Subject: Mechanical Vibrations								
Program: B.Tech. Mechanical Engineering				Subject Code: ME0705			Semester: VII	
Teaching Scheme				Examination Evaluation Scheme				
Lecture	**Tutorial**	**Practical**	**Credits**	**University Theory Examination**	**University Practical Examination**	**Continuous Internal Evaluation (CIE)-Theory**	**Continuous Internal Evaluation (CIE)-Practical**	**Total**
2	2	2	4	24/60	24/60	16/40	16/40	200

Course Objectives

1. Understand basics of vibration.
2. Understand of single degree of freedom systems- forced undamped and damped vibrations.
3. Understand Multi-Degree of freedom systems and Natural Frequency calculations.
4. Understand continuous system.

CONTENTS

UNIT-I

[10 hours]

Introduction

Vibration terminology, Harmonic and periodic motions, Beats phenomenon, uses and effects, practical applications and current research trends

Single Degree of Freedom Systems – Free Undamped and Damped Vibrations

Free undamped vibrations using Newton's second law, D'Alemberts principles, Energy method, Rayleigh's method, free damped vibrations, logarithmic decrement, under damped, over damped and critically damped conditions**.**

UNIT-II

[12 hours]

Single Degree of Freedom Systems –Forced Undamped and Damped Vibrations

Forced harmonic undamped vibration, Damped free Magnification factor, Transmissibility, Vibration Isolation, Equivalent viscous damping, Rotor unbalance, Excitation and Stability analysis

Two Degree of Freedom Systems

Generalized and Principal coordinates, derivation of equations of motion, Lagrange's equation, Coordinate coupling, Forced Harmonic vibration.

UNIT-III

[14 hours]

Multi-Degree of Freedom Systems

Derivation of equations of motion for MDOFs, influence coefficient method, Properties of undamped and damped vibrating systems: flexibility and stiffness matrices, reciprocity theorem, Modal analysis.

Natural Frequency Calculations

Rayleigh method, Stodala method, Matrix iteration method , Holzer's method and Dunkerley's method, Whirling Speed of shaft.

UNIT-IV

[14 hours]

Continuous Systems

Introduction to continuous systems, lateral vibration of string, transverse vibrations of the beam, Orthogonality of eigenvectors.

Vibration Measurement Apparatus

Vibration measuring instruments, acceleration and frequency measuring instruments, FFT analyzer.

Course outcomes

On the completion of this course, students will be able to...

1. Comprehend basic concepts of vibrations and importance of vibration with respect to machine design
2. To model any mechanical/structural components whose frequencies are required to be calculated
3. Theoretically find natural frequencies of any damped/undamped as well as free/forced vibration system
4. Find measured natural frequencies of any structure, component

List of Experiments

Sr. No.	Title	Learning Outcomes
1	To study frequency of simple pendulum.	After studying this experiment, student will able to understand, • Theoretical derivation of angular vibrations • Find experimental frequencies of simple pendulum • Reasons for why theoretical frequencies are deviating from experimental one
2	To study frequency of compound pendulum.	After studying this experiment, student will able to understand, • Theoretical derivation of compound pendulum/bar • Find experimental frequencies of simple pendulum • Reasons for why theoretical frequencies are deviating from experimental one

3	To study frequency of sprig mass system.	After studying this experiment, student will able to understand, • Theoretical derivation of Linear vibrations • Find experimental frequencies of simple pendulum • Reasons for why theoretical frequencies are deviating from experimental one
4	To study frequency of lateral vibration system.	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of Lateral vibrations • Reasons for why theoretical frequencies are deviating from experimental one
5	To study frequency of torsion vibration system (single Rotor).	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of torsional vibrations • Reasons for why theoretical frequencies are deviating from experimental one
6	To study free damped vibration system.	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of damped torsional vibrations • Reasons for why theoretical frequencies are deviating from experimental one

7	To study whirling speed of shaft.	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of whirling shaft • Reasons for why theoretical frequencies are deviating from experimental one
8	To study forced damped vibration system.	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of forced damped vibration systems • Reasons for why theoretical frequencies are deviating from experimental one
9	To study frequency of roller rolls without slip inside cylinder	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of Lateral vibrations • Reasons for why theoretical frequencies are deviating from experimental one
10	To study frequency of U tube filled with liquid	After studying this experiment, student will able to understand, • Theoretical and experimental frequencies of liquid filled in U tube • Reasons for why theoretical frequencies are deviating from experimental one

Reference Books

1. Mechanical Vibration by Singiresu S. Rao, Pearson Education
2. Mechanical Vibrations by G. K. Groover, Nemchand & Bro
3. Theory of Vibration with Application by Willium T Thomson, Pearson Education
4. Theory and Problems of Mechanical Vibrations by Graham Kelly, schaum series
5. Fundamental of Mechanical Vibrations by Graham Kelly Mcgraw hill

Web resources

1. Mechanical Vibrations (http://nptel.ac.in/courses/112103111/)

Chapter 1: Fundamentals of Vibration

* Importance of Study of Vibration

- Most human activities involve vibration:
  - We hear because our eardrums vibrate
  - We see because lightwaves undergo vibration
  - Breathing is associated with the vibration of lungs
  - Walking involves oscillatory motion of legs and hands
  - etc.
- Engineering applications of vibration include design of machines, foundations, structures, engines, turbines, and control systems.
- Problems caused by vibration:
  - Structures and machines with an unbalance in their system can be subjected to vibration and fail because of material fatigue. (Rotating parts in engines and/or turbines.)
  - Vibration can cause more rapid wear of machine parts such as bearings and gears, and also creates excessive noise.
  - In machines, vibration can loosen fasteners such as nuts.
  - "Resonance" can occur whenever the natural frequency of vibration of a machine or structure coincides with the frequency of the external excitation. Resonance can have devastating effects.
  - etc.

- So, researchers have tried to understand "vibration", and mathematical theories have been developed to describe "vibration".

- One of the important purposes of vibration study is to reduce vibration through proper design of machines and their mountings. In addition, proper design of structures for vibration is also quite important. Proper design can minimize imbalance and control the effects of the imbalance.

- Another important purpose of vibration study can be to utilize vibration profitably. Vibration can be put to work in vibratory conveyors, hoppers, sieves, compactors, washing machines, clocks, etc. Vibration can be also used in simulating earthquakes and conducting studies on the seismic design and performance assessment of structures.

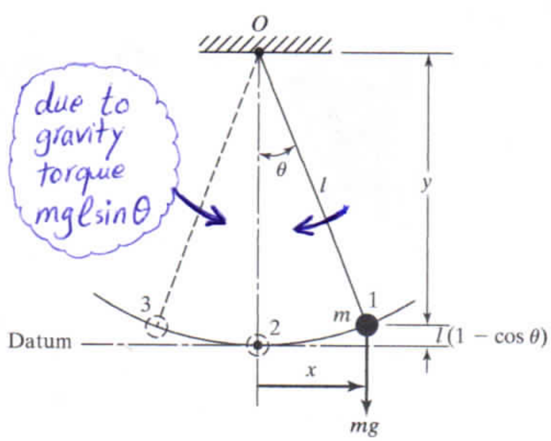
* Basic Concepts of Vibration

- Vibration: Any motion that repeats itself after an interval of time is called "vibration" or "oscillation".

