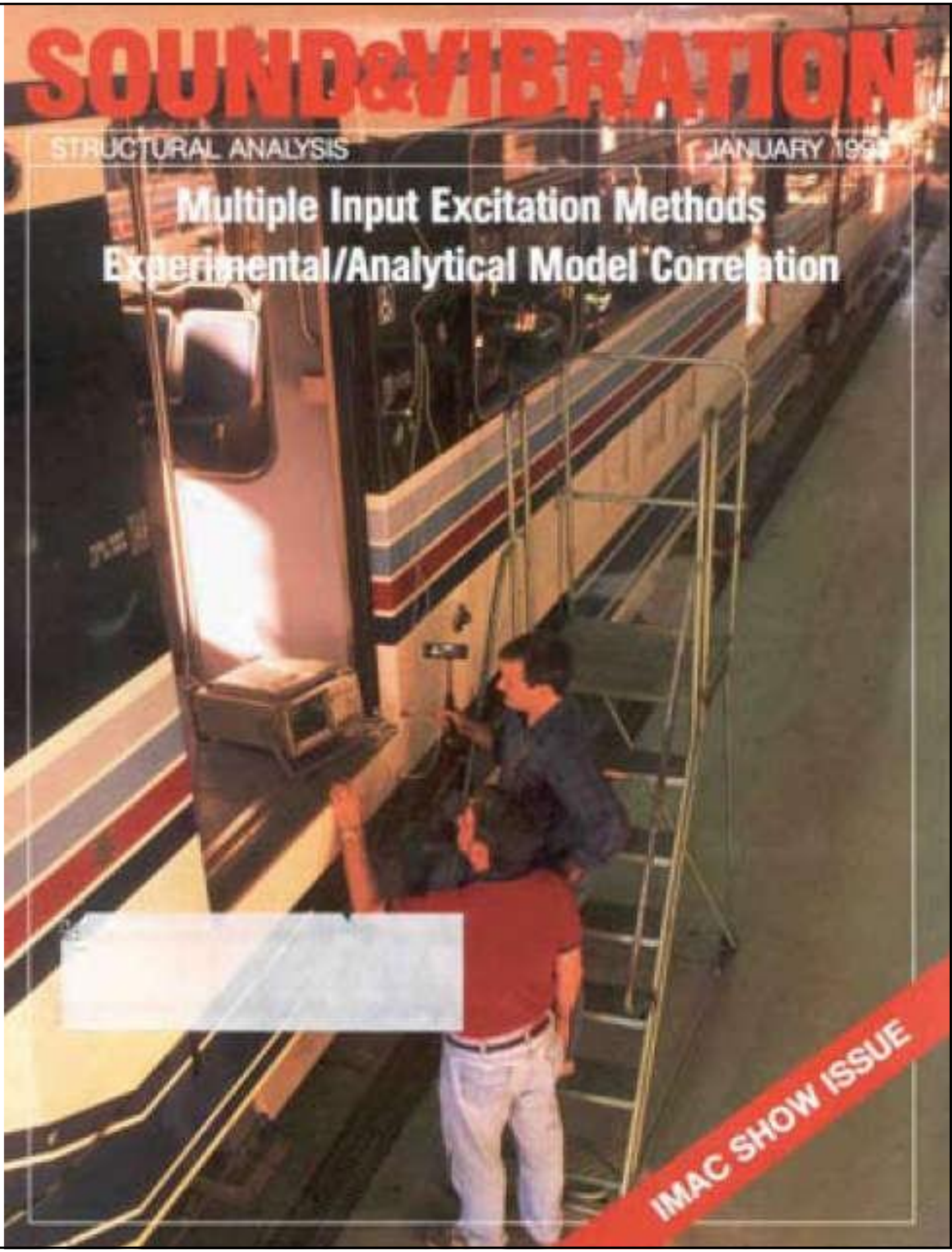
Vibration in Recreation



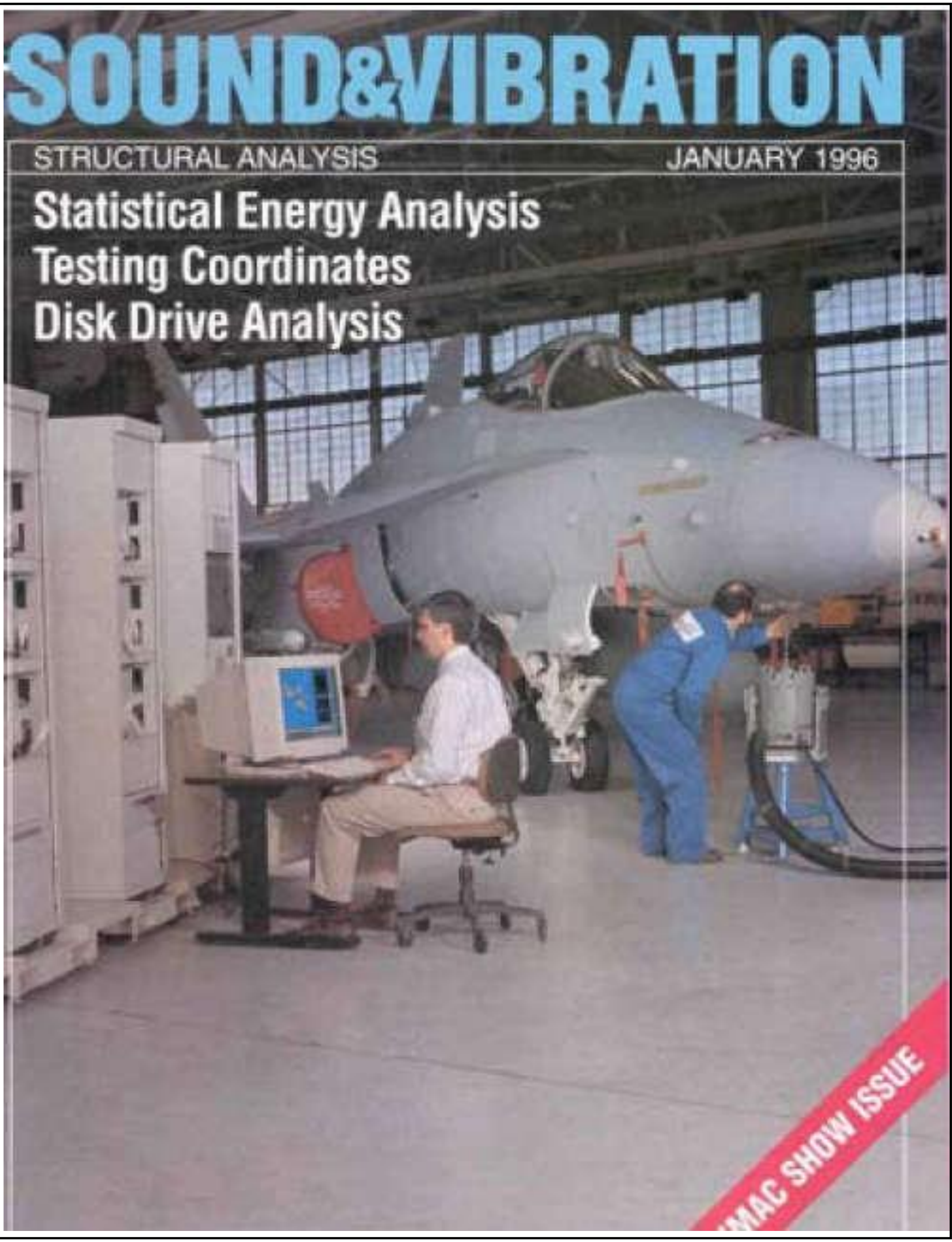
Vibration in Defense



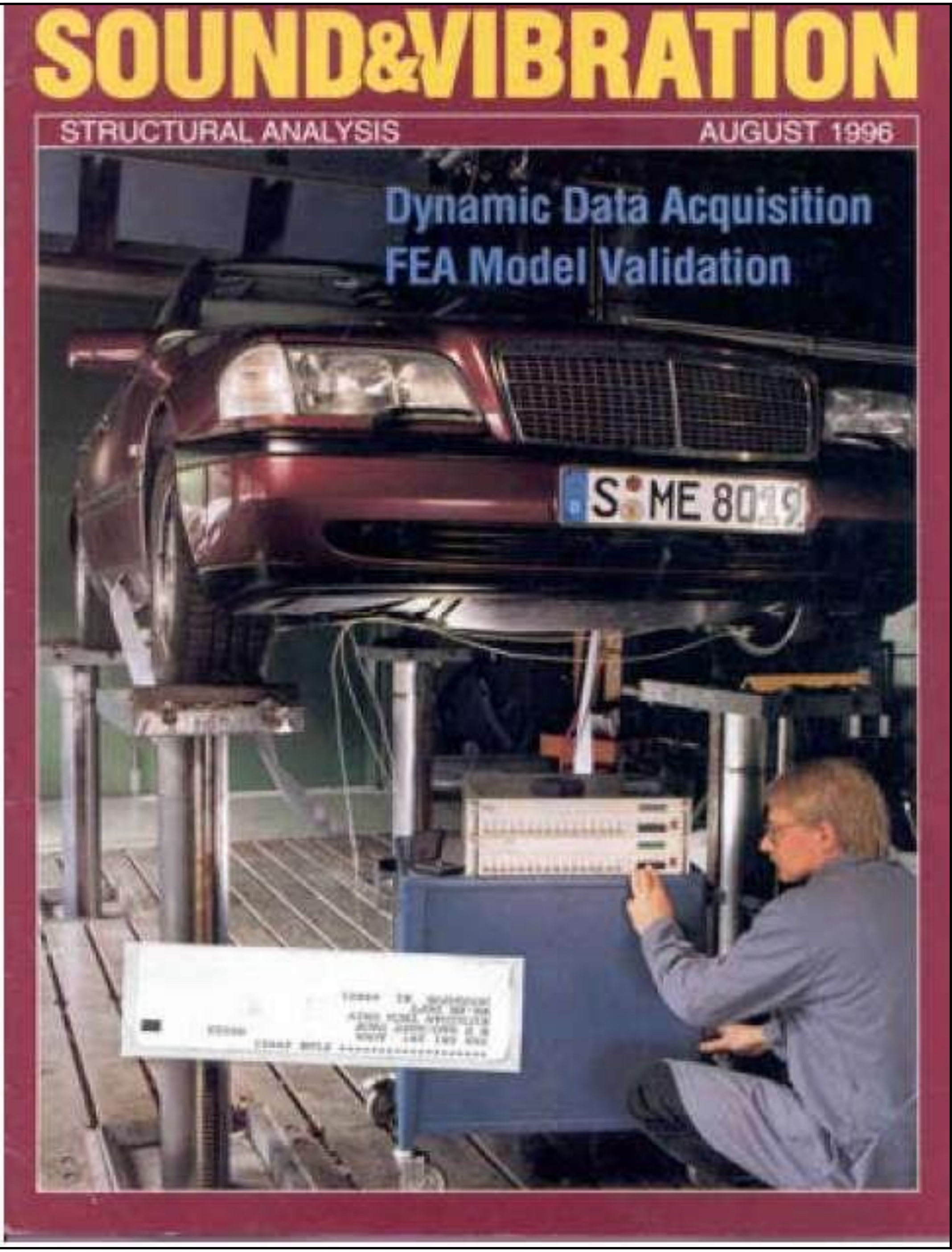
Vibration in Transportati on



Vibration in Aerospace

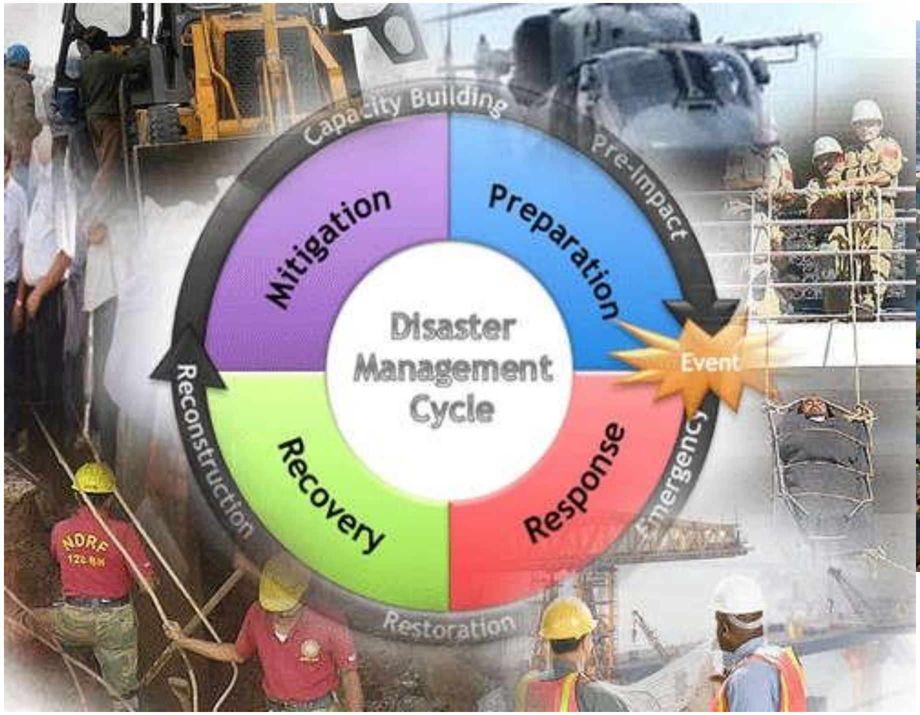


Vibration in Automobile



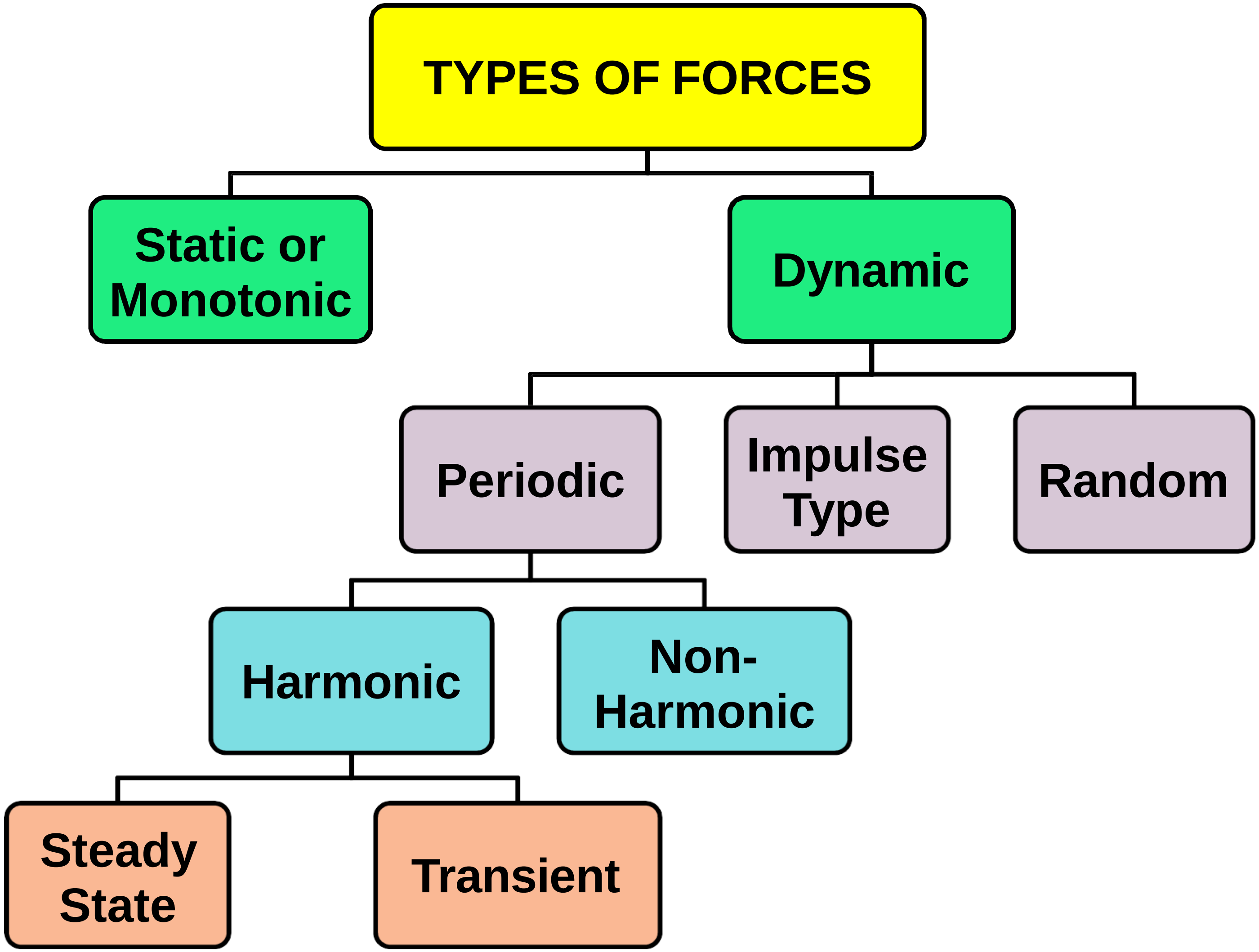
Vibration in Health Care

Vibration in Structures



Vibration during Disasters

Basic Definitions



Basic Definitions

Periodic Motion: A motion that repeats itself after equal interval of time.

Time Period: Time taken for one complete cycle.

Simple Harmonic Motion: Motion of particle with time that moves round a circle uniform with angular velocity. Trigonometric functions can be used to represent such motion.

Basic Definitions

Amplitude (Z or 2Z): The maximum displacement of a vibrating body from its mean position. The amplitude can either be single amplitude (Z) when the distance from mean position to maximum displacement is measured or double amplitude (2Z) when the distance from negative maximum to positive maximum displacement (motion) is measured.

Frequency: It is the number of cycles per unit time. Frequency and time period are inversely proportional to each other. A vibratory motion can have either a very high frequency or a very low frequency. Frequency can be expressed either as angular (circular) frequency (ω) or oscillatory frequency (f). ω is expressed in radians per second and f is expressed in cycles per second or Hertz.

Basic Definitions

Natural frequency: It is the frequency of free vibration of a system. It is constant for a system. In fact, it is an inherent property of a system. It depends on the elastic properties, mass and stiffness of the system.

Resonance: Vibration of a system when the frequency of external force is equal to the natural frequency of the system. The amplitude of vibration at resonance becomes excessive. During resonance, with minimum input, there will be a maximum output. Hence both displacement and the stresses in the vibrating body become very high.

Basic Definitions

Damping: It is the resistance to motion. It is also the sluggishness. Hence it is the delay in response to any action. Damping is observed only under fast loading, and not during static loading.

Degree of freedom: The number of independent coordinate systems required to specify a motion. If the motion is in one direction due to the vibration of a single spring, then it is a Single degree of freedom system. If a particle is likely to vibrate in space, it will have six degrees of freedom, namely three translations and three rotations along three axis. A continuum can have infinite degrees of freedom.

Basic Definitions

Phase difference : The angle between two rotating vectors representing Simple Harmonic Motion, In time domain, it can be represented as the delay in one motion compared with the other.