- Elementary parts of vibratory systems are:
  - A means for storing potential energy $\rightarrow$ spring or elasticity
  - A means for storing kinetic energy $\longrightarrow$ mass or inertia
  - A means for dissipating energy $\longrightarrow$ damper

- The vibration of a system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration.

Consider the following simple pendulum:



Position 1: $\begin{cases} \text{kinetic: } 0 \\ \text{potential: } mgl(1-\cos\theta) \end{cases}$

converted

Position 2: $\begin{cases} \text{kinetic: All} \\ \text{potential: } 0 \end{cases}$

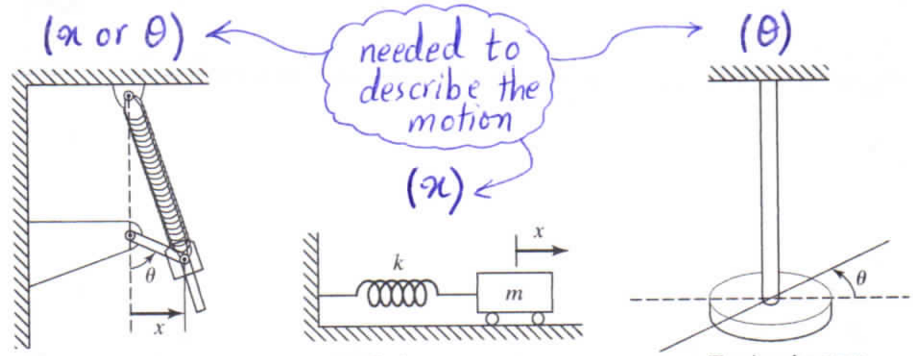
converted

Position 3: $\begin{cases} \text{kinetic: } 0 \\ \text{potential: All} \end{cases}$

It is noted that θ gradually decreases due to the resistance (damping) offered by the surrounding medium (air). Some energy is dissipated in each cycle of vibration and the pendulum ultimately stops.

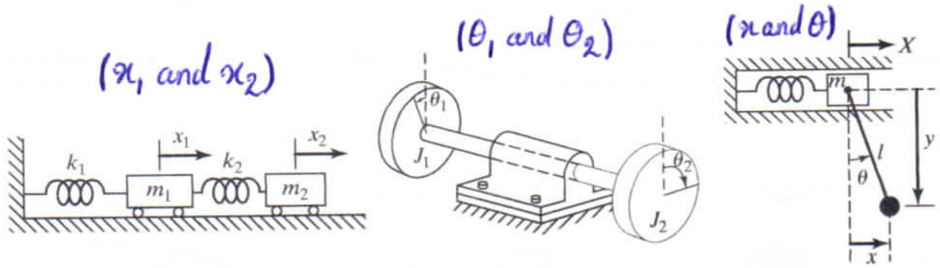
- Number of degrees of freedom: The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time.

single-degree-of-freedom systems →

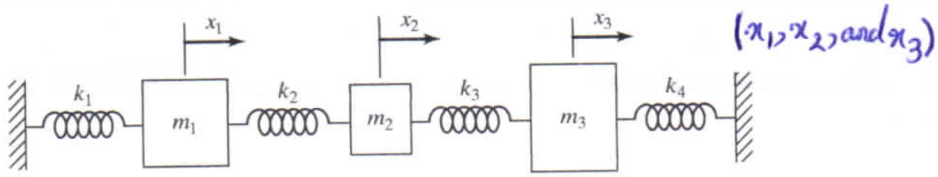


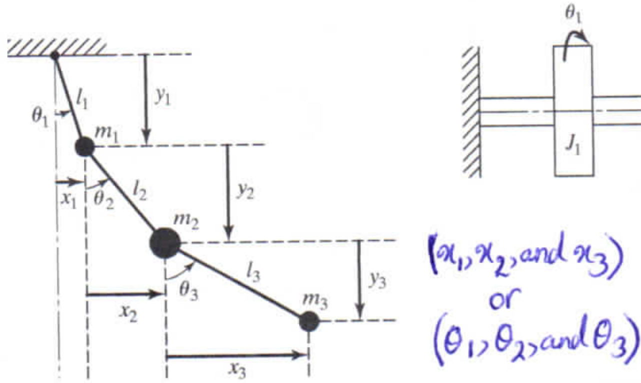
Slider-crank-spring mechanism — Spring-mass system — Torsional system

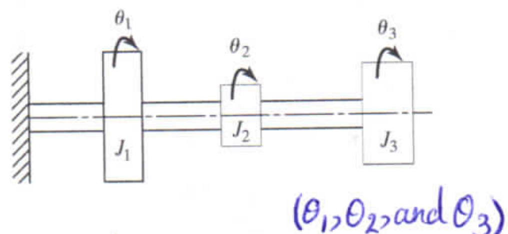
two-degree-of-freedom systems →



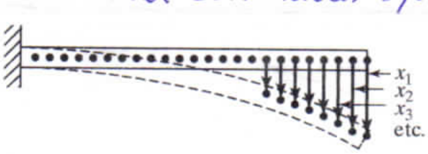
three-degree-of-freedom systems →







– Discrete and continuous systems: Systems with a finite number of degrees of freedom are called "discrete" or "lumped parameter" systems, and those with an infinite number of degrees of freedom are called "continuous" or "distributed" systems.



Continuous system (cantilever beam) approximated as discrete system

* Classification of Vibration

- Free Vibration: If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as "free vibration". No external force acts on the system.

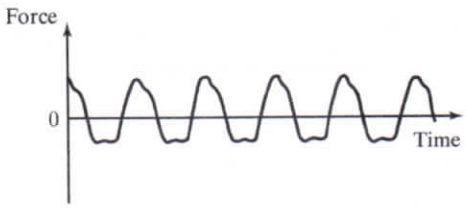
- Forced Vibration: If a system is subjected to an external force, the resulting vibration is known as "forced vibration".

  Note: If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as "resonance" occurs, and the system undergoes dangerously large oscillations.

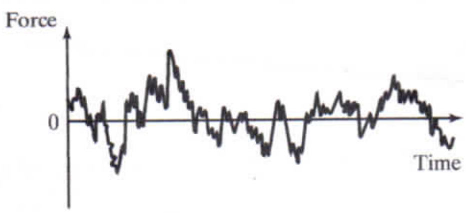
- Undamped/Damped Vibration: If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as "undamped vibration". If any energy is lost in this way, however, it is called "damped vibration".

- Linear/Nonlinear Vibration: If all the basic components of a vibratory system – the spring, the mass, and the damper – behave linearly, the resulting vibration is known as "linear vibration". If, however, any of the basic components behave nonlinearly, the vibration is called "nonlinear vibration".

- Deterministic/Random Vibration: If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called "deterministic". The resulting vibration is known as "deterministic vibration".
In some cases, the excitation is nondeterministic or random; the value of the excitation at a given time can not be predicted. If the excitation is random, the resulting vibration is called "random vibration". In this case, the vibratory response of the system is also random. Ground motion during earthquakes is an example of random excitation.



A deterministic (periodic) excitation



A random excitation

* Vibration Analysis Procedure

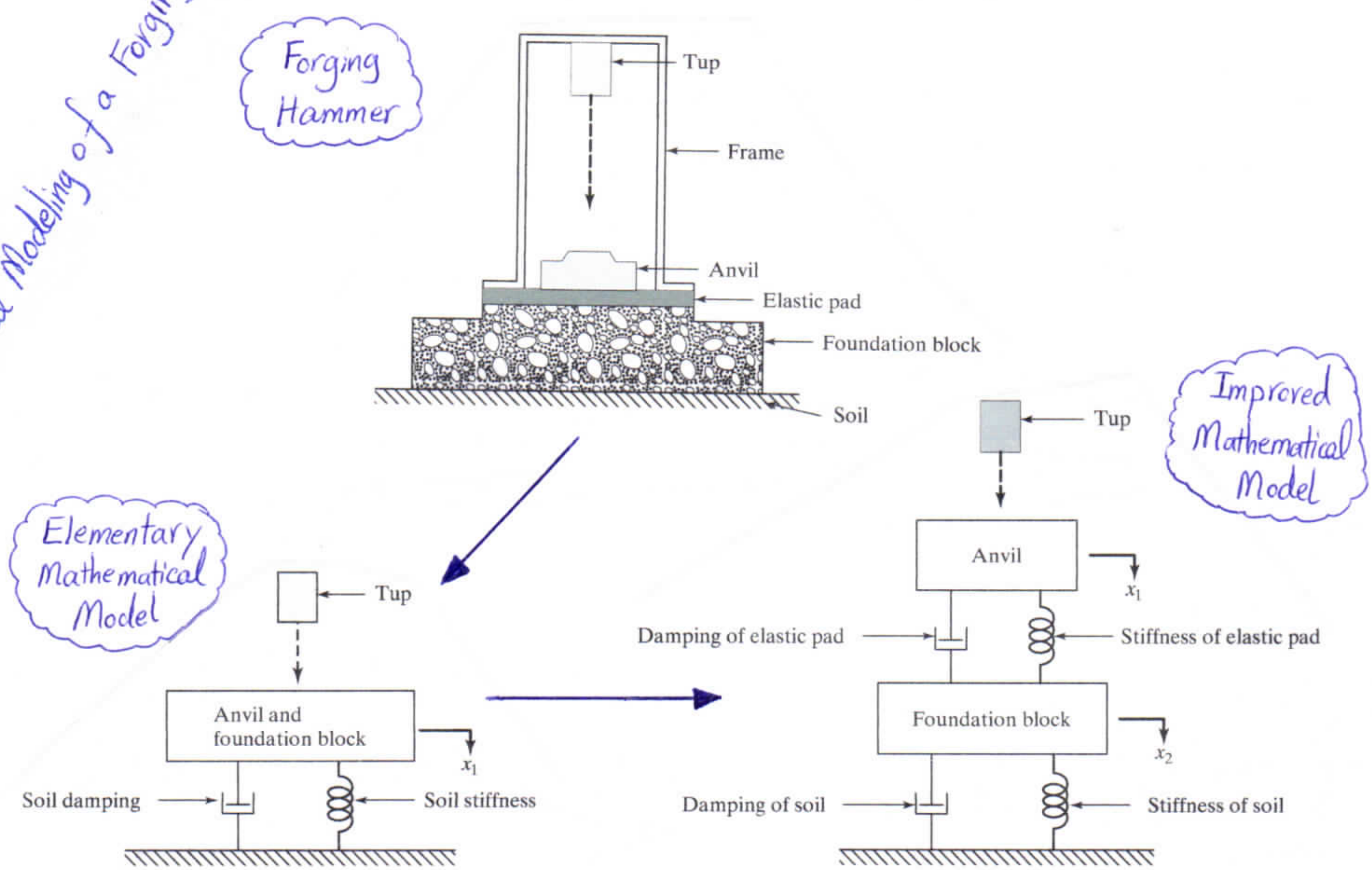
1. "Mathematical Modeling" to represent all the important features of the system for the purpose of deriving the mathematical equations governing the system's behavior.

2. "Derivation of Governing Equations": Free-body diagrams of all masses are drawn, and the equations are derived using principles of dynamics by Newton's second Law of motion, D'Alembert's principle, and principle of conservation of energy.

3. "Solution of the Governing Equations" to find the response of the vibrating system. Depending on the nature of the problem, standard methods for solving differential equations, Laplace transform methods, matrix methods, and numerical methods can be used to find the solution.

4. "Interpretation of the Results": The solution of governing equations gives the displacements, velocities, and accelerations of the various masses of the system. These results must be interpreted properly.

Mathematical Modeling of a Forging Hammer



* Spring Elements

– Linear Spring: A spring is said to be linear if the elongation or reduction in length "x" is related to the applied force F as

$$F = kx$$

where "k" is a constant, known as the "spring constant" or "spring stiffness" or "spring rate".

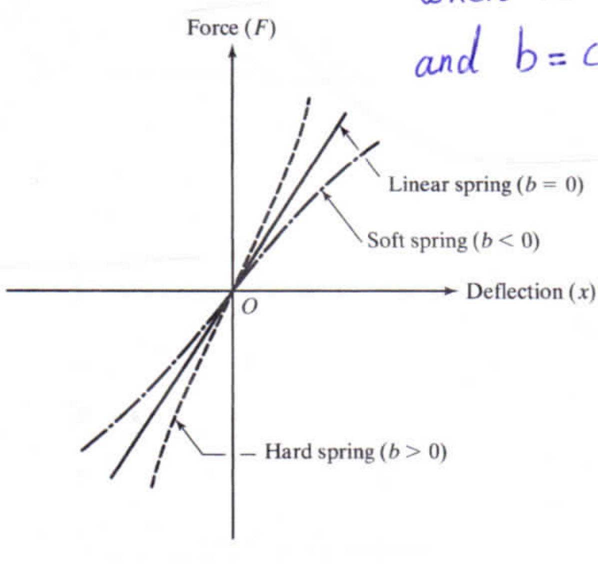
The work done in deforming a spring is stored as strain or potential energy in spring, and is given by:

$$U = \frac{1}{2} k x^2$$

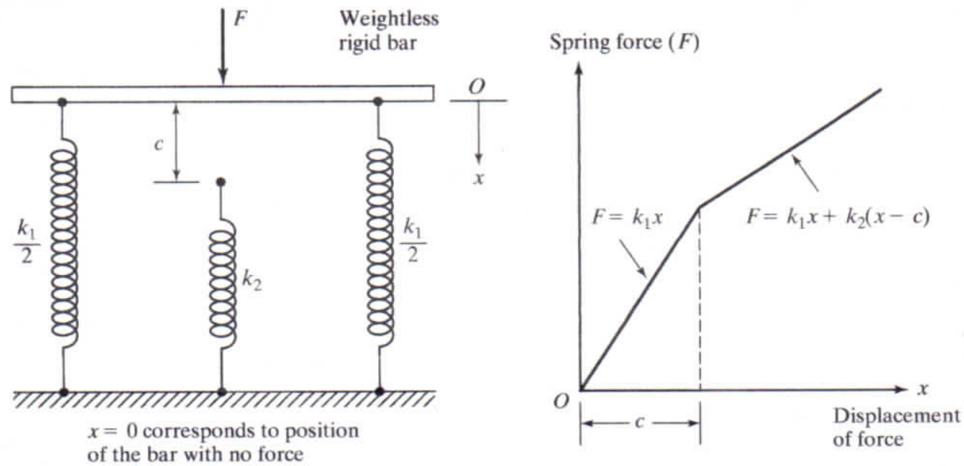
– Nonlinear Springs: Springs with nonlinear force-deflection relation. For example, consider

$$F = ax + bx^3$$

where a = constant associated with linear part $(a>0)$ and b = constant associated with nonlinearity.

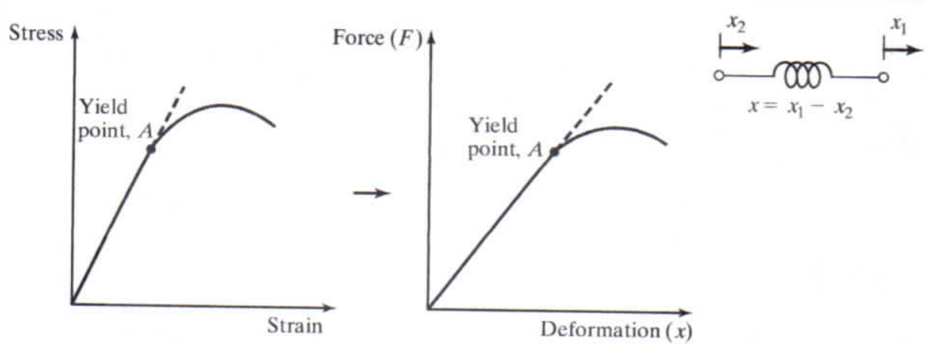


– Systems with two or more 'Linear' springs may exhibit 'nonlinear' force–displacement relationship.



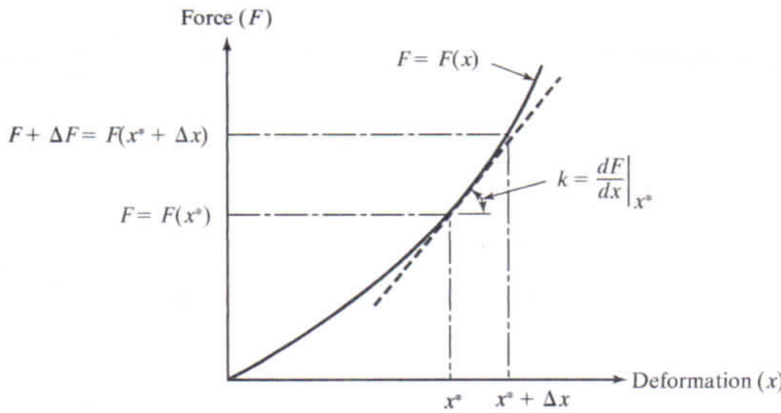
– Linearization of a Nonlinear Spring:

Springs act linearly ($F = kx$) up to a certain limit. Once the stress exceeds the yield point of the material, the force–deformation relation becomes nonlinear.



Nonlinearity beyond proportionality limit.

In practice, we assume that deflections are small and make use of the linear relation $F = kx$. Even, if the force–deflection relation is nonlinear, we approximate it as a linear one by using a "Linearization" process. Consider,



Linearization process.

assume F causes x^*. If ΔF is added to F, then Δx will be added to x^*, i.e. $F + \Delta F = F(x^* + \Delta x)$. Then, we can express the spring force $F + \Delta F$ using Taylor's series expansion.

$$F + \Delta F = F(x^* + \Delta x) = F(x^*) + \frac{dF}{dx}\Big|_{x^*}(\Delta x) + \frac{1}{2!}\frac{d^2F}{dx^2}\Big|_{x^*}(\Delta x)^2 + \cdots$$

If $\Delta x \to 0$ and we neglect the higher-order derivative terms, then

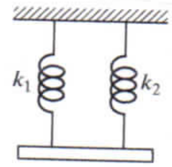
$$F + \Delta F = F(x^*) + \frac{dF}{dx}\Big|_{x^*}(\Delta x)$$

Since $F = F(x^*)$ and $\Delta F = k\Delta x$, the linearized spring constant k at x^* is given by:

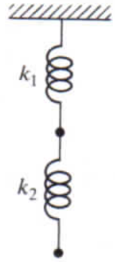
$$\boxed{k = \frac{dF}{dx}\Big|_{x^*}}$$

– Combination of Springs

Springs in Parallel



Springs in Series



If we have n springs with spring constants $k_1, k_2, \ldots, k_n$ in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \cdots + k_n$$

If we have n springs with spring constants $k_1, k_2, \ldots, k_n$ in series, then the equivalent spring constant k_{eq} can be obtained:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$

– Example: A machine, weighing 1000 lb, is supported on a rubber mount. The force-deflection relationship of the rubber mount is given by

$$F = 2000x + 200x^3$$

where the force 'F' and the deflection 'x' are measured in pounds and inches, respectively. Determine the equivalent Linearized spring constant of the rubber mount at its static equilibrium position.

Solution: At the static equilibrium position (x^*), we have

$$1000 = 2000x^* + 200x^{*3} \rightarrow \begin{cases} x^* = 0.4884 \\ x^* = -0.2442 + 3.1904i \\ x^* = -0.2442 - 3.1904i \end{cases}$$

Note: In MATLAB

```
f=[200 0 2000 -1000]
r=roots(f)
```

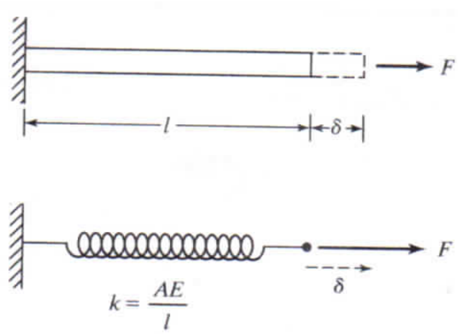
So, $x^* = 0.4884$ is considered.

$$k_{eq} = \left.\frac{dF}{dx}\right|_{x^*} = 2000 + 600(0.4884)^2 = \underline{\underline{2143.1207 \text{ lb/in}}}$$

Note that k_{eq} predicts the static deflection as:

$$F = k_{eq} \cdot x \rightarrow x = \frac{F}{k_{eq}} = \frac{1000}{2143.1207} = 0.4666 \text{ in} \xleftrightarrow[\text{with}]{\text{compare}} 0.4884 \text{ in}$$

- Example: Find the equivalent spring constant of a uniform rod of Length 'l', cross-sectional area 'A', and Young's modulus 'E' subjected to an axial tensile force 'F'.