Wave : It is the vibratory motion of a body or a particle represented in time domain or space domain. For representing a one dimensional wave mathematically, the partial differential equation is given by,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

Basic Definitions

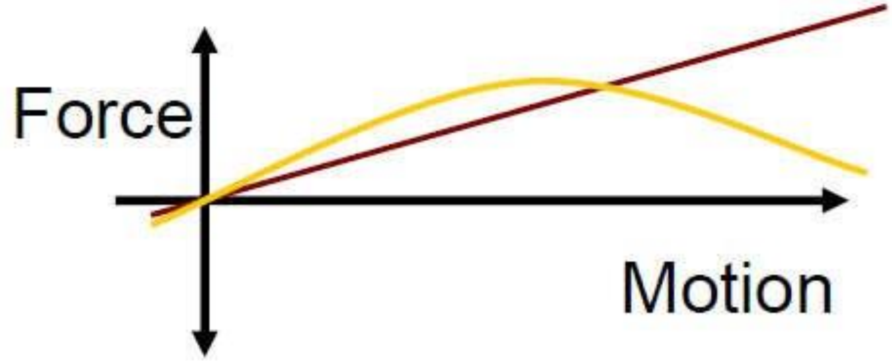
$$T = \frac{2\pi}{\omega} = \frac{1}{f} \qquad \lambda = \frac{2\pi}{k} = \frac{v}{f} = vT$$

T = Time period (in sec)
ω = Angular or Circular velocity (in rad/sec)
f = Frequency of oscillations (in cycles/sec or Hz)
λ = Wave length (in m)
k = Wave number = ω/v (in rad/m) v = Wave velocity (in m/sec)

Dynamics

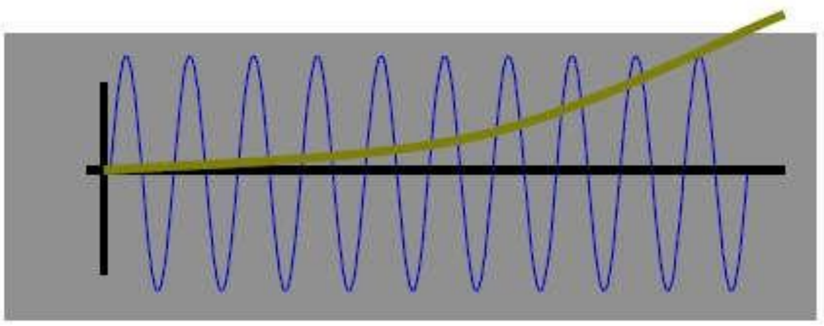
Linear

Non-Linear



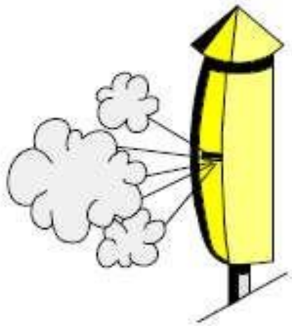
Arbitrary motion

Harmonic Motion

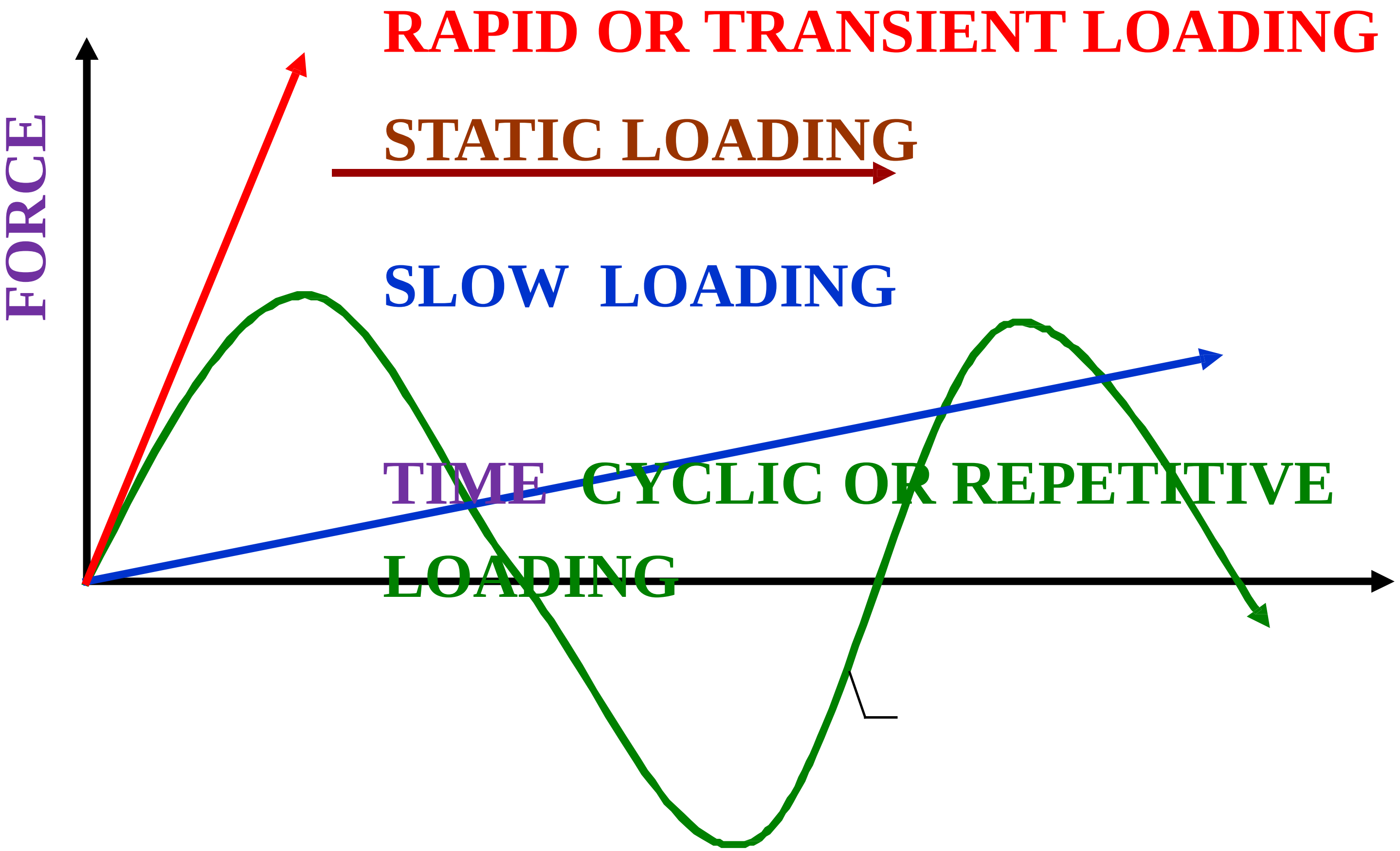


Mechanical Vibrations

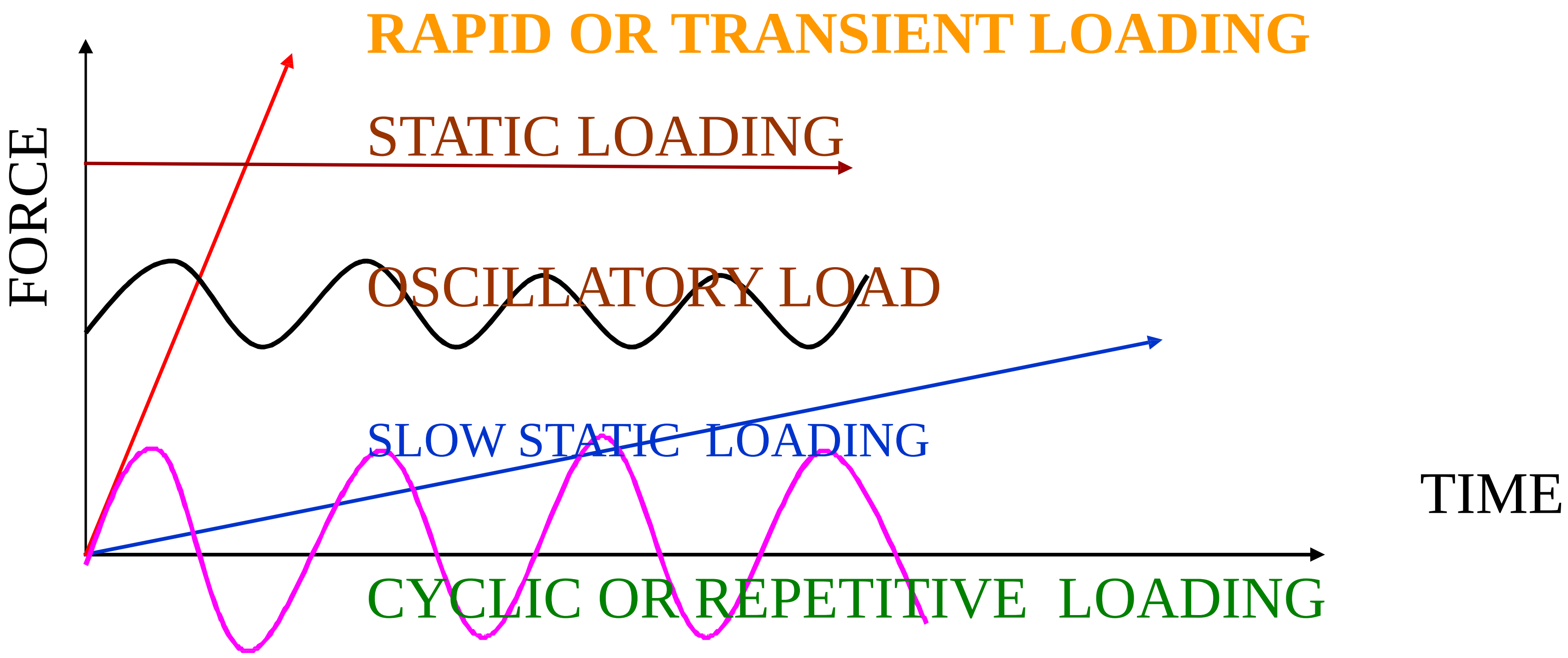
Sound (Acoustics)



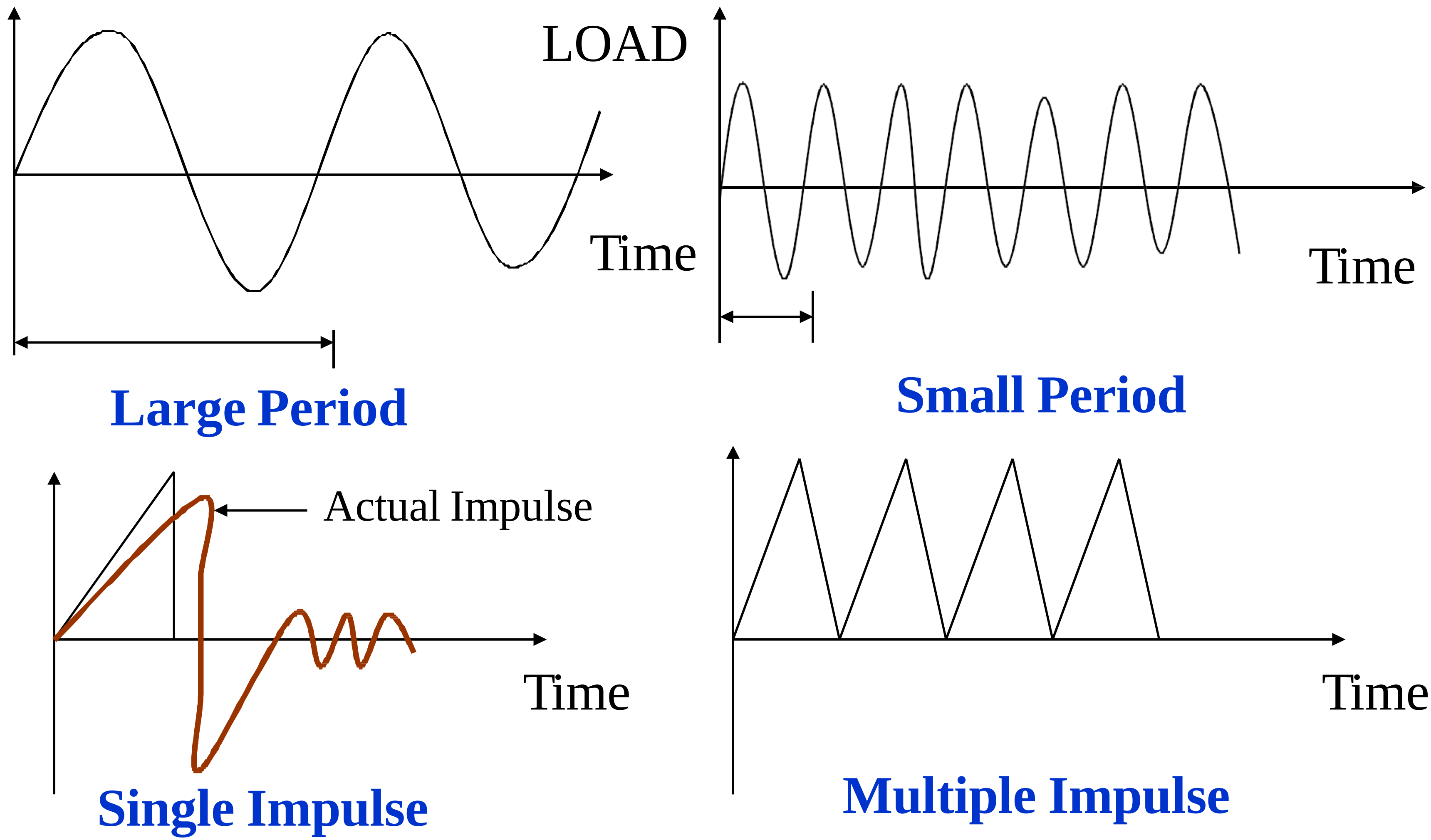
TYPES OF LOADING



TYPES OF LOADING

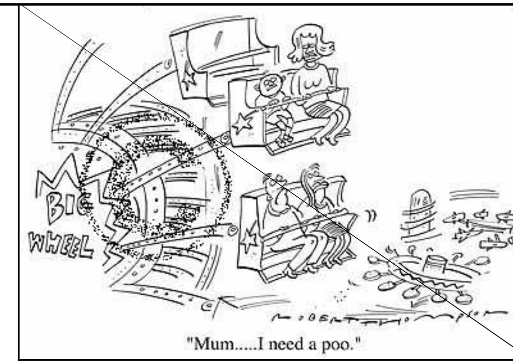


WHAT IS DYNAMIC FORCE ?

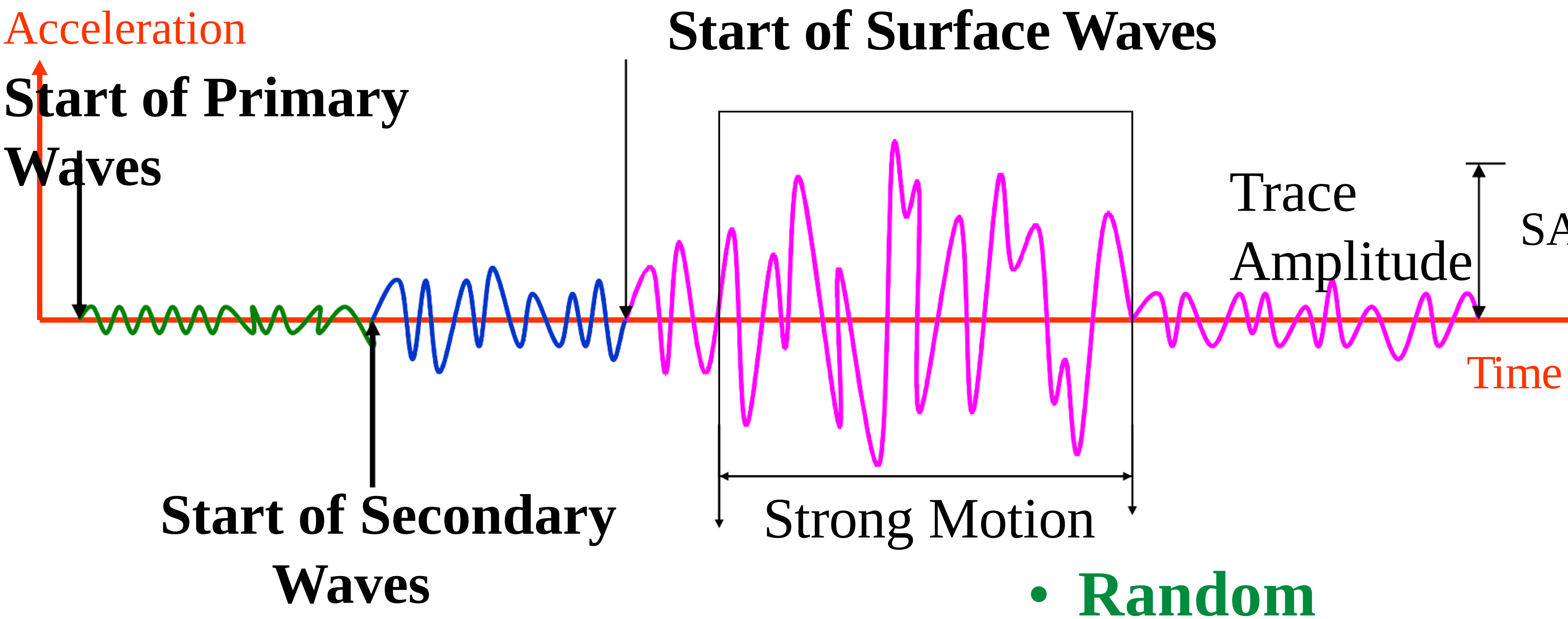


Typical Seismogram

- **PGA**
- **Predominant Frequency**
- **Duration of Strong Motion**



"Mum.....I need a poo."



- **Random**
- **Time Dependent**
- **Cyclic**

Basic Definitions

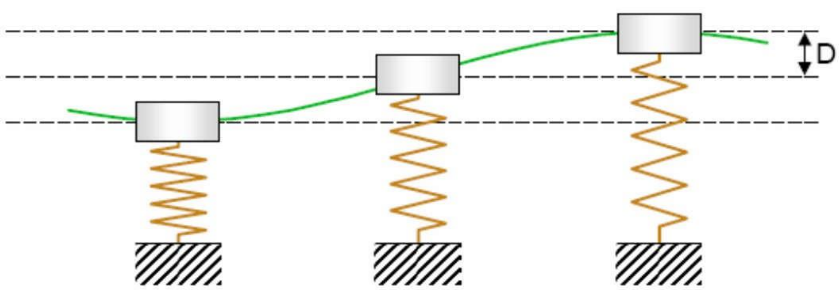
Free Vibration: Vibration of a system because of its own elastic property. No external force is required for this vibration and only initiation of vibration may be necessary.

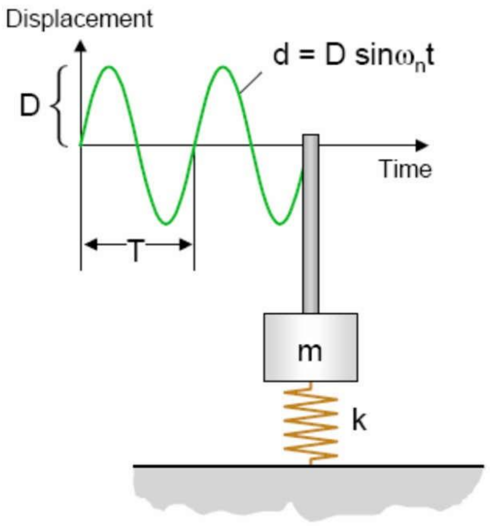
Forced Vibration: A system that vibrates under an external force at the same frequency as that of external force.

Free vibration

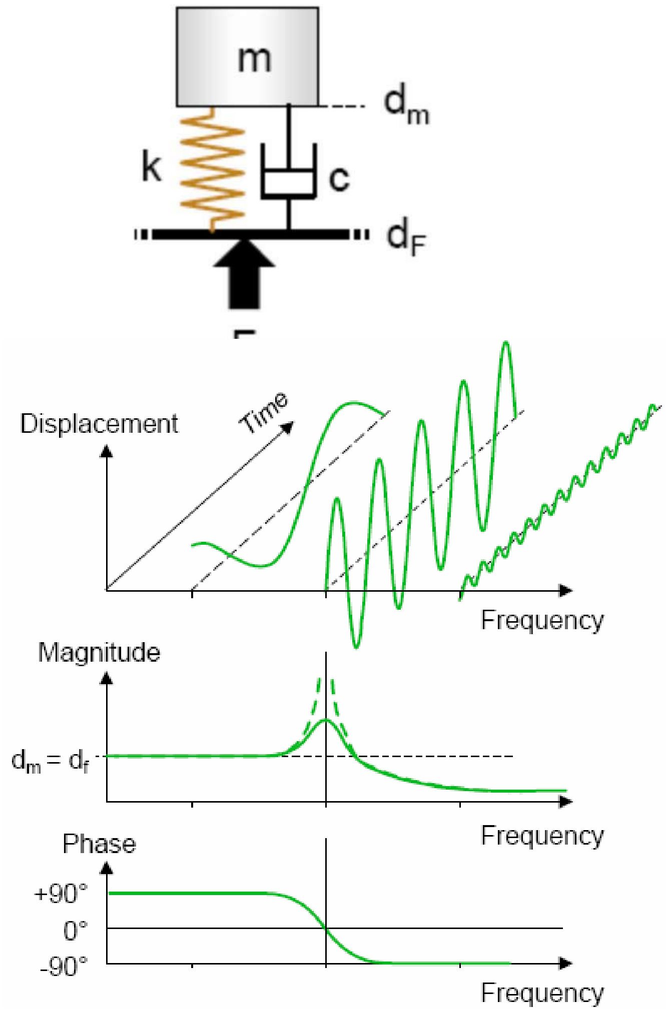
- When a system is initially disturbed by a displacement, velocity or acceleration, the system begins to vibrate with a constant amplitude and frequency depending on its stiffness and mass.
- This frequency is called as **natural frequency,** and the form of the vibration is called as **mode shapes**

Equilibrium pos.