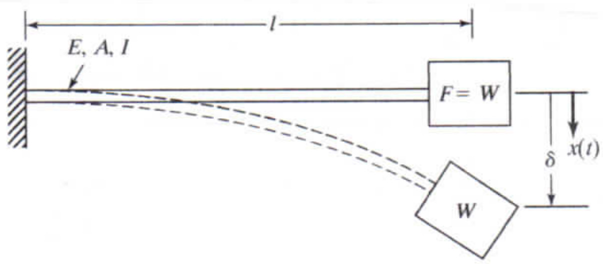


Solution: $\delta = \frac{\delta}{l} l = \varepsilon l = \frac{\sigma}{E} l = \frac{Fl}{EA}$

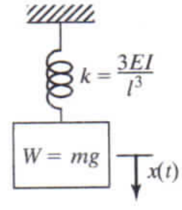
Note: $\varepsilon = \frac{\delta}{l}$, $\sigma = E\varepsilon$, $\sigma = \frac{F}{A}$

So, $k = \frac{F}{\delta} = \frac{F}{\frac{Fl}{EA}} = \underline{\underline{\frac{EA}{l}}}$

- Example: Find the equivalent spring constant of a cantilever beam subjected to a concentrated Load F at its end.



Cantilever with end force

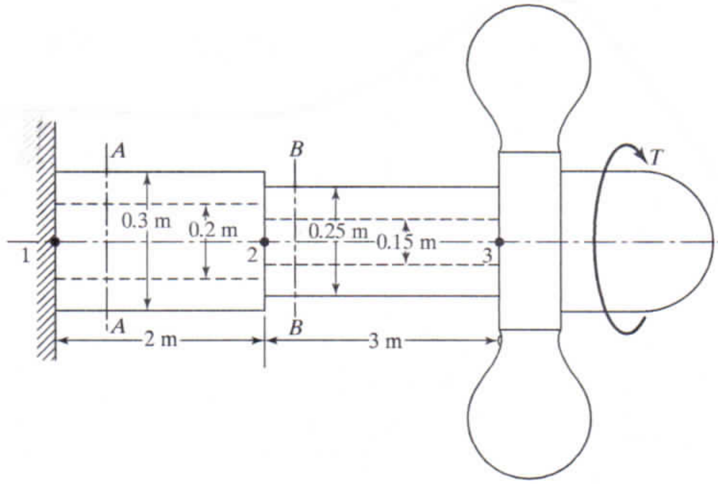


Equivalent spring

Solution: We know, $\delta = \frac{Wl^3}{3EI}$

So, $k = \frac{W}{\delta} = \underline{\underline{\frac{3EI}{l^3}}}$

- Example: Determine the torsional spring constant of the steel propeller shaft.



Solution: The spring constants of the two segments are:

$$k_{t_{12}} = \frac{GJ_{12}}{\ell_{12}} = \frac{(80\times10^9)(\pi(0.3^4-0.2^4)/32)}{2} = 25.5255\times10^6 \text{ N.m/rad}$$

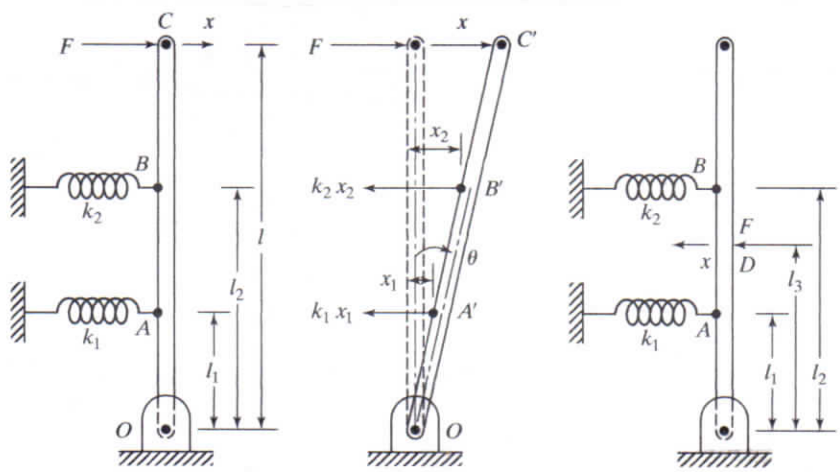
$$k_{t_{23}} = \frac{GJ_{23}}{\ell_{23}} = \frac{(80\times10^9)(\pi(0.25^4-0.15^4)/32)}{3} = 8.9012\times10^6 \text{ N.m/rad}$$

The springs are in series, so

$$\frac{1}{k_{t_{eq}}} = \frac{1}{k_{t_{12}}} + \frac{1}{k_{t_{23}}} \rightarrow k_{t_{eq}} = \frac{(25.5255\times10^6)(8.9012\times10^6)}{25.5255\times10^6 + 8.9012\times10^6}$$

$$\underline{\underline{k_{t_{eq}} = 6.5997\times10^6 \text{ N.m/rad}}}$$

– Example: A hinged rigid bar of length 'ℓ' is connected by two springs of stiffnesses 'k_1' and 'k_2' and is subjected to a force 'F' as shown. Assuming that the angular displacement of the bar (θ) is small, find the equivalent spring constant of the system that relates the applied force 'F' to the resulting displacement 'x'.



Note: Although springs are parallel, but $k_{eq} = k_1 + k_2$ does not work, since displacements of springs are not equal!

Solution:

Equilibrium Method

$$\begin{cases} x_1 = \ell_1 \sin\theta \approx \ell_1\theta \\ x_2 = \ell_2 \sin\theta \approx \ell_2\theta \\ x = \ell \sin\theta \approx \ell\theta \end{cases}$$

Reactions of Springs:

$k_1 x_1$

$k_2 x_2$

$$\Sigma M_O = 0 \longrightarrow k_1 x_1(\ell_1) + k_2 x_2(\ell_2) = F\ell$$

$$F = k_1\left(\frac{x_1 \ell_1}{\ell}\right) + k_2\left(\frac{x_2 \ell_2}{\ell}\right) = k_{eq}\, x$$

$$k_{eq}\, x = k_1 \frac{\ell_1 \cancel{\theta} \cdot \ell_1}{\ell} + k_2 \frac{\ell_2 \cancel{\theta} \cdot \ell_2}{\ell} = k_{eq} \cdot \ell \cancel{\theta}$$

$$\underline{\underline{k_{eq} = k_1\left(\frac{\ell_1}{\ell}\right)^2 + k_2\left(\frac{\ell_2}{\ell}\right)^2}}$$

If force F was applied at point D, then we would have:

$$k_{eq} = k_1\left(\frac{\ell_1}{\ell_3}\right)^2 + k_2\left(\frac{\ell_2}{\ell_3}\right)^2$$

Energy Method

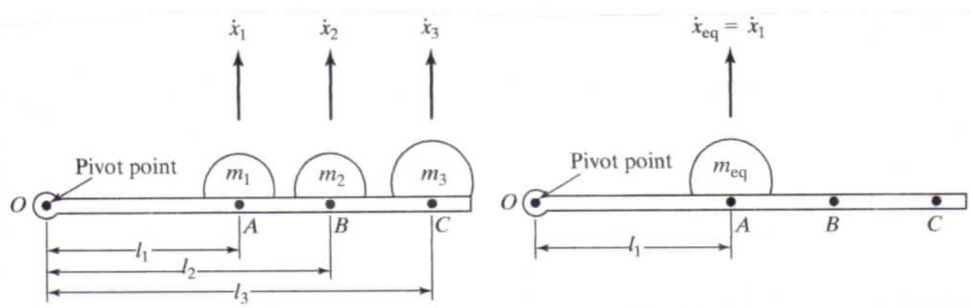
Work done by the applied force F = Strain energy stored in springs k_1 and k_2

$$\frac{1}{2}\, \overset{(k_{eq} \cdot x)}{F}\, x = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$

$$\cancel{\frac{1}{2}}\,(k_{eq} \cdot \ell\theta)\,\ell\theta = \cancel{\frac{1}{2}}\, k_1 (\ell_1\theta)^2 + \cancel{\frac{1}{2}}\, k_2 (\ell_2\theta)^2$$

* Mass or Inertia Elements

- Mass or Inertia element is assumed to be a rigid body.
- It can gain or lose kinetic energy when velocity changes.
- From Newton's second law of motion: $\underbrace{\vec{F}}_{\text{force applied to the mass}} = \underbrace{m}_{\text{mass}}\underbrace{\vec{a}}_{\text{acceleration}}$
- Work done on a mass is stored in the form of the mass's kinetic energy.
- Combination of Masses
  - Masses that are in combination can be replaced by a 'single equivalent mass'.
  - Masses can be 'translational' and/or 'rotational'.
  - Translational masses connected by a rigid bar:



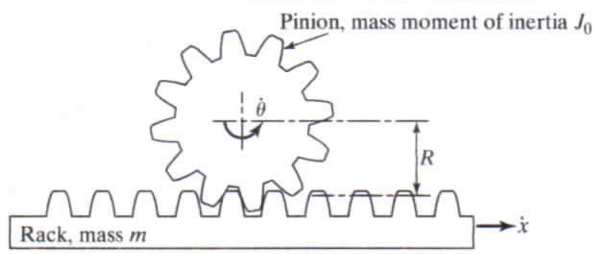
In order to find the equivalent mass of the system, we can write

$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1$ and $\dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1$ $\rightarrow$ assume: $\dot{x}_{eq} = \dot{x}_1$ $\rightarrow$

$$\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}m_2\dot{x}_2^2+\frac{1}{2}m_3\dot{x}_3^2=\frac{1}{2}m_{eq}\dot{x}_{eq}^2 \longrightarrow \text{will result in:}$$

$$m_{eq}=m_1+\left(\frac{\ell_2}{\ell_1}\right)^2 m_2+\left(\frac{\ell_3}{\ell_1}\right)^2 m_3$$

- Translational and rotational masses coupled together:



① In order to find the equivalent translational mass of the system, we can write

kinetic energy $T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}J_0\dot{\theta}^2$

equivalent kinetic energy $T_{eq}=\frac{1}{2}m_{eq}\dot{x}_{eq}^2$

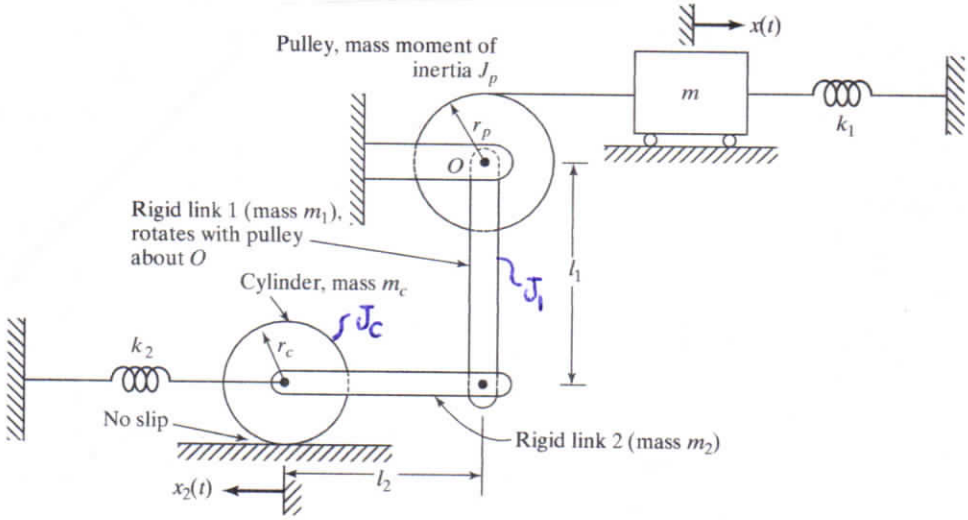
If $\dot{x}_{eq}=\dot{x}$ and $\dot{\theta}=\dot{x}/R \rightarrow T=T_{eq}$ gives $\rightarrow$ $m_{eq}=m+\frac{J_0}{R^2}$

② In order to find the equivalent rotational mass of the system, we can write

$\dot{\theta}_{eq}=\dot{\theta}$ and $\dot{x}=R\dot{\theta} \rightarrow \frac{1}{2}J_{eq}\dot{\theta}^2=\frac{1}{2}m(R\dot{\theta})^2+\frac{1}{2}J_0\dot{\theta}^2 \rightarrow$

$$J_{eq}=J_0+mR^2$$

– Example: Find the equivalent mass of the system shown below, where the rigid link 1 is attached to the pulley and rotates with it.