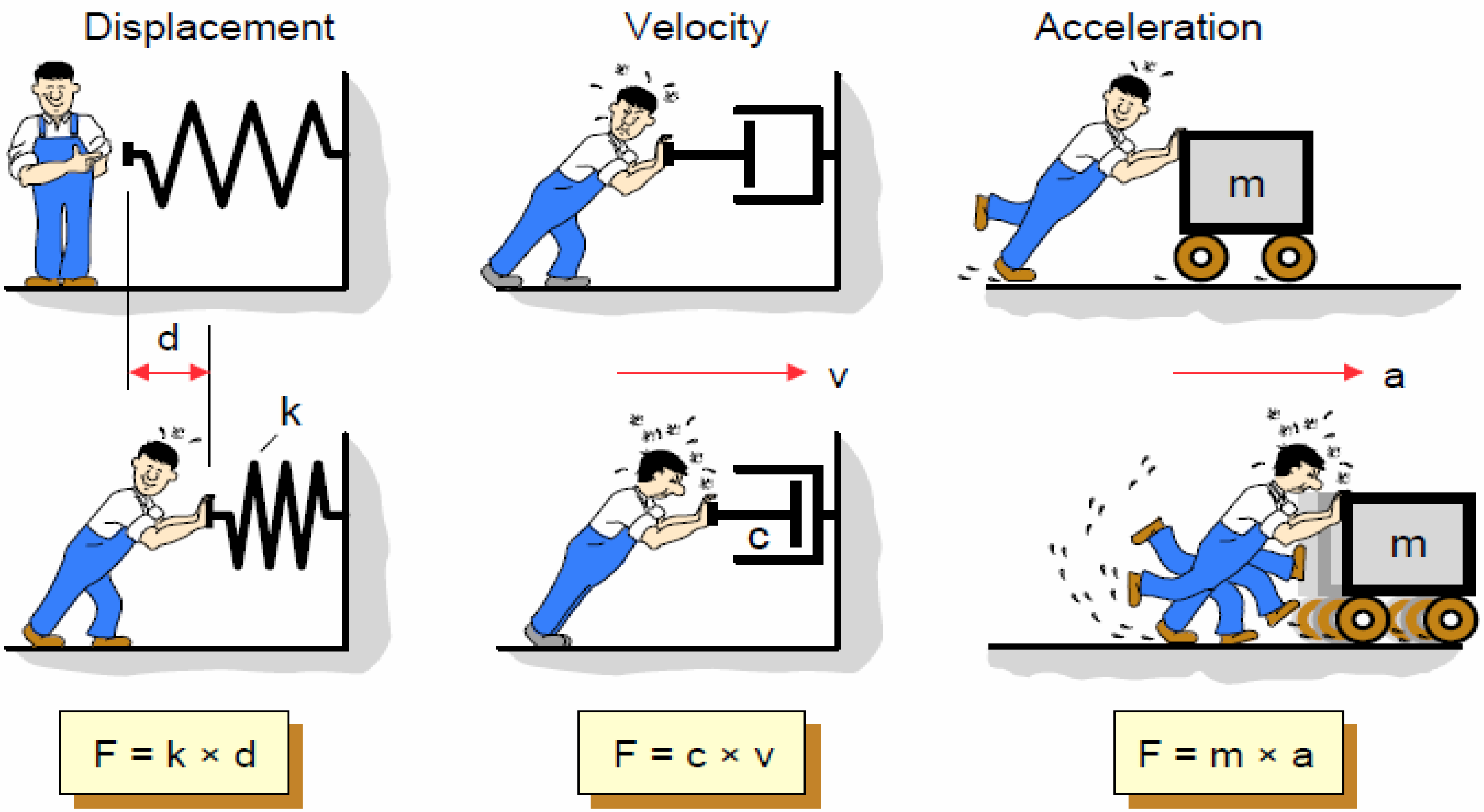
Forced Vibration



If an external force is applied to a system, the system will follow the force with the same frequency.

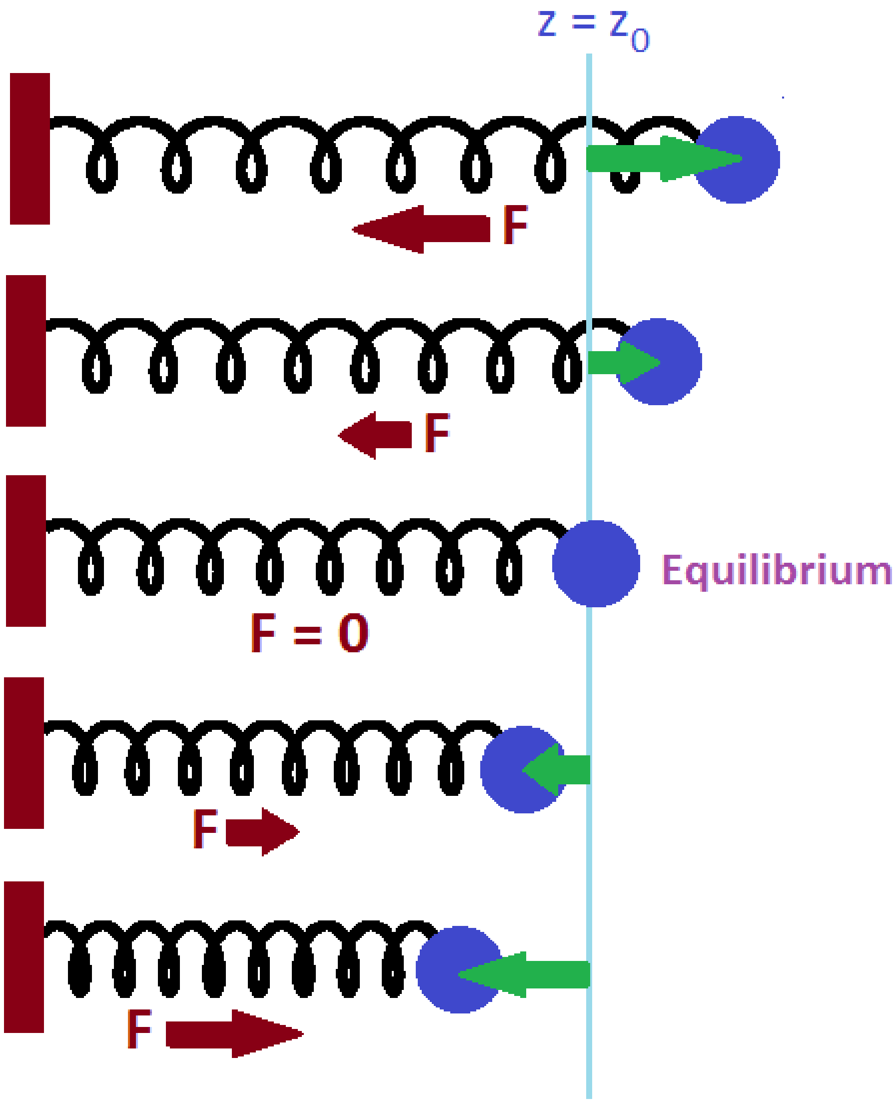
However, when the forcing frequency is increased to the system's natural frequency, amplitudes will dangerously increase in this region. This phenomenon called as **"Resonance"**

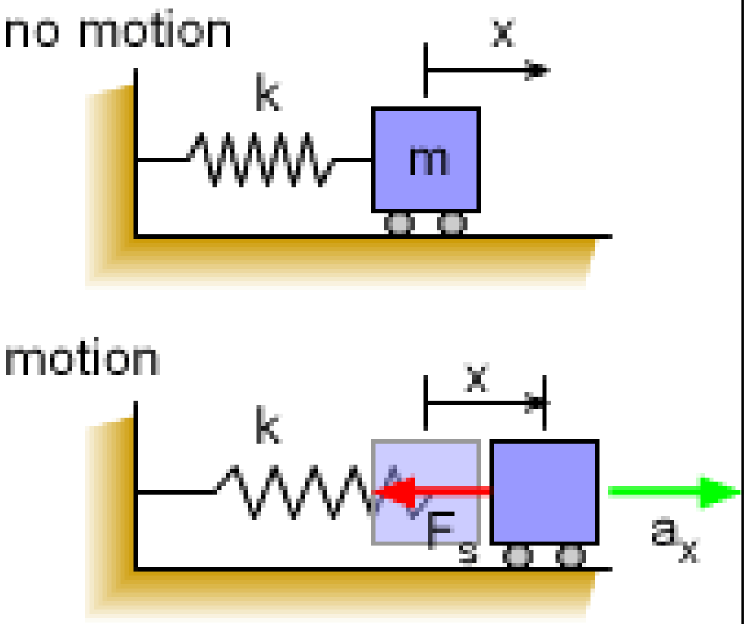
Mechanical Parameters and Components



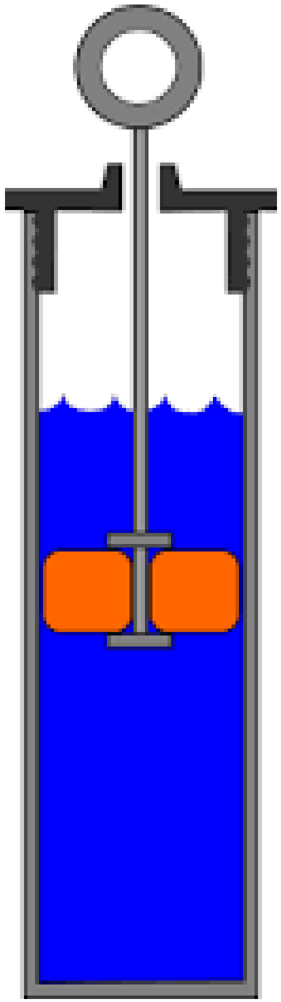
All mechanical systems contain the three basic components: spring, damper, and mass. When each of these in turn is exposed to a constant force they react with a constant displacement, a constant velocity and a constant acceleration respectively.

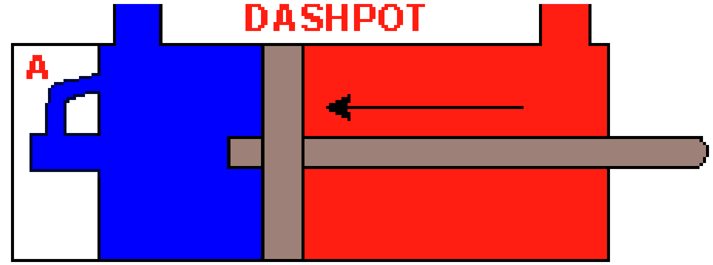
Spring in vibration

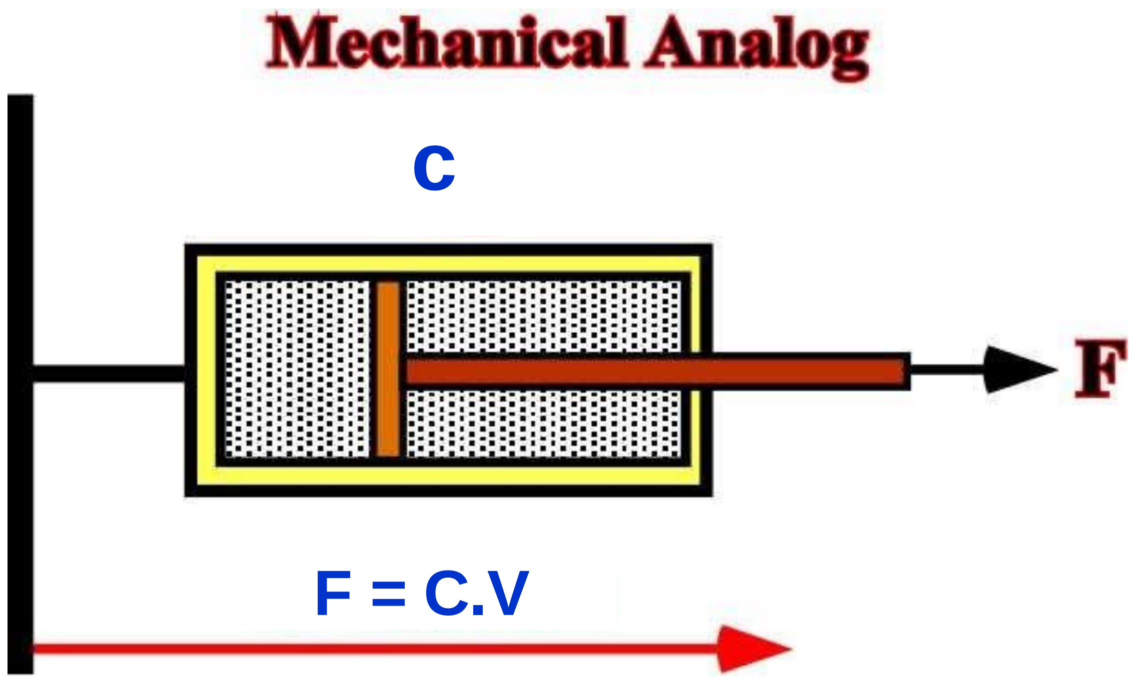




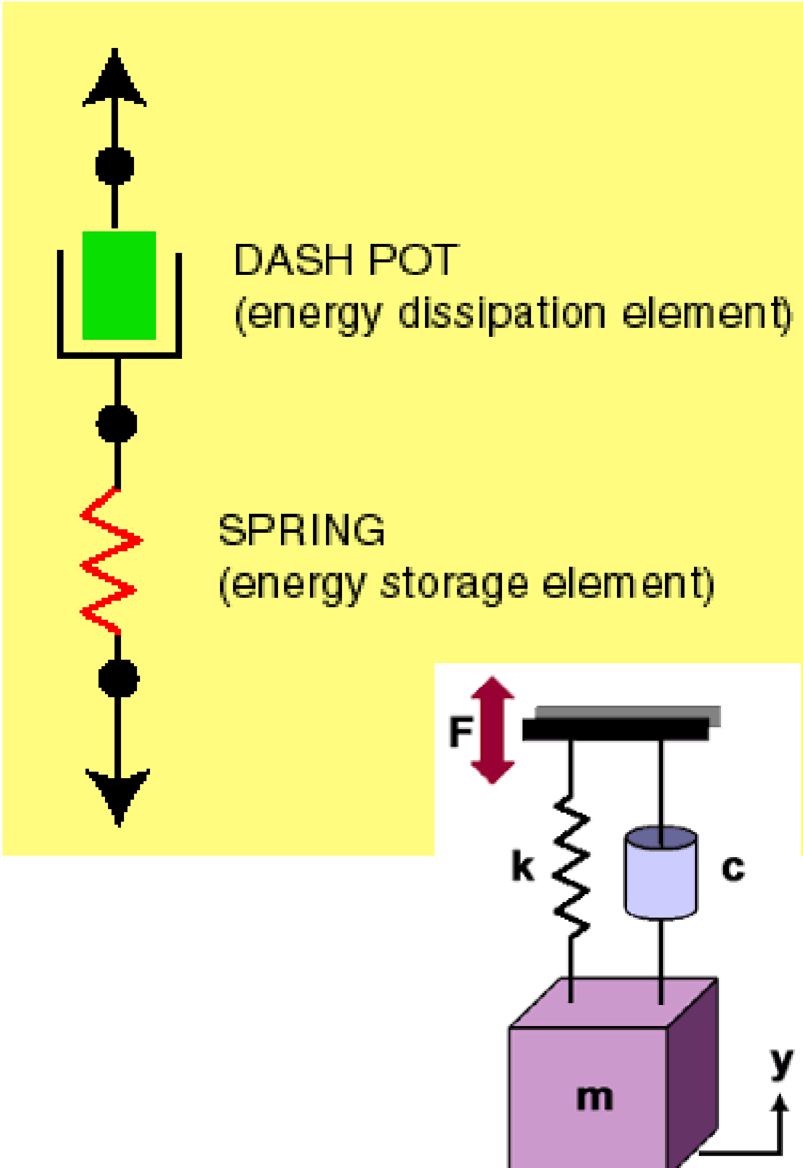
Damper - Dashpot

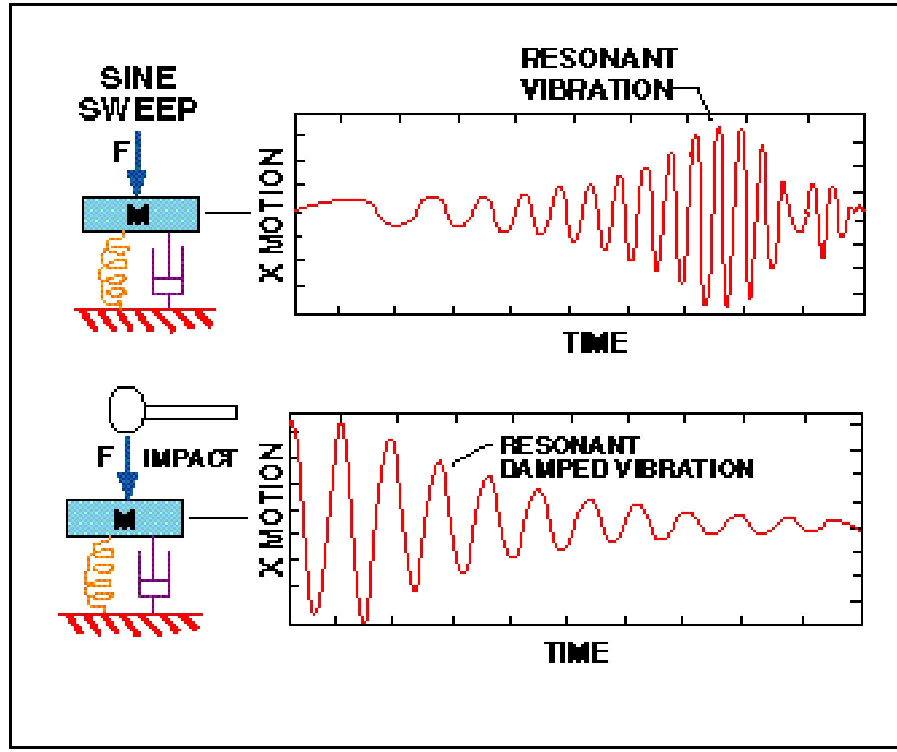


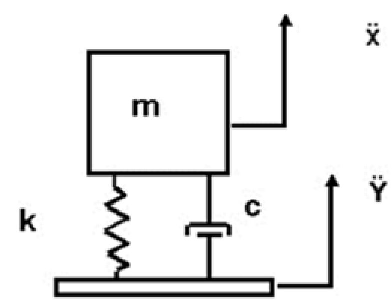




Vibration System

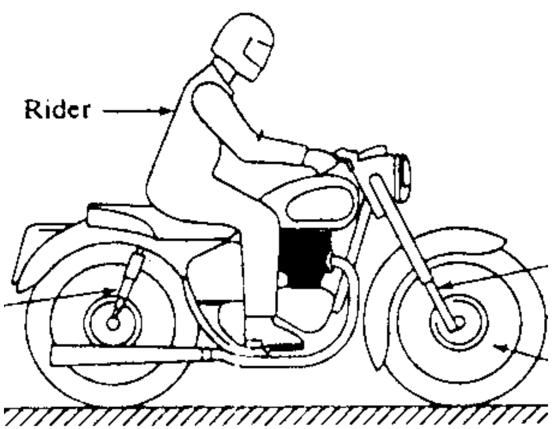


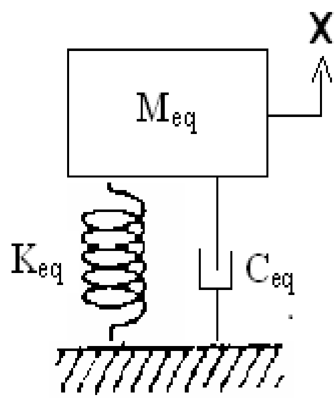


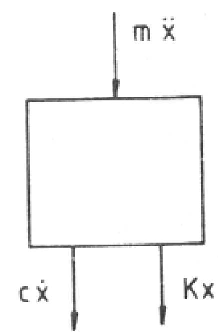


Fundamentals of Vibrations: Vibration Analysis

PHYSICAL DYNAMIC SYSTEM → MATHEMATICAL MODEL → GOVERNING EQUATIONS → SOLVE GOVERNING EQUATIONS → INTERPRETE RESULTS







FBD

$$M_{eq}\ddot{x} + C_{eq}\dot{x} + K_{eq}x = 0$$

Fundamentals of Vibrations

Modeling of the systems

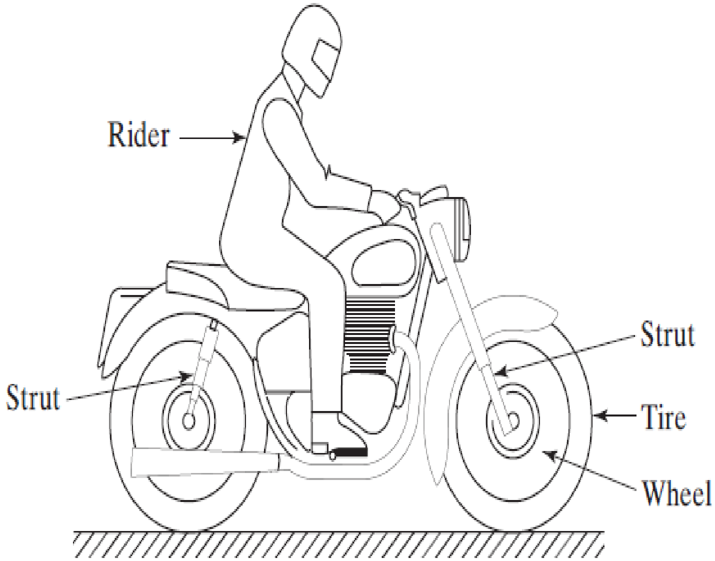
- Single degree of freedom (DOF)
- Two DOF
- Multi DOF
- Continuous system