Solution: Note!
- $m \rightarrow$ translation
- pulley $\rightarrow$ rotation
- Link 1 $\rightarrow$ rotation
- Link 2 $\rightarrow$ translation
- cylinder $\rightarrow$ rotation and translation

Assumption: small displacements

When 'm' moves 'x' $\rightarrow$ pulley and rigid Link 1 rotate
$\theta_p = \theta_1 = x/r_p \rightarrow$ rigid Link 2 displaces $x_2 = l_1\theta_p = xl_1/r_p$
$\rightarrow$ Cylinder rotates $\theta_c = x_2/r_c = xl_1/r_pr_c$

kinetic energy of the system

$$T = \underbrace{\frac{1}{2}m\dot{x}^2}_{m} + \underbrace{\frac{1}{2}J_p\dot{\theta}_p^2}_{\text{pulley}} + \underbrace{\frac{1}{2}J_1\dot{\theta}_1^2}_{\text{Link 1}} + \underbrace{\frac{1}{2}m_2\dot{x}_2^2}_{\text{Link 2}} + \underbrace{\frac{1}{2}J_c\dot{\theta}_c^2 + \frac{1}{2}m_c\dot{x}_2^2}_{\text{cylinder}}$$

kinetic energy of the equivalent system

$$T_{eq} = \frac{1}{2}m_{eq}\dot{x}^2$$

Note that $J_c = m_c r_c^2/2$ and $J_1 = m_1 \ell_1^2/3$

So, $T = T_{eq}$ results in

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_P \left(\frac{\dot{x}}{r_P}\right)^2 + \frac{1}{2}\left(\frac{m_1 \ell_1^2}{3}\right)\left(\frac{\dot{x}}{r_P}\right)^2 + \frac{1}{2} m_2 \left(\frac{\dot{x}\ell_1}{r_P}\right)^2 +$$

$$\frac{1}{2}\left(\frac{m_c r_c^2}{2}\right)\left(\frac{\dot{x}\ell_1}{r_P r_c}\right)^2 + \frac{1}{2} m_c \left(\frac{\dot{x}\ell_1}{r_P}\right)^2 = \frac{1}{2} m_{eq} \dot{x}^2 \longrightarrow$$

$$m_{eq} = m + \frac{J_P}{r_P^2} + \frac{1}{3}\frac{m_1 \ell_1^2}{r_P^2} + \frac{m_2 \ell_1^2}{r_P^2} + \frac{1}{2}\frac{m_c \ell_1^2}{r_P^2} + m_c \frac{\ell_1^2}{r_P^2}$$

* Damping Elements

- Damping: Mechanism by which the vibrational energy is gradually converted into heat or sound.

- Reduction in energy can result in decrease in system response, e.g. displacement.

- Consideration of damping is important for accurate prediction of system response.

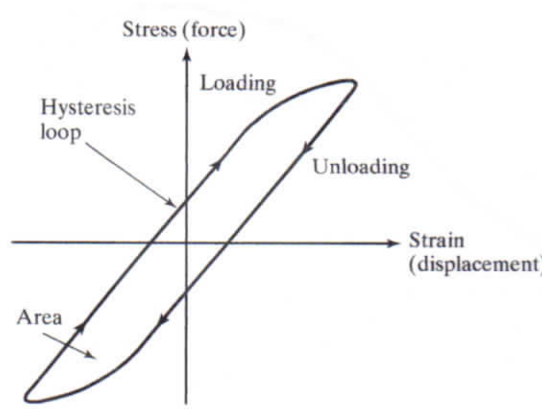
- A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. ()

- Damping is modeled as one or more of the following types:
    - Viscous Damping
    - Coulomb or Dry-Friction Damping
    - Material or Solid or Hysteretic Damping

- Viscous Damping:
  - most commonly used
  - resistance offered by fluid, e.g. air, gas, water, oil, etc., to a moving body
  - damping force proportional to velocity of vibrating body
  - examples: fluid film between sliding surfaces
    fluid flow around a piston in a cylinder
    fluid flow through an orifice
    fluid film around a journal in a bearing

- Coulomb or Dry-Friction Damping:
  - Constant damping force but in opposite direction to the motion direction
  - caused by friction between rubbing surfaces that are dry or not having enough lubrication

- Material or Solid or Hysteretic Damping:
  - caused by the friction between the internal planes, which slip or slide when a body is deformed
  - when a body is vibrated, the stress-strain diagram shows a hysteresis loop



  - Note that loop area denotes the energy lost per unit volume of the body per cycle due to damping

- Linearization of a Nonlinear Damper

  If the force (F)-velocity (v) relationship of a damper is nonlinear $[F=F(v)]$, a linearization process can be used about the operating velocity (v^*), as in the case of a nonlinear spring. The linearization process gives the equivalent damping Constant as

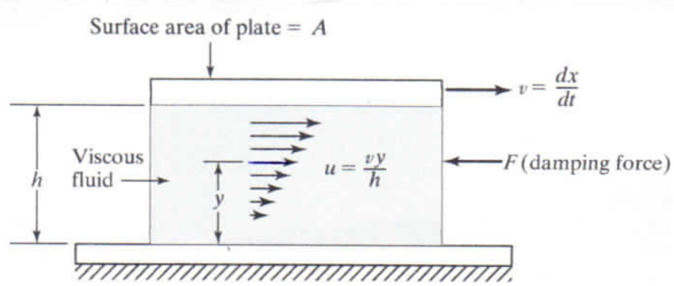
$$c = \left.\frac{dF}{dv}\right|_{v^*}.$$

- Combination of Dampers

  Parallel Dampers: $c_{eq} = c_1 + c_2 + \cdots + c_n$

  Series Dampers: $\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}$

- Example: Consider two parallel plates separated by a distance 'h', with a fluid of viscosity 'μ' between the plates. Derive an expression for the damping Constant when one plate moves with a velocity 'v' relative to the other as shown.



$$\left[u = \frac{vy}{h}\right]$$

Solution: Consider the velocity profile/variation. According to Newton's Law of viscous flow, the shear stress (τ) developed in the fluid layer at a distance 'y' from the fixed plate is given by:

$$\tau = \mu \frac{du}{dy}$$

if velocity gradient $\frac{du}{dy} = \frac{v}{h} \rightarrow \tau = \mu \frac{v}{h}$.

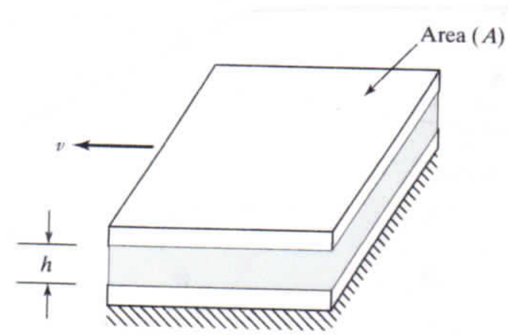
The resisting force (F) developed at the bottom surface of the moving plate is

$F = \tau A = \mu \frac{Av}{h}$ where A = surface area of moving plate

We can also write: $F = cv$

So, $\underline{\underline{c = \frac{\mu A}{h}}}$

– Example: A bearing which can be approximated as two flat plates separated by a thin film of lubricant, is found to offer a resistance of 400 N when SAE 30 oil is used as the lubricant and the relative velocity between the plates is 10 m/s. If the area of the plates (A) is 0.1 m^2, determine the clearance between the plates. (Assume μ = 0.3445 Pa-s)



Solution: Resisting force can be expressed as $F = cv \rightarrow$

$$c = \frac{F}{v} = \frac{400}{10} = 40 \ N\text{-}s/m$$

if the bearing is modeled as a flat-plate-type damper, then $c = \frac{\mu A}{h} \rightarrow h = \frac{\mu A}{c} = \frac{0.3445 \times 0.1}{40} \rightarrow$

$\underline{\underline{h = 0.86125 \ mm}}$

- Example: A precision milling machine is supported on four shock mounts, as shown. The elasticity and damping of each shock mount can be modeled as a spring and a viscous damper, as shown. Find the equivalent spring constant, k_{eq}, and the equivalent damping constant, c_{eq}, of the machine tool support in terms of the spring constants (k_i) and damping constants (c_i) of the mounts.