Each system can be under

- Free
- Forced
- Damped
- Undamped

Or a combination of these modes

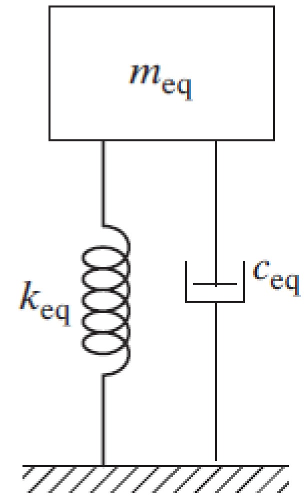
Mathematical Modeling System



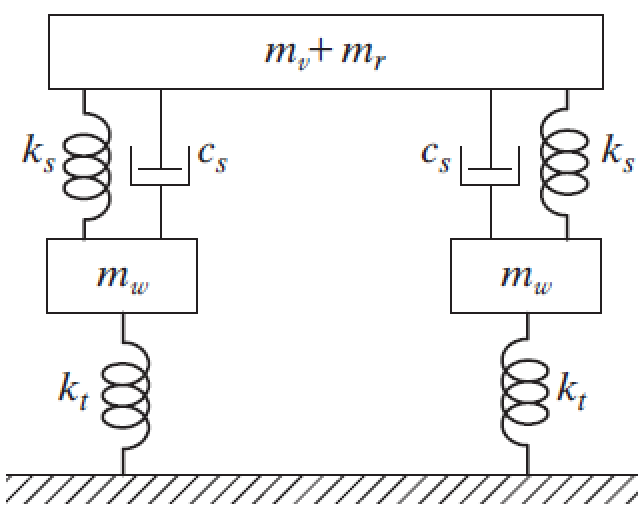
Subscripts

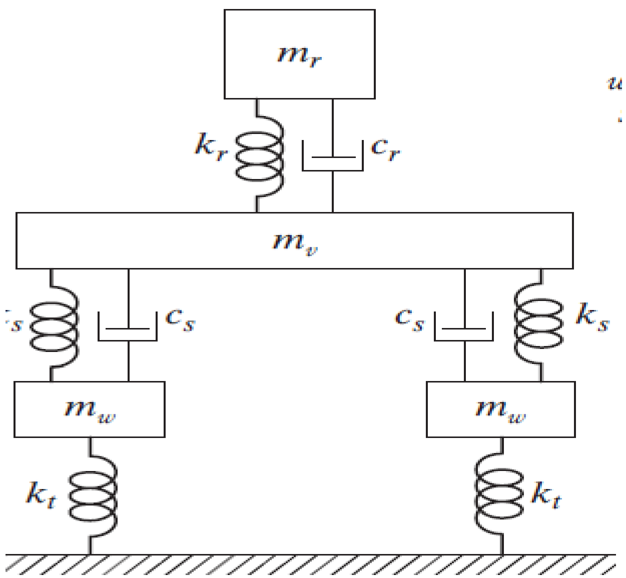
t : tire v : vehicle
w : wheel r : rider
s : strut   eq : equivalent

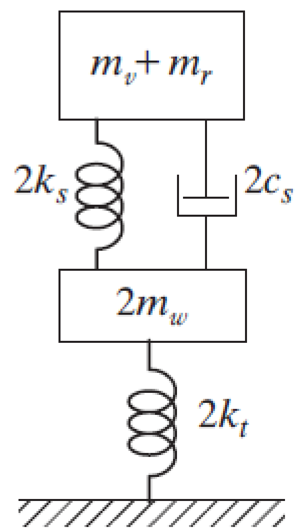
SDOF Reduction Of Bike Ride System



Most Accurate (MDOF) Modeling

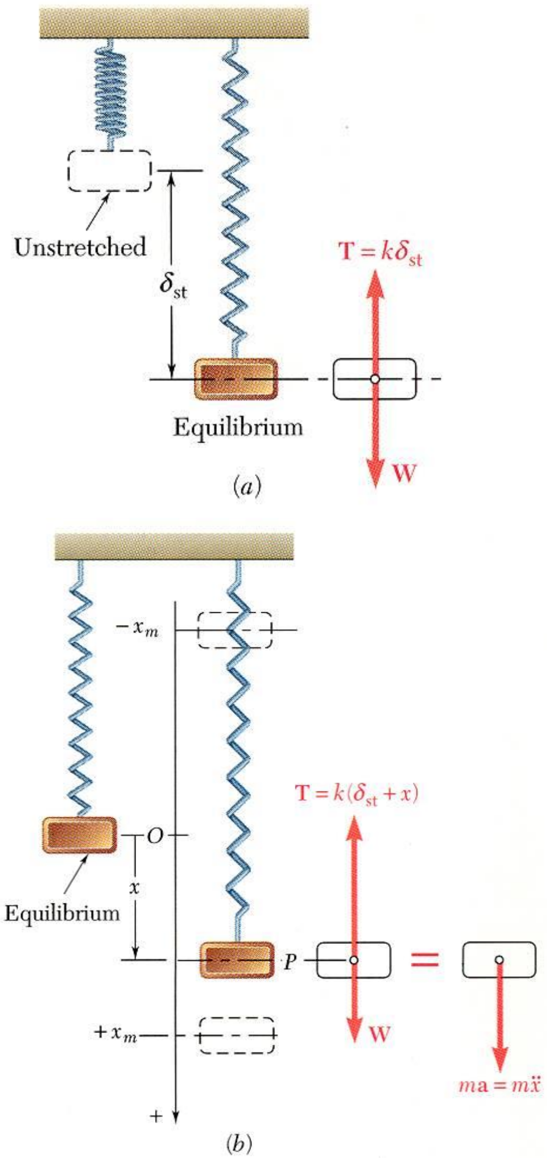






Single Degree of Freedom System

Free Vibrations of Particles: (Vertical, Undamped) Simple Harmonic



If a particle is displaced through a distance x_m from its equilibrium position and released with no velocity, the particle will undergo simple harmonic motion,

$$ma = F = W - k(\delta_{st} + x) = -kx$$

$$m\ddot{x} + kx = 0$$

General solution is the sum of two particular solutions,

$$x = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$= C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

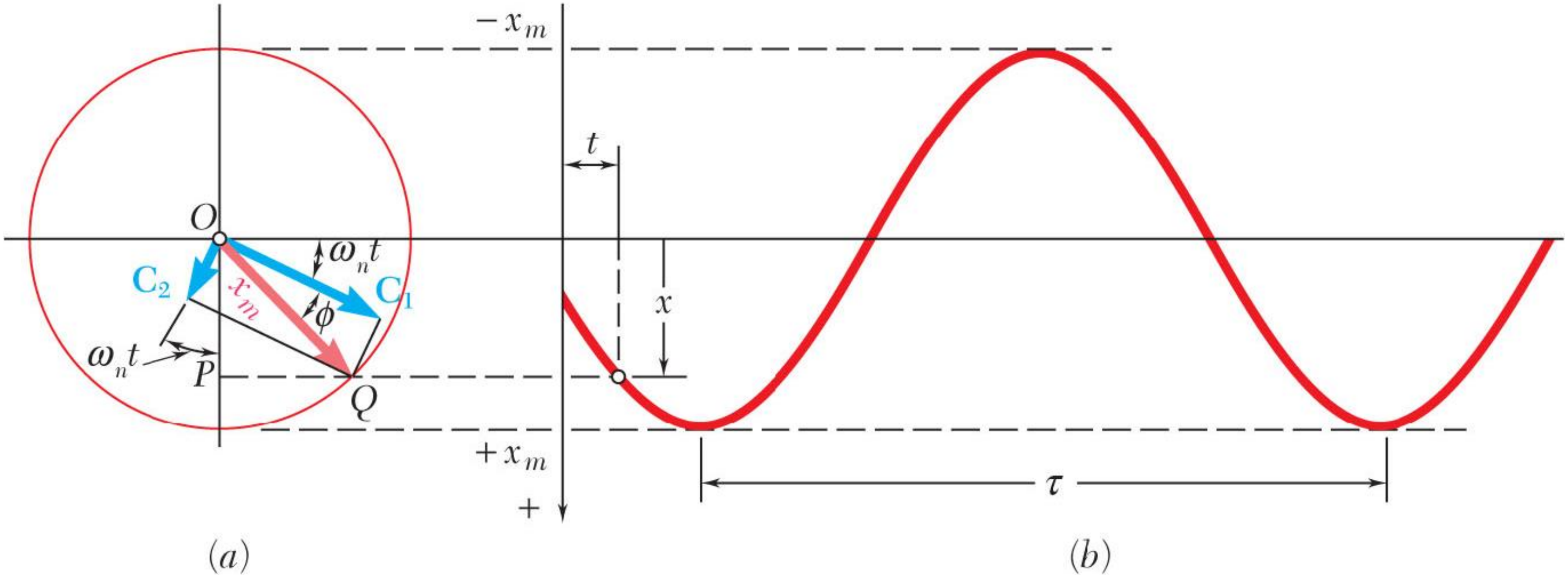
x is a periodic function and ω_n is the natural circular frequency of the motion.

C_1 and C_2 are determined by the initial conditions:

$$x = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t) \qquad C_2 = x_0$$

$$v = \dot{x} = C_1\omega_n \cos(\omega_n t) - C_2\omega_n \sin(\omega_n t) \qquad C_1 = v_0/\omega_n$$

Free Vibrations of Particles: Simple Harmonic Motion



$$x = x_m \sin(\omega_n t + \phi)$$

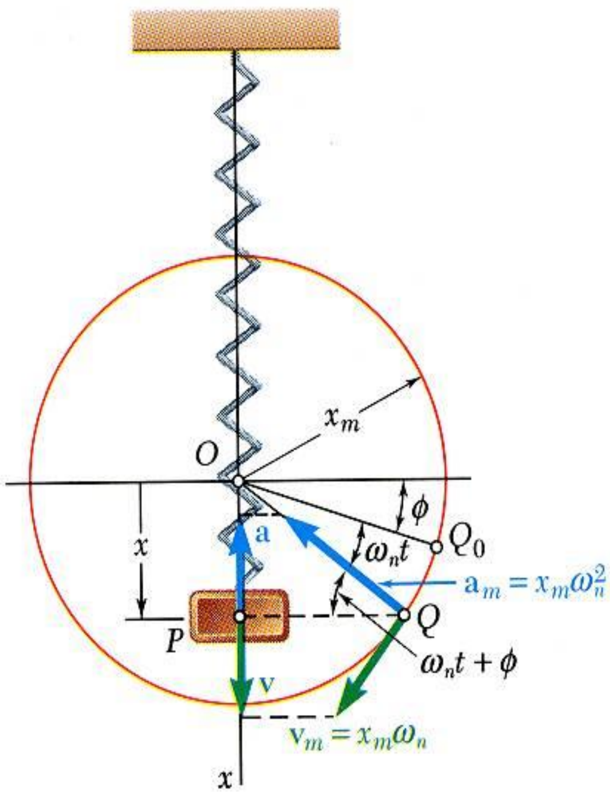
$$x_m = \sqrt{(v_0/\omega_n)^2 + x_0^2} = \text{ amplitude}$$

$$\phi = \tan^{-1}(v_0/x_0\omega_n) = \text{ phase angle}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \text{ period}$$

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} = \text{ natural frequency}$$

Free Vibrations of Particles: Simple Harmonic Motion

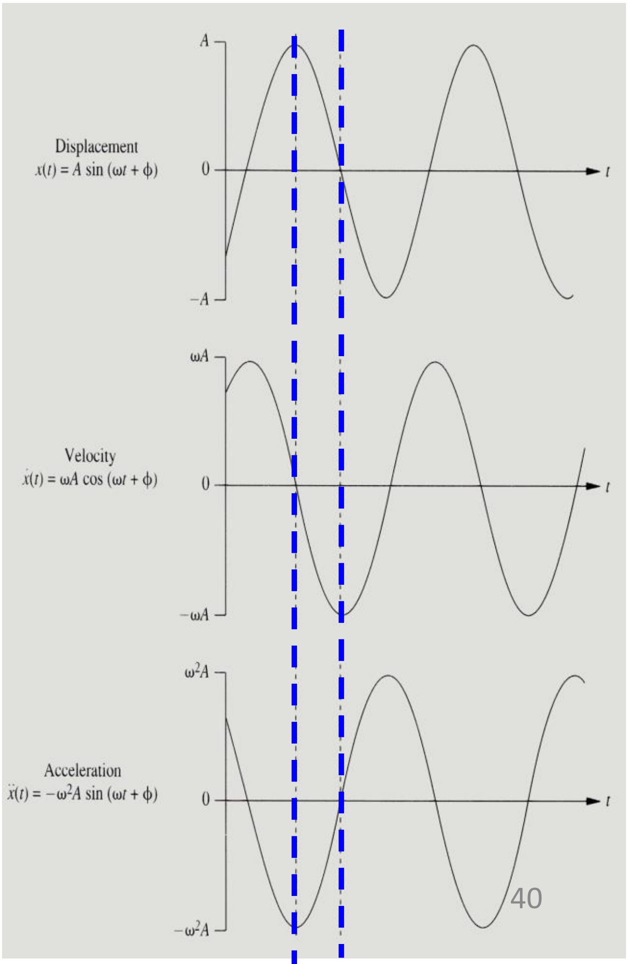


Velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve but different phase angles.

$$x = x_m \sin(\omega_n t + \phi)$$

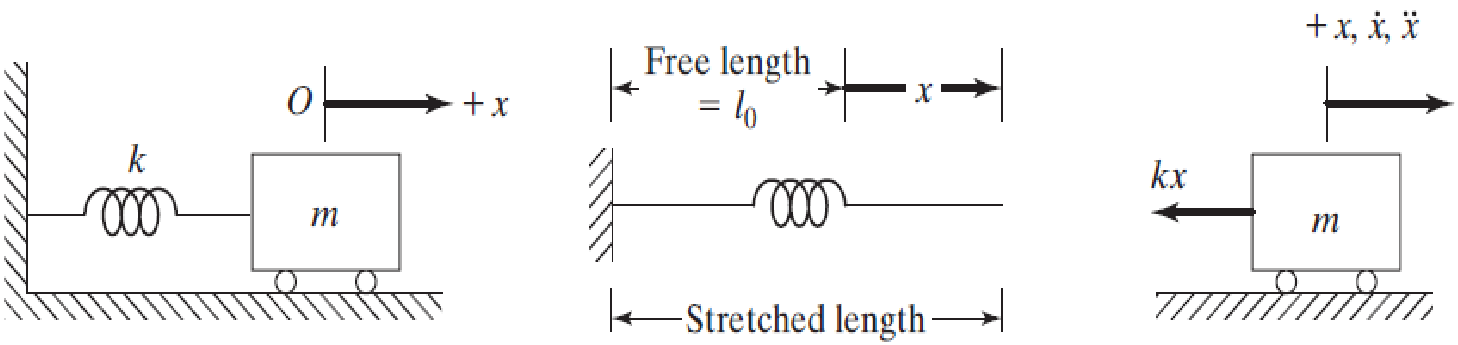
$$\begin{aligned} v &= \dot{x} \\ &= x_m\omega_n \cos(\omega_n t + \phi) \\ &= x_m\omega_n \sin(\omega_n t + \phi + \pi/2) \end{aligned}$$

$$\begin{aligned} a &= \ddot{x} \\ &= -x_m\omega_n^2 \sin(\omega_n t + \phi) \\ &= x_m\omega_n^2 \sin(\omega_n t + \phi + \pi) \end{aligned}$$



Free Vibrations of Particles: (Horizontal, Undamped)

- One coordinate *(x)* is sufficient to specify the position of the mass at any time …. SDOF
- No external force applied to the mass; hence the motion resulting from an initial disturbance ….. free vibration.
- No element that causes dissipation of energy during the motion of the mass, so the amplitude of motion remains constant with time .. *Un-damped* system

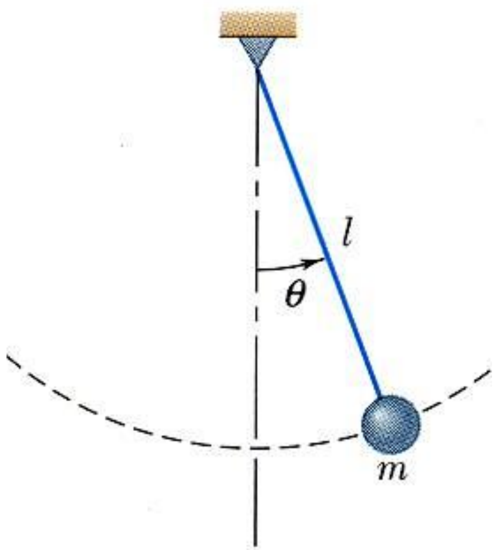


(a) (b) (c)

$$F(t) = -kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

Free Vibrations: Simple Pendulum (Approximate Solution)

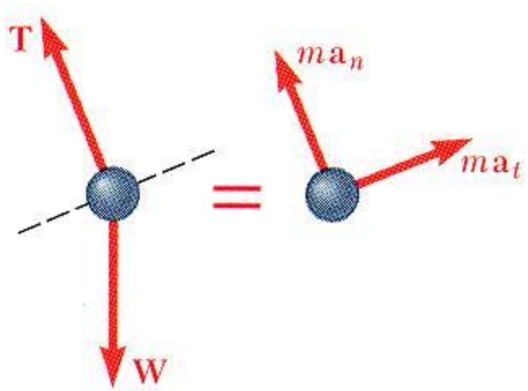


Results obtained for the spring-mass system can be applied whenever the resultant force on a particle is proportional to the displacement and directed towards the equilibrium position.

Consider tangential components of acceleration and force for a simple pendulum,

$$\sum F_t = ma_t : \quad -W\sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$



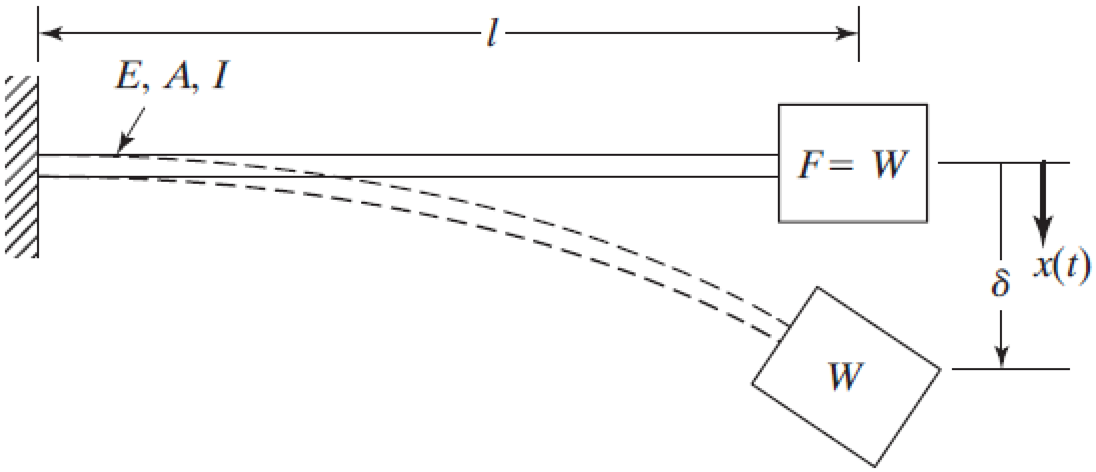
For small angles,

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

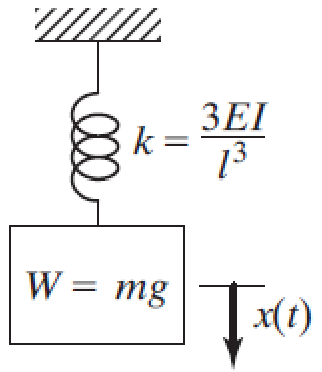
$$\theta = \theta_m \sin(\omega_n t + \phi)$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$$

Spring Constant of a Cantilever Beam



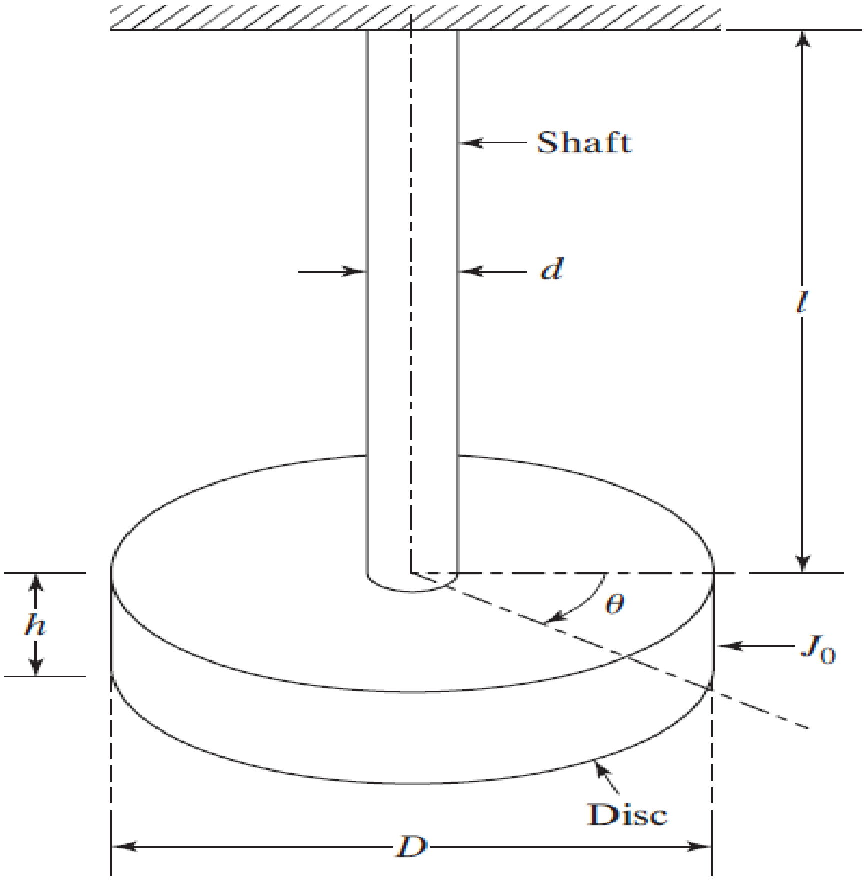
Cantilever with end force

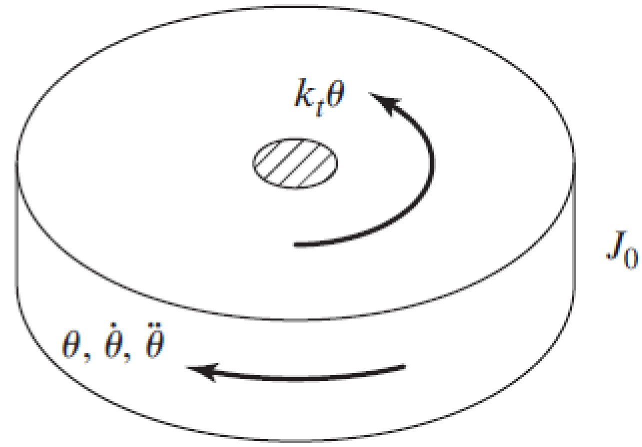


Equivalent spring

$$\delta = \frac{Wl^3}{3EI} \qquad k = \frac{W}{\delta} = \frac{3EI}{l^3}$$

Free Vibration of an Undamped Torsional System





$$M_t = \frac{GI_0}{l}\theta$$

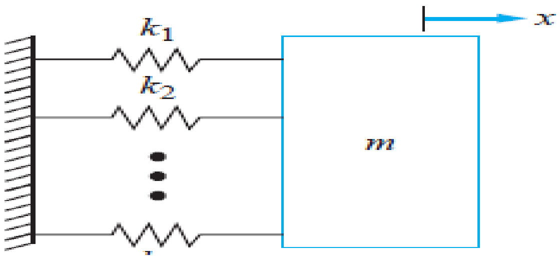
$$I_0 = \frac{\pi d^4}{32}$$

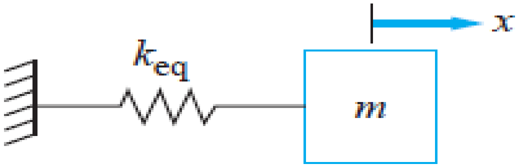
$$k_t = \frac{M_t}{\theta} = \frac{GI_0}{l} = \frac{\pi G d^4}{32 l}$$

$$J_0\ddot{\theta} + k_t\theta = 0$$

Free Vibrations: Springs in Combinations:

Parallel Combination

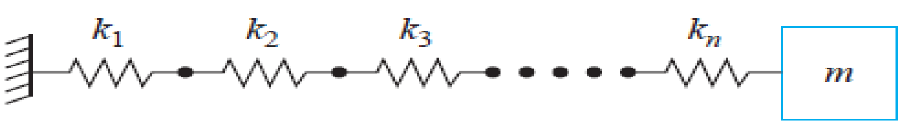


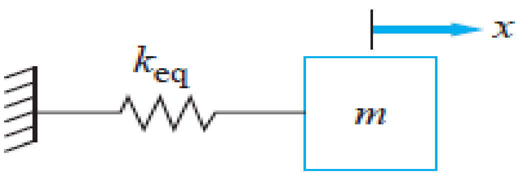


$$F = k_1 x + k_2 x + \cdots + k_n x = \left(\sum_{i=1}^{n} k_i \right) x$$

$$k_{eq} = \sum_{i=1}^{n} k_i$$

Series combination

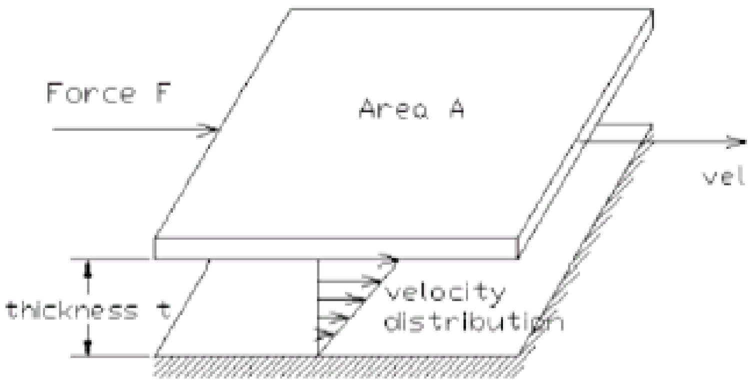


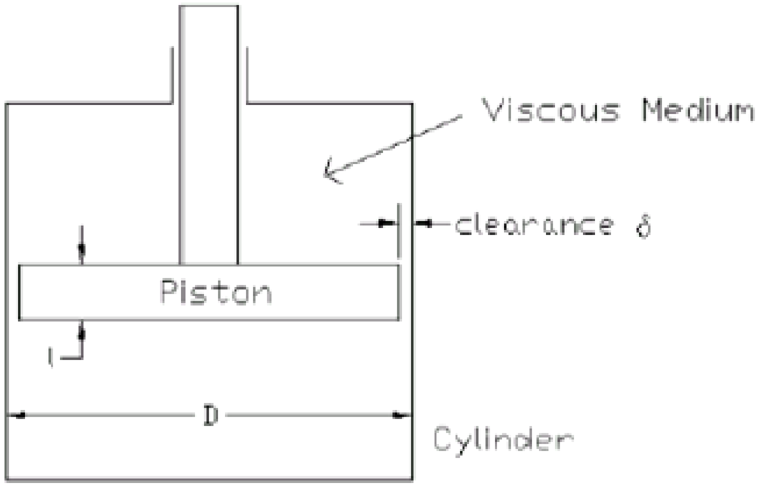


$$x = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i$$

$$x = \sum_{i=1}^{n} \frac{F}{k_i} \qquad k_{eq} = \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i}}$$

Viscous Damping Principle





$$\tau = \mu \frac{dv}{dz}$$

$$F = (\pi D t)\mu \frac{v}{\delta} = c\dot{x}$$

$$c = \frac{\pi D t \mu}{\delta} \ \mathrm{N\text{-}s/m}$$

- Damping force is proportional to velocity and = Damping Coefficient C times Velocity dx/dt – dissipates energy

- Dashpots can de designed as in shock absorbers or the equivalent effect of energy dissipating capacity determined from tests to find the value of this coefficient c

Free Motion of a Damped SDoF System

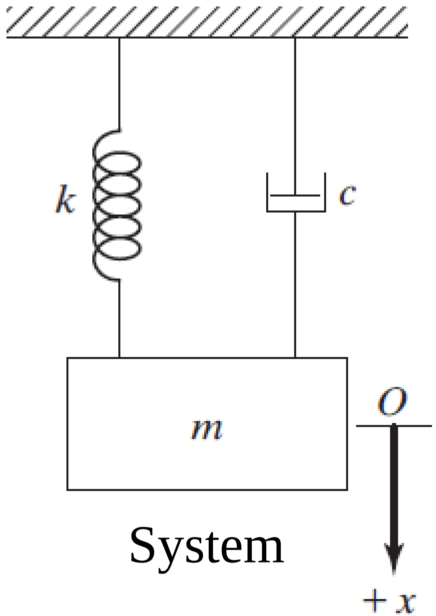
Free Vibration with Viscous Damping

Equation of motion : $m\ddot{x} + c\dot{x} + kx = 0$

Dividing by m : $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

where $\omega_n = \sqrt{\dfrac{k}{m}}$ is the undamped natural frequency,

$\zeta = \dfrac{c}{c_0} = \dfrac{c}{2\sqrt{km}}$ is the viscous critical damping ratio.



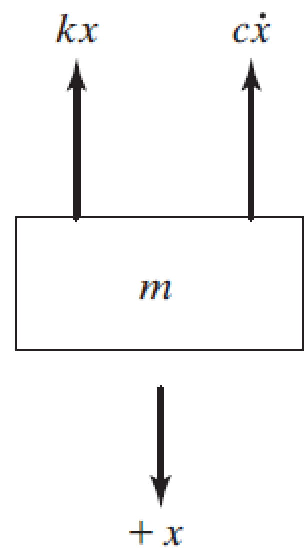
System

Solution of the equation of motion

Eq of motion : $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

Solution is of the form : $x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$

A & B are two constants depending on initial conditions.



FBD

Possibilities depending on the value of ζ

To find α_1 & α_2, insert $x(t) = Ae^{\alpha t}$ into EOM.

$$Ae^{\alpha t}(\alpha^2 + 2\zeta\omega_n\alpha + \omega_n^2) = 0$$

$$\rightarrow \alpha_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$$

Case 1. ζ<1. Underdamped case with oscillatory motion

Both roots are complex: $\alpha_1 = (-\zeta + i\sqrt{1-\zeta^2})\omega_n$

$$\alpha_2 = (-\zeta - i\sqrt{1-\zeta^2})\omega_n$$

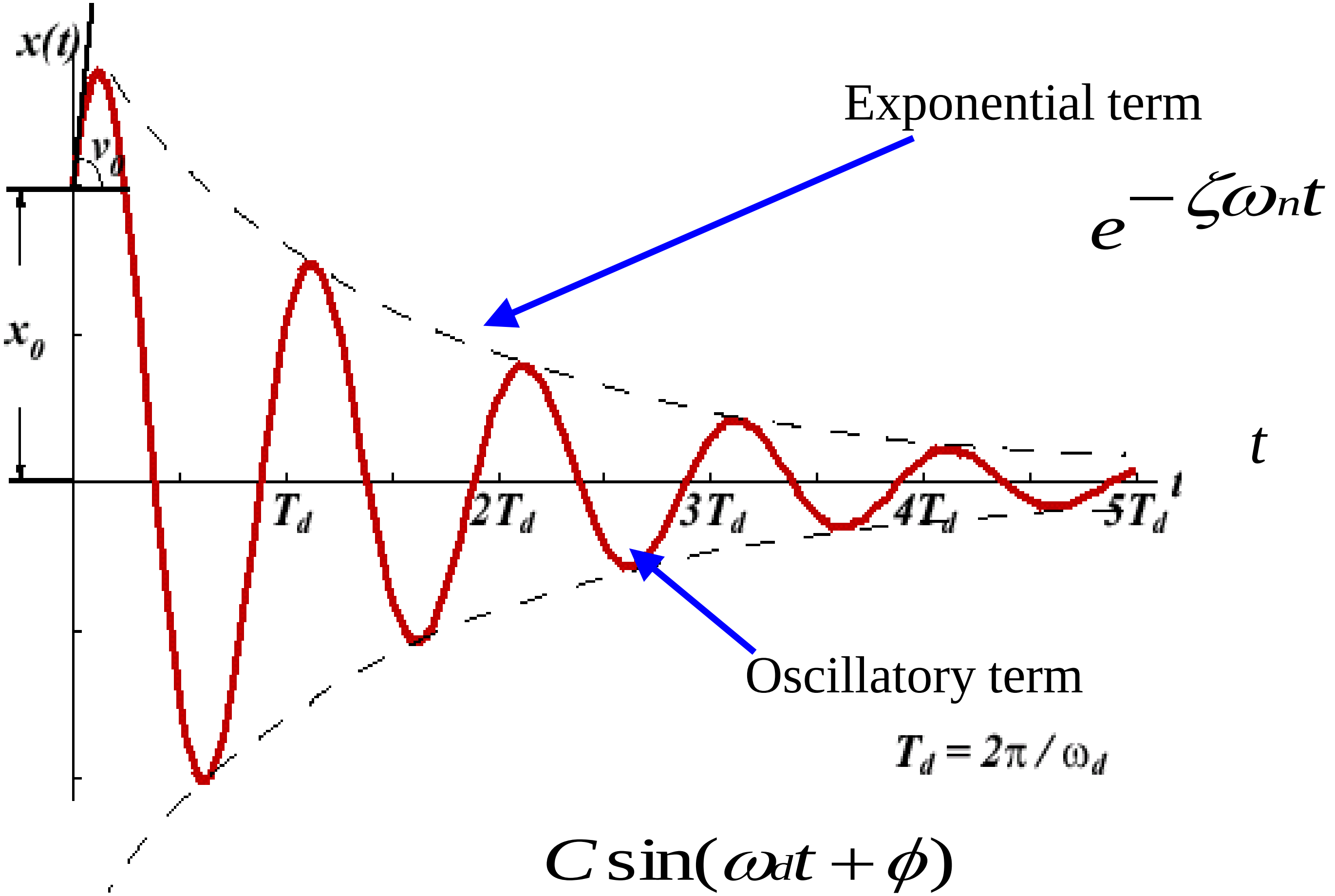
The general solution becomes:

$$x(t) = e^{-\zeta\omega_n t}[A\cos(\sqrt{1-\zeta^2}\,\omega_n t) + B\sin(\sqrt{1-\zeta^2}\,\omega_n t)]$$

$$= e^{-\zeta\omega_n t}C\sin(\sqrt{1-\zeta^2}\,\omega_n t + \phi) = e^{-\zeta\omega_n t}C\sin(\omega_d t + \phi)$$

where $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ is the damped natural frequency.

Time history for oscillatory motion



Case 2. ζ>1. Overdamped case with oscillatory motion

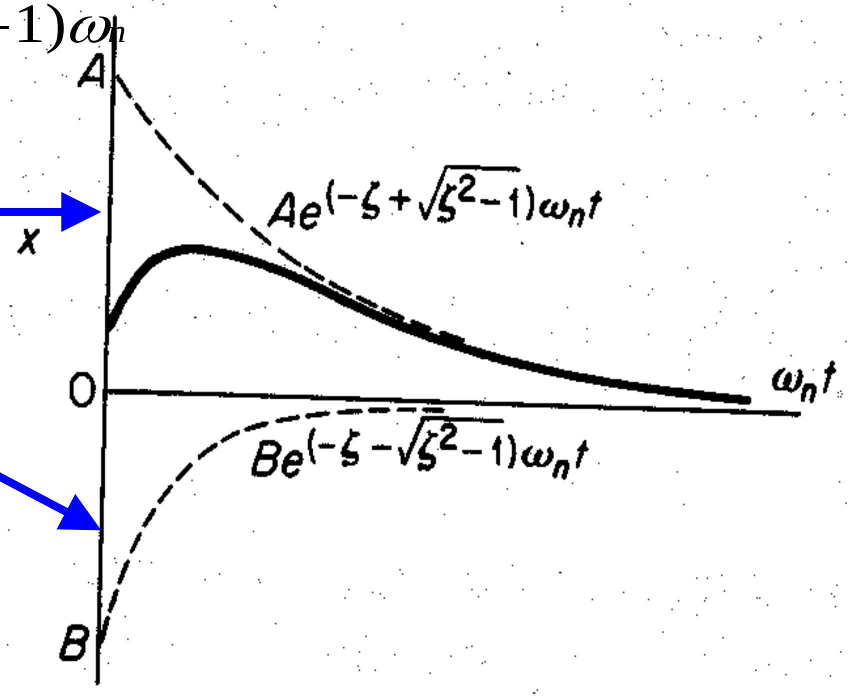
Both roots are real: $\alpha_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n$

$$\alpha_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

The solution becomes:

$$x(t) = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t}$$

$$+ Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

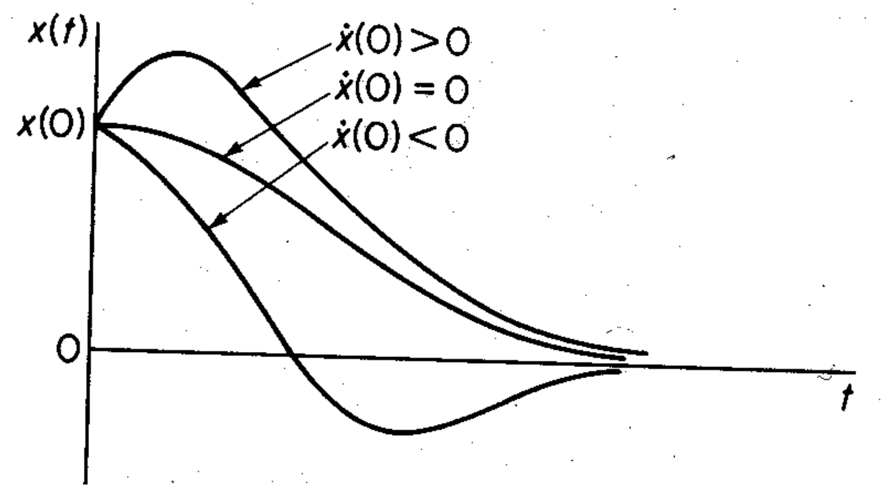


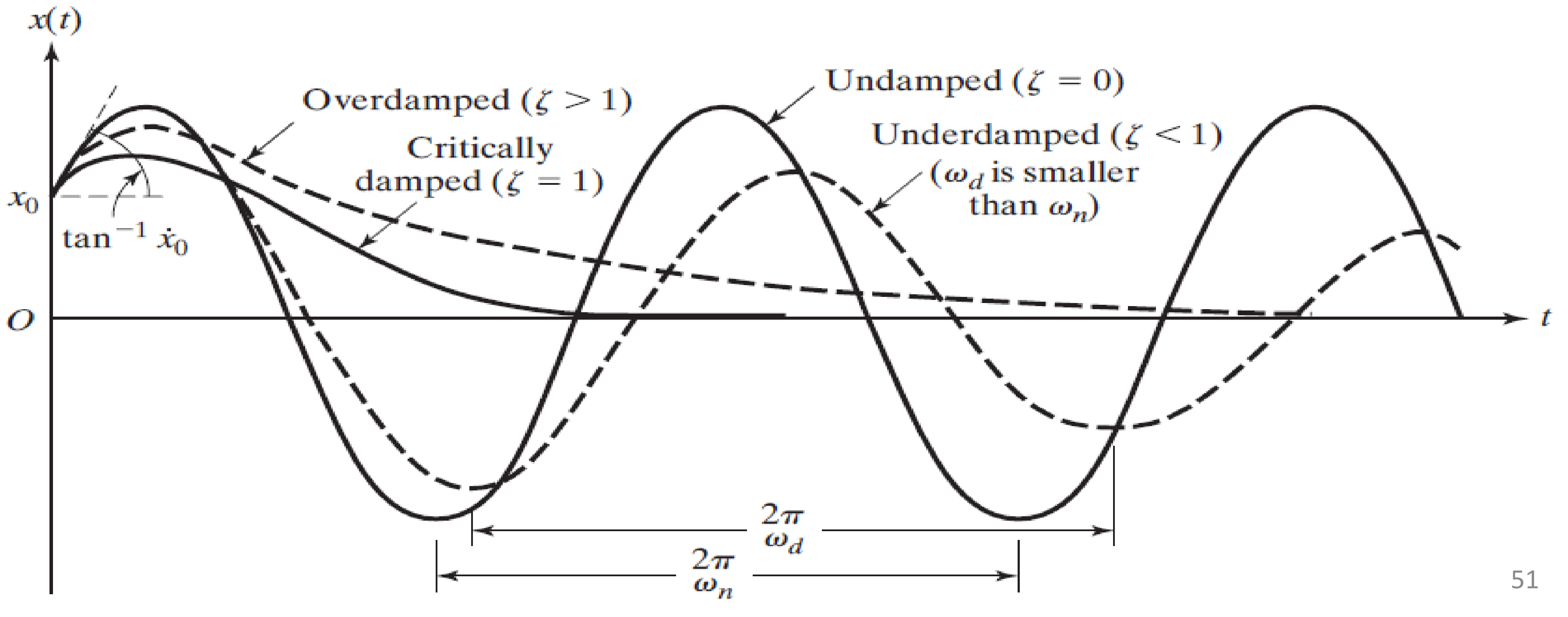
Case 3. ζ=1. Critically damped motion case with Max rate of decay

Double root: $\alpha_1 = \alpha_2 = -\omega_n$

The solution becomes:

$$x(t) = (A + Bt)e^{-\omega_n t}$$





- A critically damped system will have the smallest damping required for aperiodic motion; hence the mass returns to the position of rest in the shortest possible time without overshooting.

- The property of critical damping is used in many practical applications.

- For example, large guns have dashpots with critical damping value, so that they return to their original position after recoil in the minimum time without vibrating. If the damping provided were more than the critical value, some delay would be caused before the next firing.

Logarithmic Decrement

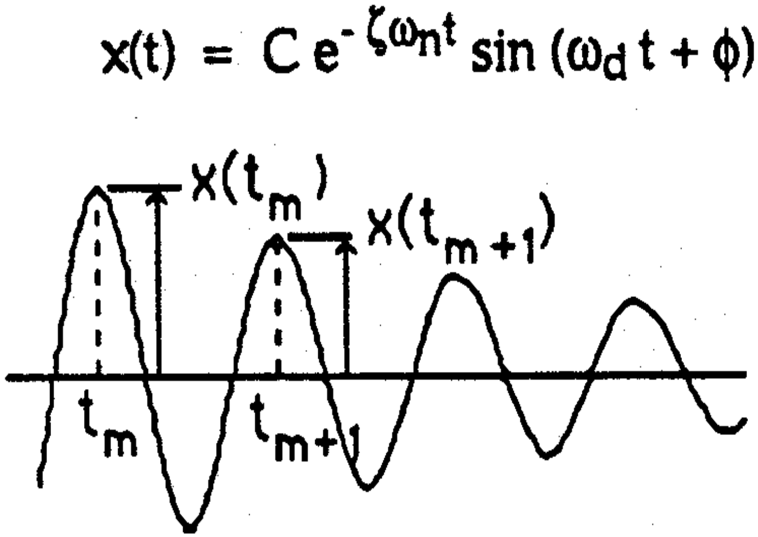
The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.

The amplitude ratio between two successive cycles:

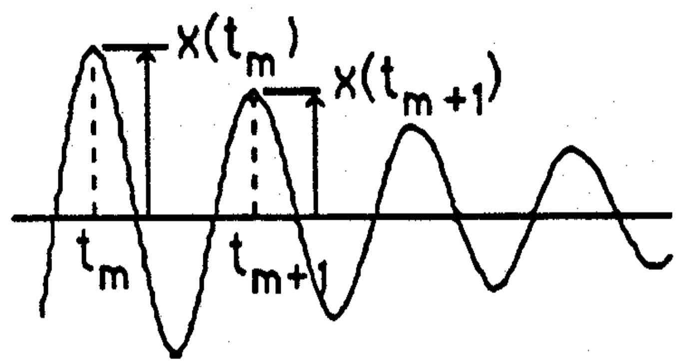
$$\frac{x(t_m)}{x(t_{m+1})} = \frac{e^{-\zeta\omega_n t_m} C \sin(\omega_d t_m + \phi)}{e^{-\zeta\omega_n t_{m+1}} C \sin(\omega_d t_{m+1} + \phi)}$$

$$\left|\frac{x(t_m)}{x(t_{m+1})}\right|_{AMP} = e^{-\zeta\omega_n (t_m - t_{m+1})}$$

$$= e^{\zeta\omega_n T} = e^{\zeta\omega_n \frac{2\pi}{\omega_d}} = e^{\zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}}$$



Logarithmic Decrement



$$\left|\frac{x(t_m)}{x(t_{m+1})}\right| = e^{\zeta\frac{2\pi}{\sqrt{1-\zeta^2}}}$$

Taking logarithms of both sides :

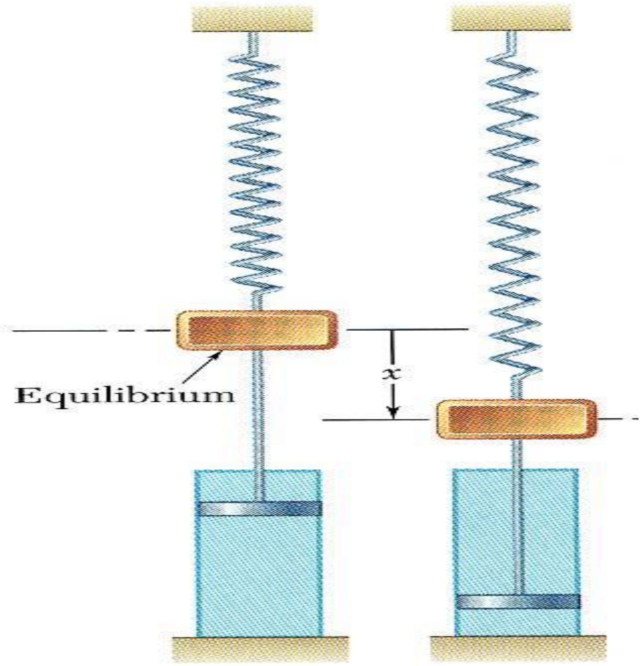
$$\text{logarithmic decrement } \delta = \ln\left|\frac{x(t_m)}{x(t_{m+1})}\right| = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta \ (\text{for } \zeta << 1)$$

If the two amplitudes are separated by (N-1) cycles:

$$\delta = \frac{1}{N}\ln\left|\frac{x(t_m)}{x(t_{m+N})}\right|$$

$$\delta = \ln\left|\frac{x_1}{x_2}\right| = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\times\frac{c}{2m}$$

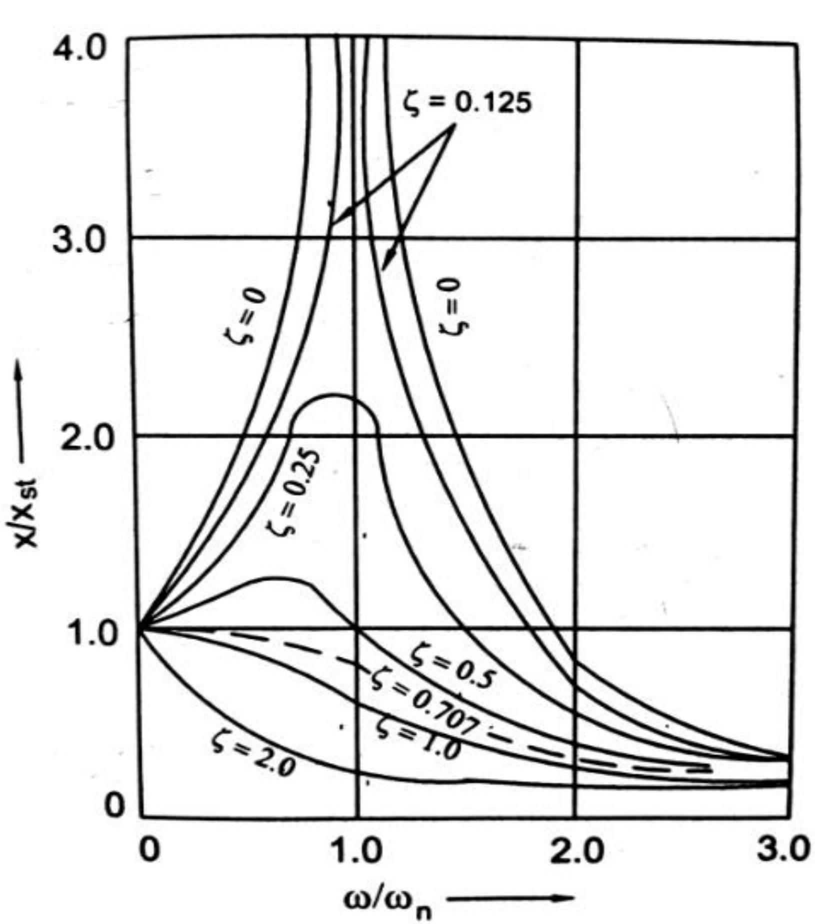
Damped Forced Vibrations

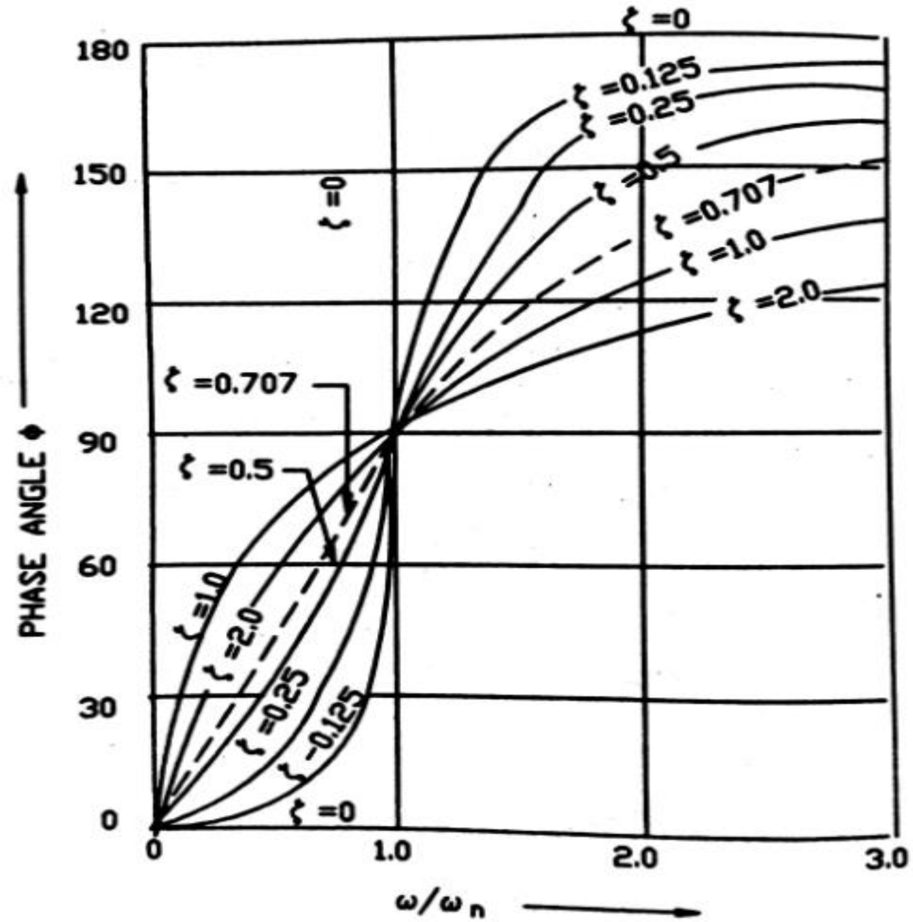


$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \qquad x = x_{complementary} + x_{particular}$$

$$\frac{x_m}{P_m/k} = \frac{x_m}{x_{st}} = \frac{1}{\sqrt{\left[1-(\omega_f/\omega_n)^2\right]^2 + \left[2(c/c_c)(\omega_f/\omega_n)\right]^2}} = \text{Magnification factor (M.F)}$$

$$\tan\phi = \frac{2(c/c_c)(\omega_f/\omega_n)}{1-(\omega_f/\omega_n)^2} = \text{phase difference between forcing and steady state response}$$





It is seen from these curves that the response of a particular system at a any particular frequency is lower for higher value of damping. In other words, the curves for higher values of damping lie below those for lower values of damping.

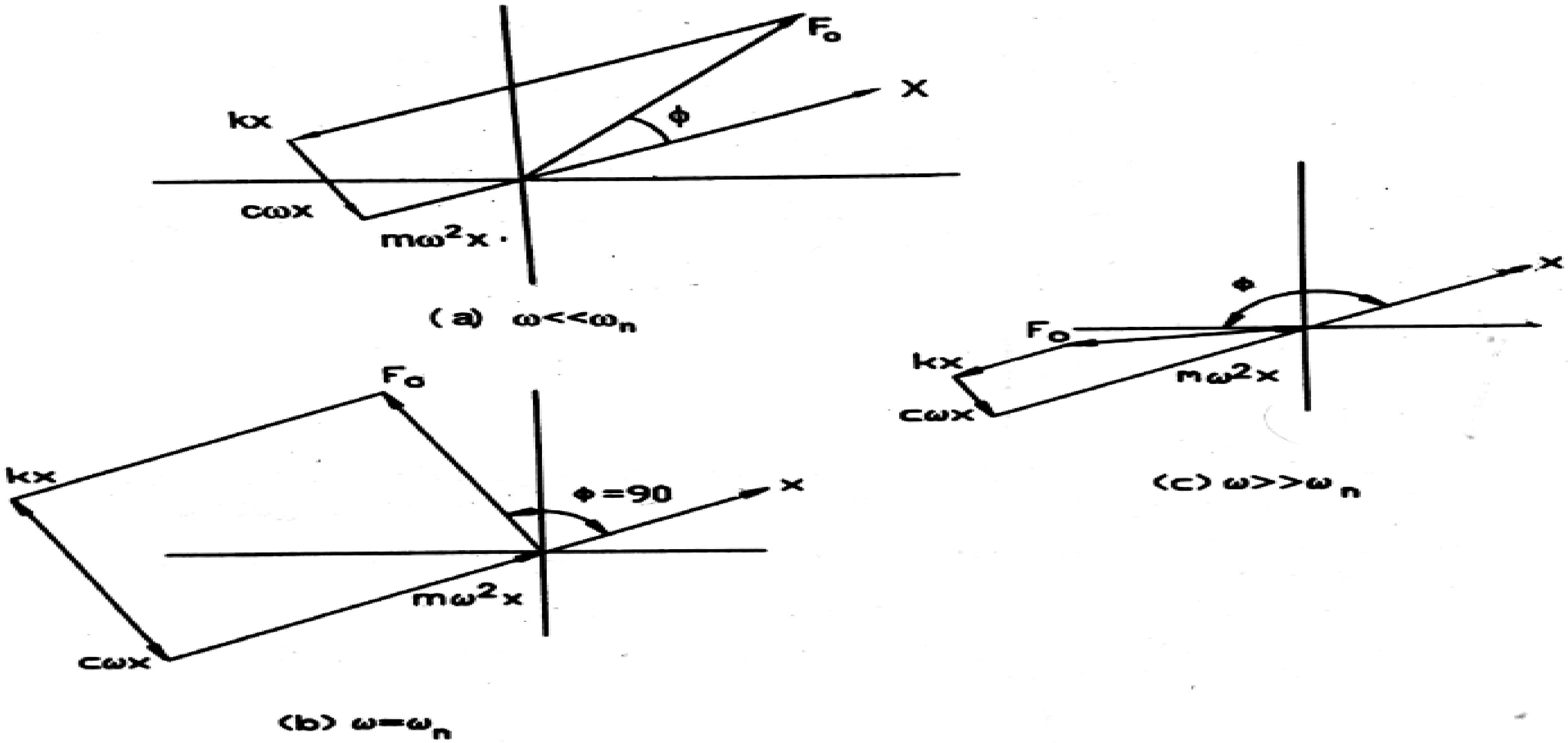
Magnification Factor

- At zero frequency the magnification is unity and is independent of the damping i.e. $X = X_{st}$ which itself is the definition of zero frequency deflection.
- At very high frequency the magnification tends to zero or the amplitude of vibration becomes very small.
- At resonance ($\omega = \omega_n$), the amplitude of vibration becomes excessive for small damping and decreases with increase in damping.
- For zero damping at resonance the amplitude is infinite theoretically.

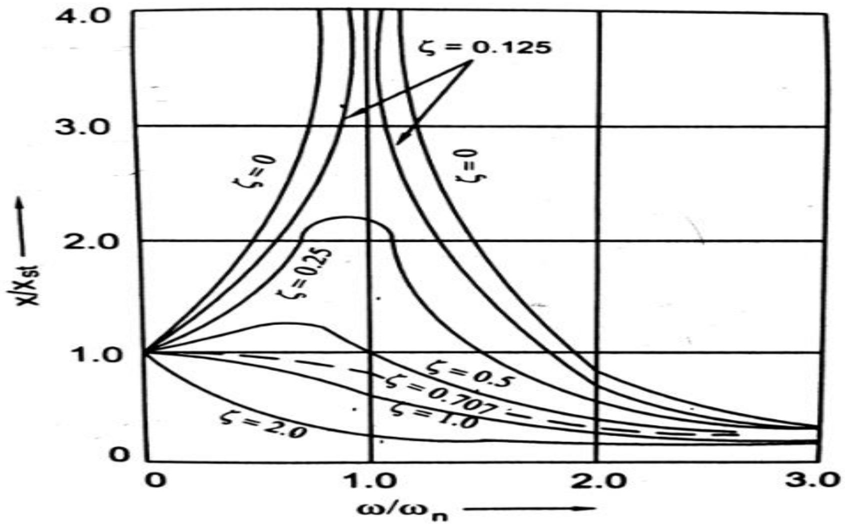
Phase angle

- The phase angle also varies from zero at low frequencies to 180 degree at very high frequencies.
- It is 90 degree at resonance and is independent of damping. Over a small frequency range containing the resonance point, the variation of phase angle is more abrupt for lower values of damping than for higher values. More abrupt the changes in phase angle about resonance, more sharp is the peak in frequency response curve.
- For zero damping the phase lag suddenly changes from zero to 180 degree at resonance.

Vector Diagram for Forced Vibrations for Various Operating Conditions



- At very low frequencies, phase angle is zero and impressed force balances the spring force.
- At resonant frequency, phase angle is 90 degree and the impressed force balances the damping force. Also spring force and the inertia force are equal and opposite at resonance.
- At very high frequencies the phase angle is 180 degree and impressed force balances the inertia force.



It should be carefully seen that the maximum amplitude occurs not at the resonant frequency but a little towards its left. This shift increases with the increase in damping. For zero damping the maximum amplitude (infinite value), of course, is obtained at the resonant frequency.

Frequency at which maximum amplitude occurs can be obtained:

$$\frac{d(MF)}{d\left(\frac{\omega}{\omega_n}\right)} = \frac{2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\left[-2\left(\frac{\omega}{\omega_n}\right)\right]+2\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right](2\zeta)}{2\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2\right\}^{3/2}} = 0$$

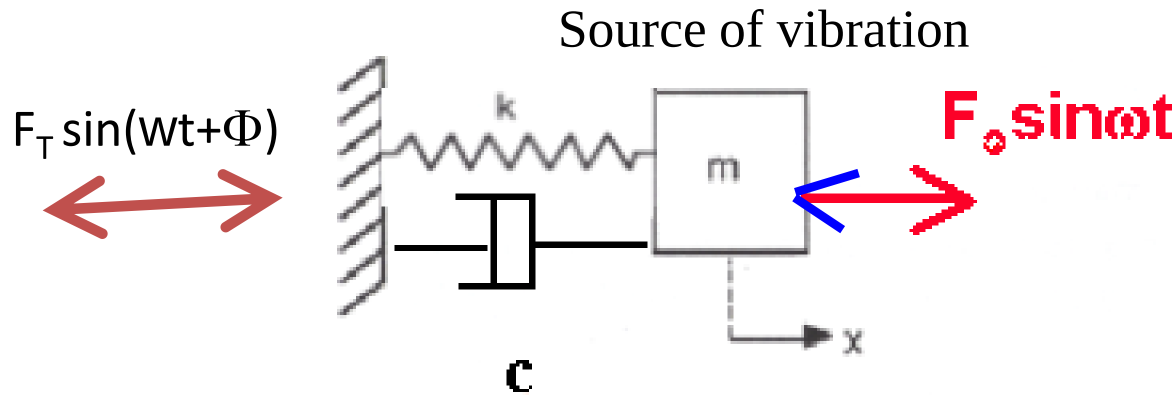
Which gives, $$\left(\frac{\omega_p}{\omega_n}\right) = \sqrt{1-2\zeta^2}$$

where ω_p means the frequency corresponding to the peak amplitude.

No maxima or peak will occur when expression within the radical sign is -ive. for $\zeta > 0.707$.

It can also be seen that response curve is below unity magnification line.

Forced Response



The response of the system to some given harmonic excitation can be found using a transfer function approach:

$$\text{OUTPUT} = \text{SYSTEM FUNCTION} \;\; \text{x} \;\; \text{INPUT}$$

$$X = H(\textit{system properties}, \ \omega) \quad \text{x} \quad F\text{orce}$$

$$\text{For forced response, we have}: ([K] - \omega^2 [M]) \ \{X\} = \{F\}$$

$$\rightarrow \{X\} = ([K] - \omega^2 [M])^{-1} \{F\} = [H]\{F\}$$

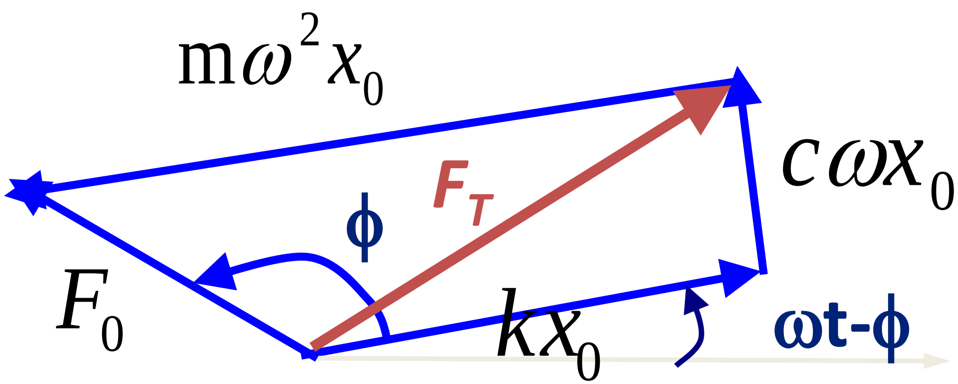
We want the normalized response to a single excitation, applied to each co-ordinate in turn so that we can obtain the total response by summation.

Basic Theory

The force transmitted to ground is due to the spring and damper :

$$F_T = kx + c\dot{x}$$

What we want to know is the ratio : $\dfrac{\text{Force transmitted}}{\text{Excitation Force}} = \dfrac{F_T}{F_0}$



From the phasor diagram :

$$F_T = x_0\sqrt{k^2 + (c\omega)^2}$$

From Lecture, we know that :

$$x_0 = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

So, transmissibility, T, is given by :

$$T = \left|\frac{F_T}{F_0}\right| = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

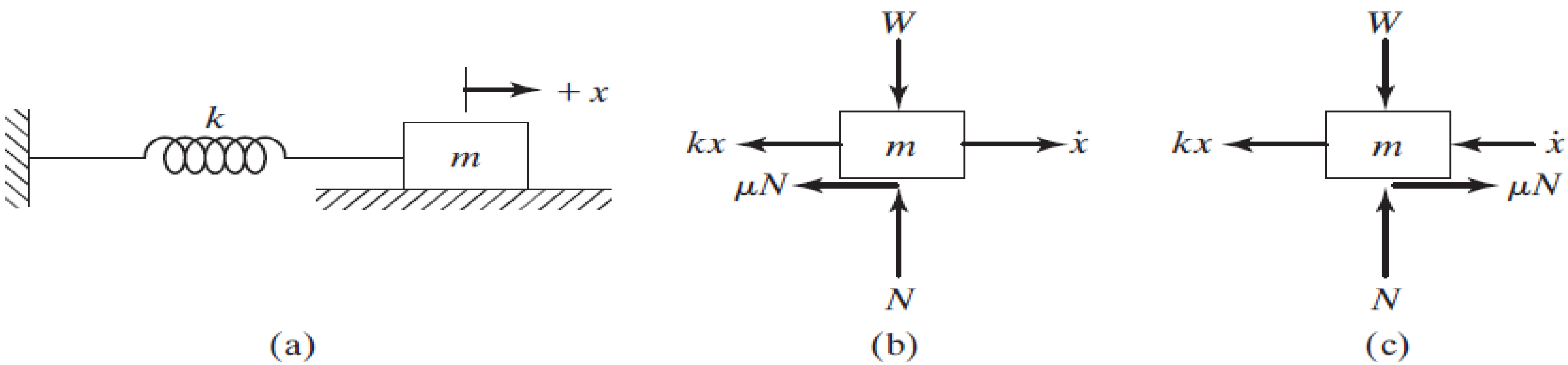
where $r = \dfrac{\omega}{\omega_n}$ and $\zeta = \dfrac{c}{2\sqrt{km}}$

How to Obtain Low Transmissibility ?

$$T = \left| \frac{F_T}{F_0} \right| = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}$$

- We want T to be as low as possible.
- T is small ……………….if $\omega >> \omega_n$.
- We want to lower ω_n
- We want low stiffness and/or high mass.

Coulomb Damping (Dry Friction)



In vibrating structures, whenever the components slide relative to each other, dry-friction damping appears internally. As stated, Coulomb damping arises when bodies slide on dry surfaces. Coulomb s law of dry friction states that, when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact. Thus the friction force F is given by

$$F = \mu N = \mu W = \mu m g$$

where N is the normal force, equal to the weight of the mass and is the coefficient of sliding or kinetic friction. The friction force acts in a direction opposite to the direction of velocity. Coulomb damping is sometimes called constant damping, since the damping force is independent of the displacement and velocity; it depends only on the normal force N between the sliding surfaces.

Equation of motion $m\ddot{x} + \mu mg\, \text{sgn}(\dot{x}) + kx = 0$

where sgn(y) is called the signum function, whose value is defined as 1 for y>0, -1 for y<0 and 0 for y=0.

To find the solution using this procedure, let us assume the initial conditions as:

$$x(t = 0) = x_0$$

$$\dot{x}(t = 0) = 0$$

Case 1.

When x is positive and dx/dt is positive or when x is negative and dx/dt is positive (i.e., for the half cycle during which the mass moves from left to right), the equation of motion can be obtained using Newton s second law:

$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N$$

The solution can be:

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k}$$

Where $\omega_n = \sqrt{k/m}$ is the frequency of vibration and A1 and A2 are constants whose values depend on the initial conditions of this half cycle.

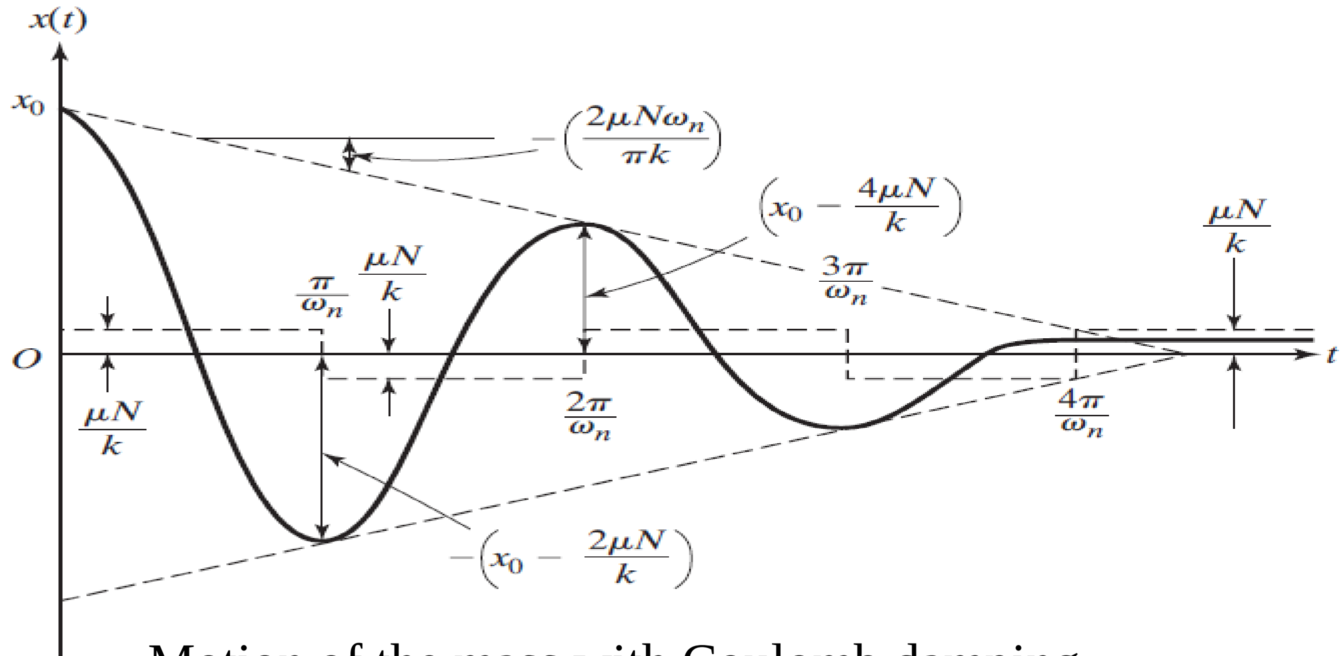
Case 2. When x is positive and dx/dt is negative or when x is negative and dx/dt is negative (i.e., for the half cycle during which the mass moves from right to left), the equation of motion can be:

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N$$

The solution of Eq.:

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k}$$

The term μN/k appearing in Equations is a constant representing the virtual displacement of the spring under the force μN, if it were applied as a static force.



Motion of the mass with Coulomb damping

Note the following characteristics of a system with coulomb damping:

1. The equation of motion is nonlinear with coulomb damping.

2. The natural frequency of the system is unchanged with the addition of coulomb damping.

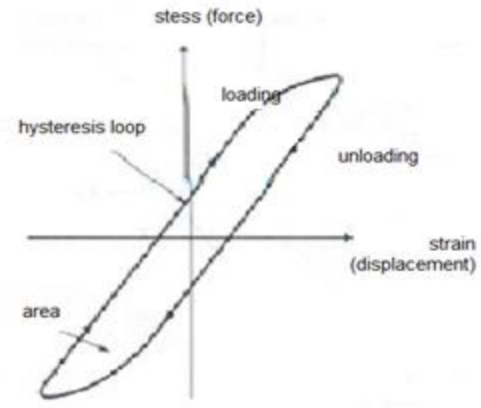
3. The motion is periodic with coulomb damping.

4. The system comes to rest after some time with coulomb damping.

5. The amplitude reduced linearly with coulomb damping.

The Structural (Hysteresis) Damping

The damping caused by the friction between the internal planes that slip or slide as the material deforms is called hysteresis (or solid or structural) damping .



The Coulomb-friction model is as a rule used to describe energy dissipation caused by rubbing friction. While as structural damping (caused by contact or impacts at joins), energy dissipation is determined by means of the coefficient of restitution of the two components that are in contact.
This form of damping is caused by Coulomb friction at a structural joint. It depends on many factors such as joint forces or surface properties .

Area of hysteresis loop is energy dissipation per cycle of motion- termed as per-unit-

$$\Delta W = \oint F\, dx = \oint_0^{\frac{2\pi}{\omega}} (kX \sin \omega t + cX \cos \omega t)(\omega X \cos \omega t) dt = \pi \omega c X^2 = d = the\ area$$

The purpose of structural damping is to dissipate vibration energy in a structure, thereby reducing the amount of radiated and transmitted sound

Free Vibration with Hysteresis Damping

By the applying the Newton's second law :

$$F = kx + c\dot{x} \quad x(t) = X\cos\omega t$$

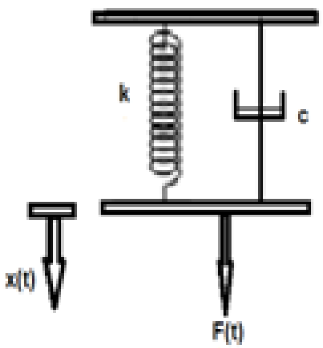
For a harmonic motion of frequency ω and amplitude X :

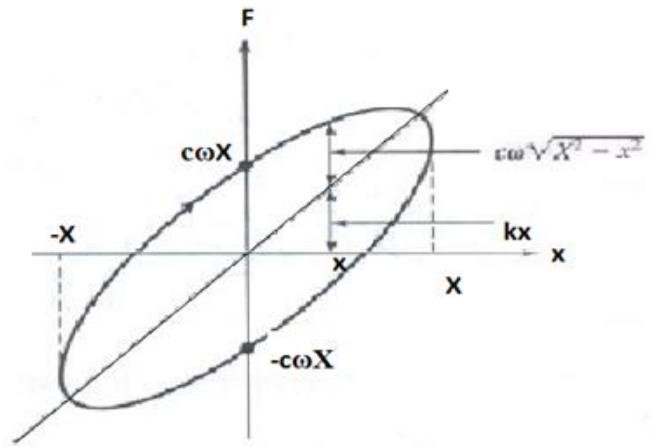
$$F(t) = kx \mp c\omega\sqrt{X^2 - x^2} \qquad c = \frac{h}{\omega}$$

The damping coefficient c is assumed to be inversely proportional to the frequency as

The energy dissipated by the damper in a cycle of motion becomes

$$\Delta W = \pi h X^2$$

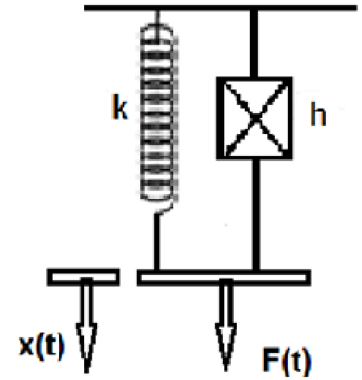




Complex Stiffness

The force displacement relation can be expressed by:

$$F = (k + ih)x = k\left(1 + i\frac{h}{k}\right)x = k(1 + i\beta)x$$

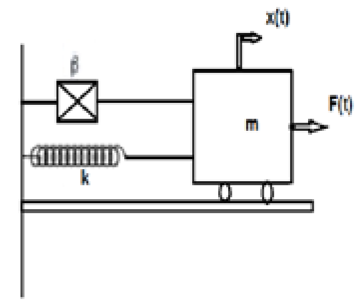


let $\beta = \frac{h}{k}$ is a constant indicating dimensionless measurement of damping.

$k(1 + i\beta)$ is called the complex stiffness of the system .

the energy loss per cycle : $\Delta W = \pi k \beta X^2$

Forced Vibration with Hysteresis Damping:



The system is subjected to harmonic force ;

$$F(t) = F_0 \cos \omega t$$

The equation of motion can be expressed as

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + k\omega = F_0 \cos \omega t$$

The particular solution is $x_p(t) = X\cos(\omega t - \phi)$ and Where

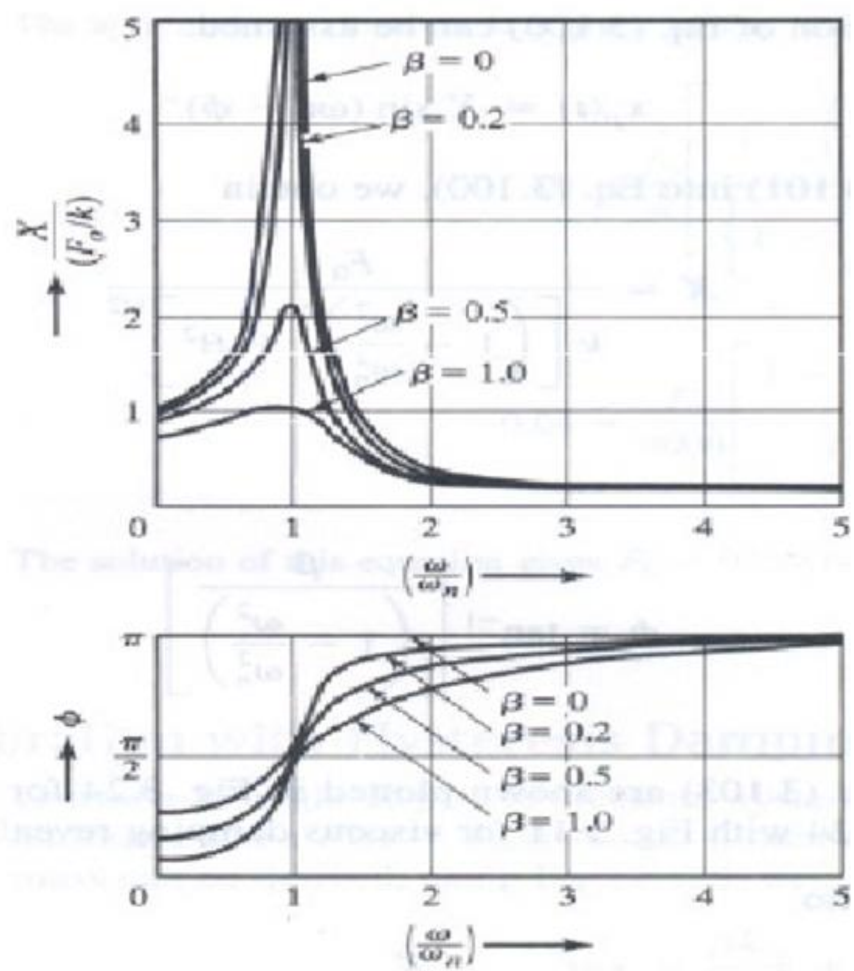
$$X = \frac{F_0}{k\left[\left(1 - \frac{\omega^2}{\omega_n^{\,2}}\right)^2 + \beta^2\right]^{\frac{1}{2}}}$$

$$\tan \phi = \frac{\beta}{\left(1 - \frac{\omega^2}{\omega_n^{\,2}}\right)}$$

The amplitude ratio $\frac{X}{F_0/k}$ reaches its maximum value of $F_0/k\beta$ at the resonant frequency in the case of hysteresis damping $(r = \frac{\omega}{\omega_n} = 1)$, while it occurs at a frequency below resonance in the case of viscous damping .

$$\tan^{-1}\beta \text{ at } \omega = 0$$

- The phase ϕ has a value in the case of hysteresis damping . This indicates that the response can never be in phase with the forcing function in the case of hysteresis damping .



Measurement of Damping

Hysteresis Loop Method:

Depending on inertial and elastic conditions the hysteresis loop will change but the work done in the conservative forces will be zero, consequently work done will be equal to energy dissipated by damping only without normalizing with respect to mass.

The energy dissipation per hysteresis loop of hysteretic damping is

$$\Delta U_v = \pi {x_0}^2 \omega c$$

And the initial max potential energy is

$$\Delta U_h = \pi {x_0}^2 h$$

the loss factor of hysteretic damping is given by :

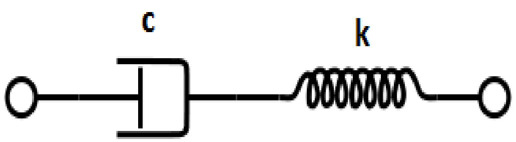
$$\eta = \frac{h}{k} = \frac{2}{\zeta}$$

Then ,the equivalent damping ratio for hysteretic damping is

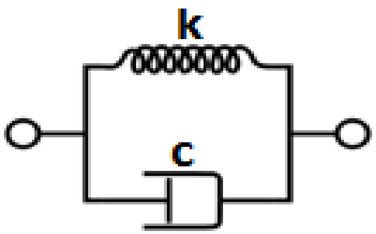
$$\zeta = \frac{h}{2k}$$

Models of Hysteresis Damping (Structural Damping):

The Maxwell Model

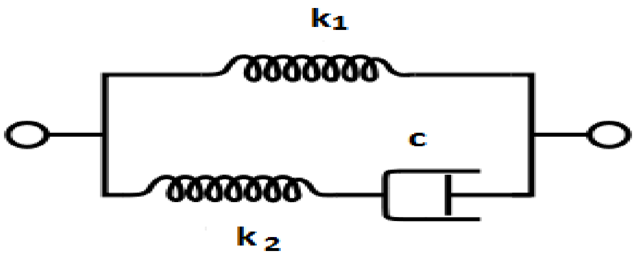


The Maxwell model can be represented by a purely viscous damper and a purely elastic spring connected in



The Kelvin–Voigt Model

Also called the Voigt model, can be represented by a purely viscous damper and purely elastic spring connected in parallel .



Standard Linear Solid Model

The standard linear solid (SLS) model, (Zener model), is a method of modeling the behavior of viscoelastic material using a linear combination of springs and damper.

Determination of Equivalent Viscous Damping from Frequency Response Curve

Where the free vibration test is not practical, damping may be obtained from the frequency- response curve of forced vibration test.

Suppose the frequency response curve as obtained for as system excited with a constant force is shown in figure below:

Magnification at resonance is given by:

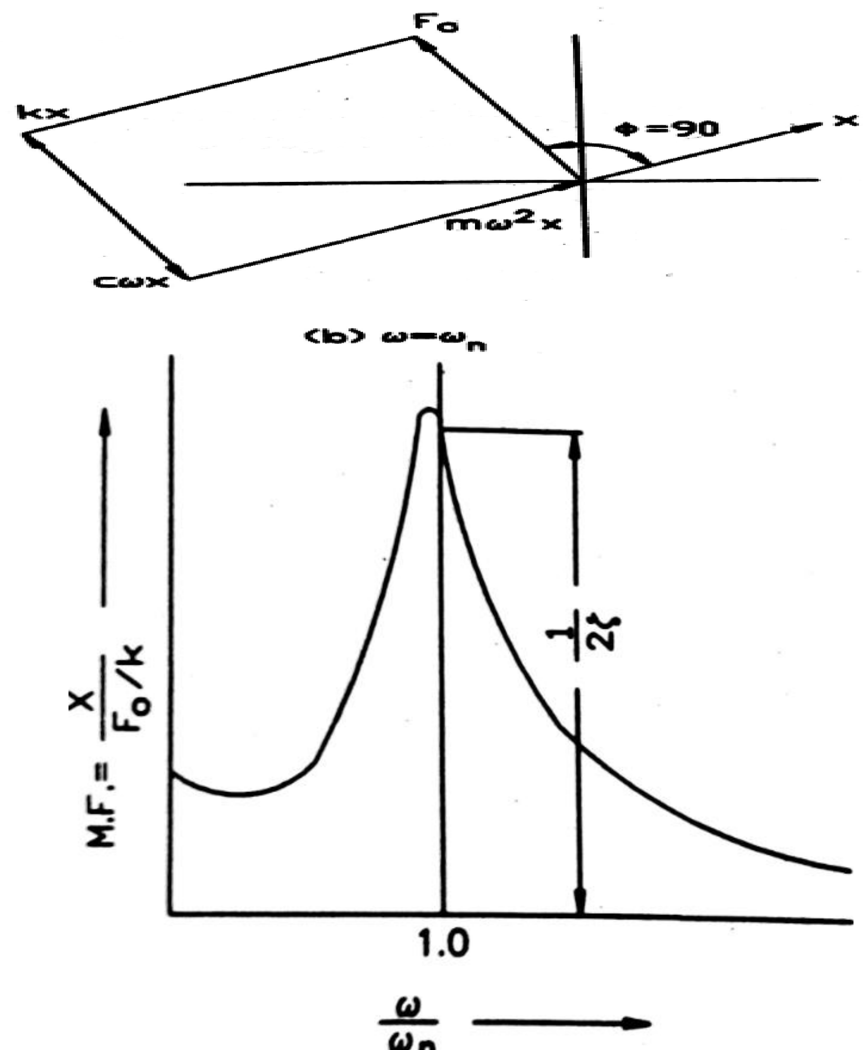
$$c\omega X = F_0$$

or the amplitude at resonance is :

$$X_r = \frac{F_0}{c\omega} = \frac{F_0/k}{c\omega/k}$$

$$= \frac{X_{st}}{2\zeta\, \omega/\omega_n} = \frac{X_{st}}{2\zeta} \quad \text{as } \omega = \omega_n \text{ at resonance}$$

$$2\zeta = \frac{X_{st}}{X_r}$$



$$\frac{X}{X_{st}} = \frac{1}{\sqrt{\left[1-(\omega/\omega_p)^2\right]^2 + \left[2\zeta(\omega/\omega_p)\right]^2}}$$

$$\frac{X_p}{X_{st}} = \frac{1}{2\zeta}$$

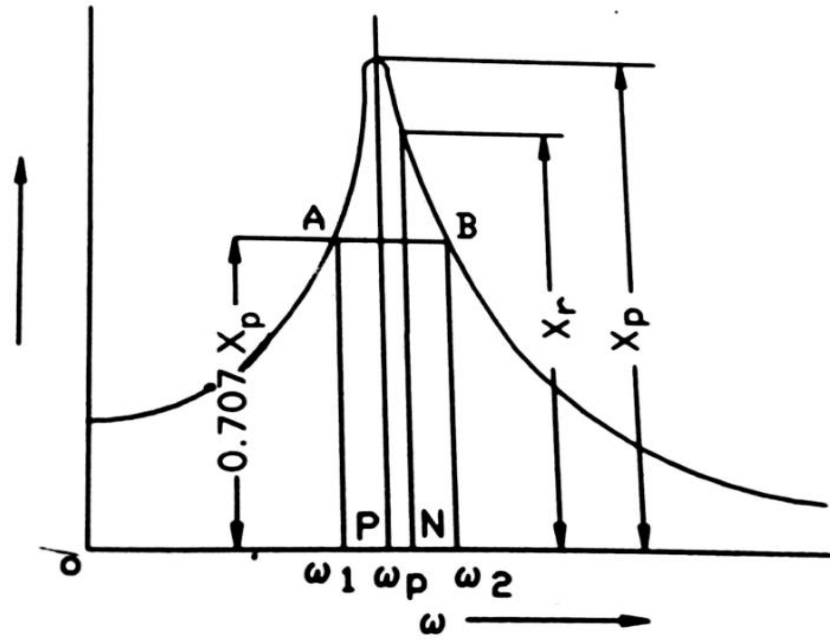
$$X = 0.707X_p$$

assuming $\zeta << 1$

$$\left(\frac{\omega}{\omega_p}\right)^2 = 1 \pm 2\zeta$$

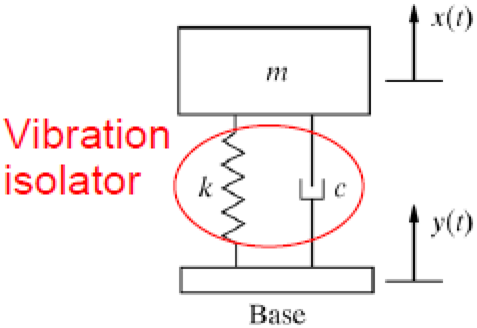
that is $\left(\frac{\omega_1}{\omega_p}\right)^2 = 1 - 2\zeta$ and $\left(\frac{\omega_2}{\omega_p}\right)^2 = 1 + 2\zeta$

$$\omega_p = \frac{\omega_1 + \omega_2}{2} \quad \text{so} \left[\frac{\omega_2 - \omega_1}{\omega_p} = 2\zeta\right]$$



Taking these measurements from the frequency response curve after making necessary construction, we get the first approximate value of ζ.

Vibration Isolation (Moving Base)



Objective:

To isolate a device from the source of vibration (moving base).
(To reduce vibration of machine transmitted through moving base.)

Application

Automobile suspension

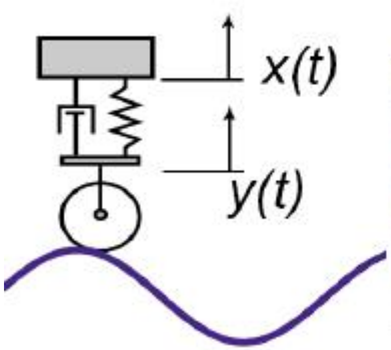
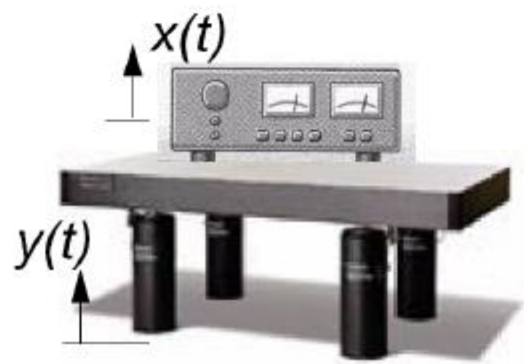
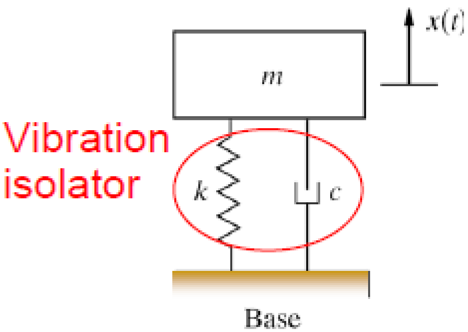


Table isolator



Vibration isolation (Fixed base)

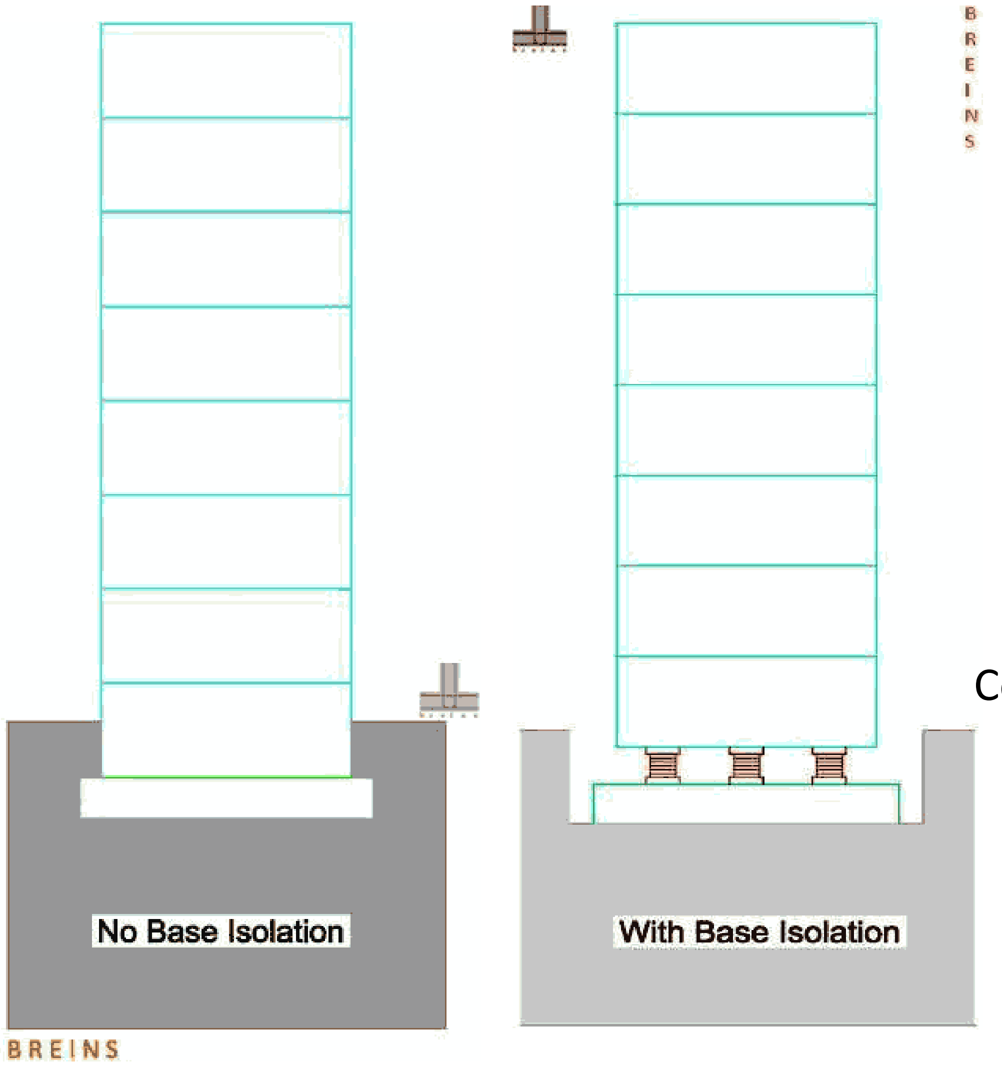


Objective:

To isolate the source of vibration (machine) from the other system.
To reduce vibration (force) transmitted from the machine to base.

Washing machine

Generator or the other machines



Coil spring isolator with integral viscous damping unit

The force experienced by the instrument or mass m (same as the force transmitted to mass m) is given by:

$$F_t(t) = m\ddot{x}(t) = k\{x(t) - y(t)\} + c\{\dot{x}(t) - \dot{y}(t)\}$$

where *y(t)* is the displacement of the base, *x(t)-y(t)* is the relative displacement of the spring, and $\dot{x}(t) - \dot{y}(t)$ is the relative velocity of the damper. In such cases, we can insert an isolator (which provides stiffness and /or damping) between the base being subjected to force or excitation and the mass to reduce the motion and/or force transmitted to the mass. Thus both displacement isolation and force isolation become important in this case also.

Vibration Isolation System with Rigid Foundation

It is assumed that the operation of the machine gives rise to a harmonically varying force $F(t) = F_0 \cos \omega t$

The equation of motion of the machine (of mass *m) is given by:*

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

Since the transient solution dies out after some time, only the steady-state solution will be left. The steady-state solution of Eq. is given by

$$x(t) = X\cos(\omega t - \phi)$$

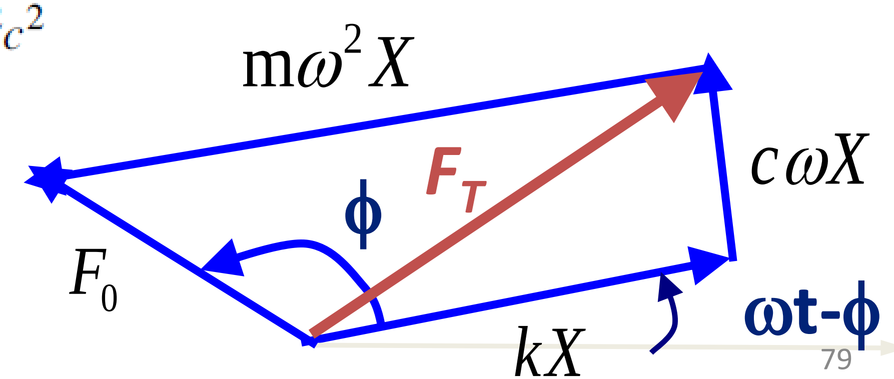
Where $X = \dfrac{F_0}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}}$ and $\phi = \tan^{-1}\left(\dfrac{\omega c}{k - m\omega^2}\right)$

The force transmitted to the foundation through the spring and the dashpot, $F_t(t)$ is given by $F_t(t) = kx(t) + c\dot{x}(t) = kX\cos(\omega t - \phi) - c\omega X\sin(\omega t - \phi)$

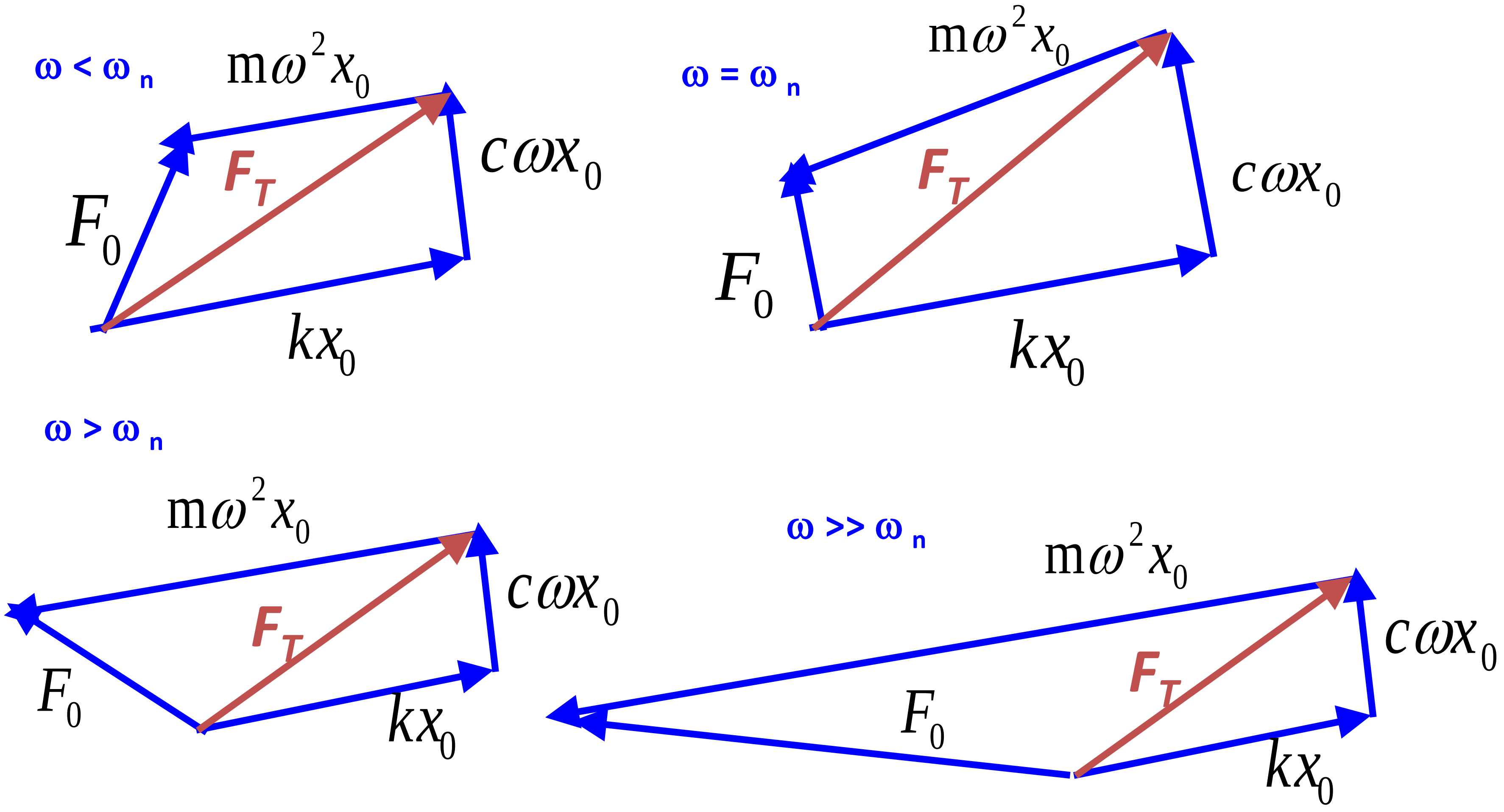
The magnitude of the total transmitted force FT is given by

$$F_T = [(kx)^2 + (c\dot{x})^2]^{1/2} = X\sqrt{k^2 + \omega^2 c^2}$$

$$= \frac{F_0(k^2 + \omega^2 c^2)^{1/2}}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}}$$



Variation of F_0 with ω

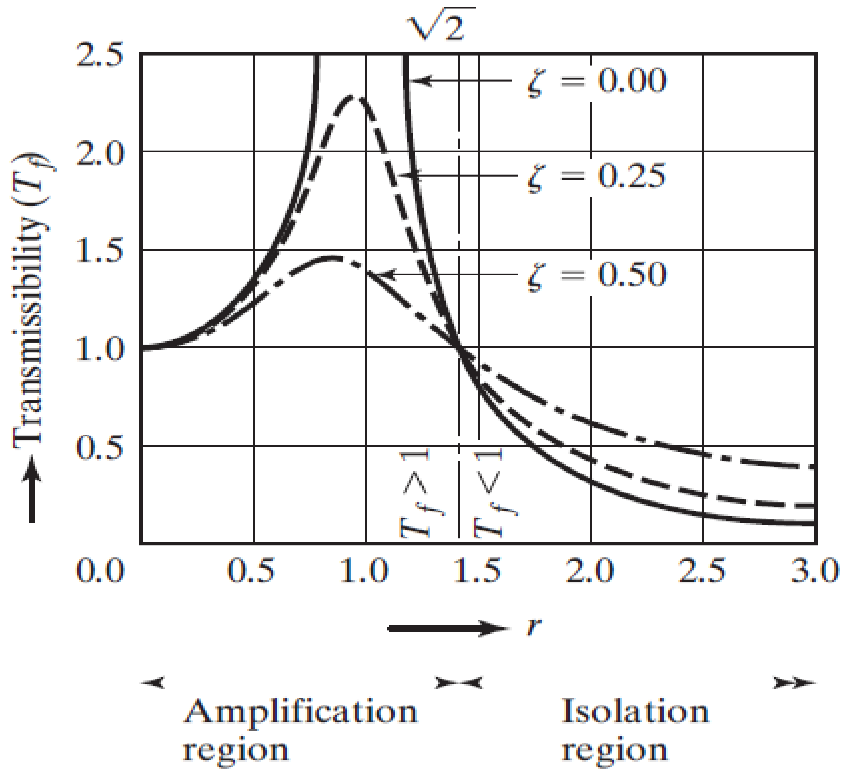


The transmissibility or transmission ratio of the isolator (T_f)is defined as the ratio of the magnitude of the force transmitted to that of the exciting force:

$$T_f = \frac{F_T}{F_0} = \left\{\frac{k^2 + \omega^2 c^2}{(k - m\omega^2)^2 + \omega^2 c^2}\right\}^{1/2}$$

$$= \left\{\frac{1 + (2\zeta r)^2}{[1 - r^2]^2 + (2\zeta r)^2}\right\}^{1/2}$$

where $r = \frac{\omega}{\omega_n}$ is the frequency ratio. The variation of T_f with the frequency ratio $r = \frac{\omega}{\omega_n}$ is shown in Fig. In order to achieve isolation, the force transmitted to the foundation needs to be less than the excitation force. It can be seen, from Fig., that the forcing frequency has to be greater than $\sqrt{2}$ times the natural frequency of the system in order to achieve isolation of vibration.



For small values of damping ratio ζ and for frequency ratio r > 1the force transmissibility, given by Eq., can be approximated as

$$T_f = \frac{F_t}{F} \approx \frac{1}{r^2 - 1} \quad \text{or} \quad r^2 \approx \frac{1 + T_f}{T_f}$$

The magnitude of the force transmitted to the foundation can be reduced by decreasing the natural frequency of the system (ω_n).

The force transmitted to the foundation can also be reduced by decreasing the damping ratio. However, since vibration isolation requires r> √2, the machine should pass through resonance during start-up and stopping. Hence, some damping is essential to avoid infinitely large amplitudes at resonance.

Although damping reduces the amplitude of the mass (X) for all frequencies, it reduces the maximum force transmitted to the foundation (F_t) only if r< √2. Above that value, the addition of damping increases the force transmitted.

If the speed of the machine (forcing frequency) varies, we must compromise in choosing the amount of damping to minimize the force transmitted. The amount of damping should be sufficient to limit the amplitude X and the force transmitted F_t while passing through the resonance, but not so much to increase unnecessarily the force transmitted at the operating speed.

Reduction of the Vibratory Motion of the Mass. In many applications, the isolation is required to reduce the motion of the mass (machine) under the applied force. The displacement amplitude of the mass m due to the force F(t), given by Eq., can be expressed as:

Displacement transmissibility or amplitude ratio:

$$T_d = \frac{X}{\delta_{st}} = \frac{kX}{F_0} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

It indicates the ratio of the amplitude of the mass, X, to the static deflection under the constant force F0.