Solution: Figures are shown on the next page.

All springs are subjected to the same displacement (x).

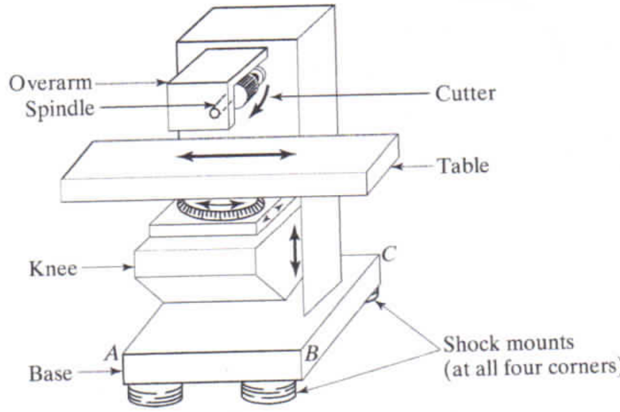
All dampers are subjected to the same relative velocity ($\dot{x}$).

Note: x and $\dot{x}$ are the displacement and velocity of the center of mass, respectively.

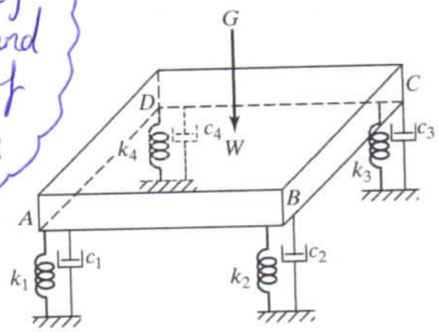
The forces acting on the springs can be expressed as

$$F_{si} = k_i x \qquad i = 1, 2, 3, 4$$

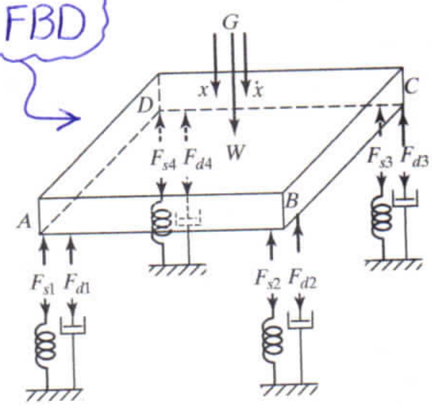
Precision Milling Machine



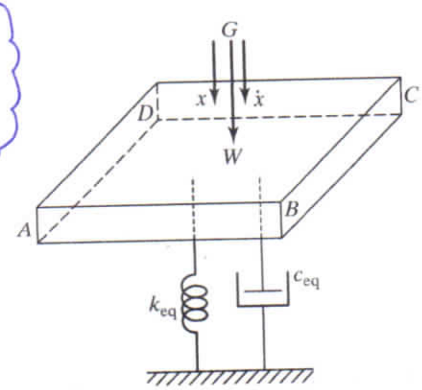
Modeling of elasticity and damping of each shock mount



FBD



System with single spring and damper



The forces acting on the dampers can be expressed as

$$F_{di} = c_i \dot{x} \qquad i = 1, 2, 3, 4$$

The total forces acting on all springs and all dampers are

$$F_s = F_{s1} + F_{s2} + F_{s3} + F_{s4} \quad \text{and} \quad F_d = F_{d1} + F_{d2} + F_{d3} + F_{d4}$$

We can write $F_s + F_d = W$ where W is the total vertical force acting on the milling machine.

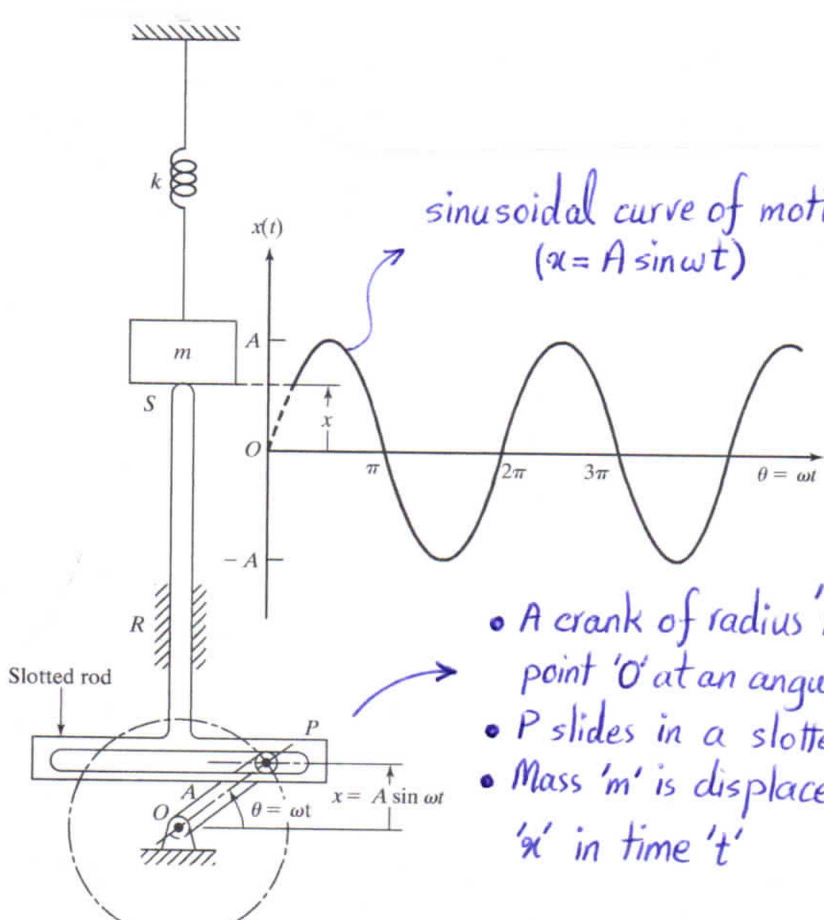
we, also, know that $\begin{cases} F_s = k_{eq}\,x \\ F_d = c_{eq}\,\dot{x} \end{cases}$, So

$\underline{\underline{k_{eq} = k_1 + k_2 + k_3 + k_4 = 4k}}$ and $\underline{\underline{c_{eq} = c_1 + c_2 + c_3 + c_4 = 4c}}$

* Harmonic Motion

– Periodic Motion: This is a motion that is repeated after equal intervals of time. The simplest type of periodic motion is 'harmonic motion'. (See below)

sinusoidal curve of motion
($x = A\sin\omega t$)



- A crank of radius 'A' rotates about point 'O' at an angular velocity 'ω'
- P slides in a slotted rod
- Mass 'm' is displaced by an amount 'x' in time 't'

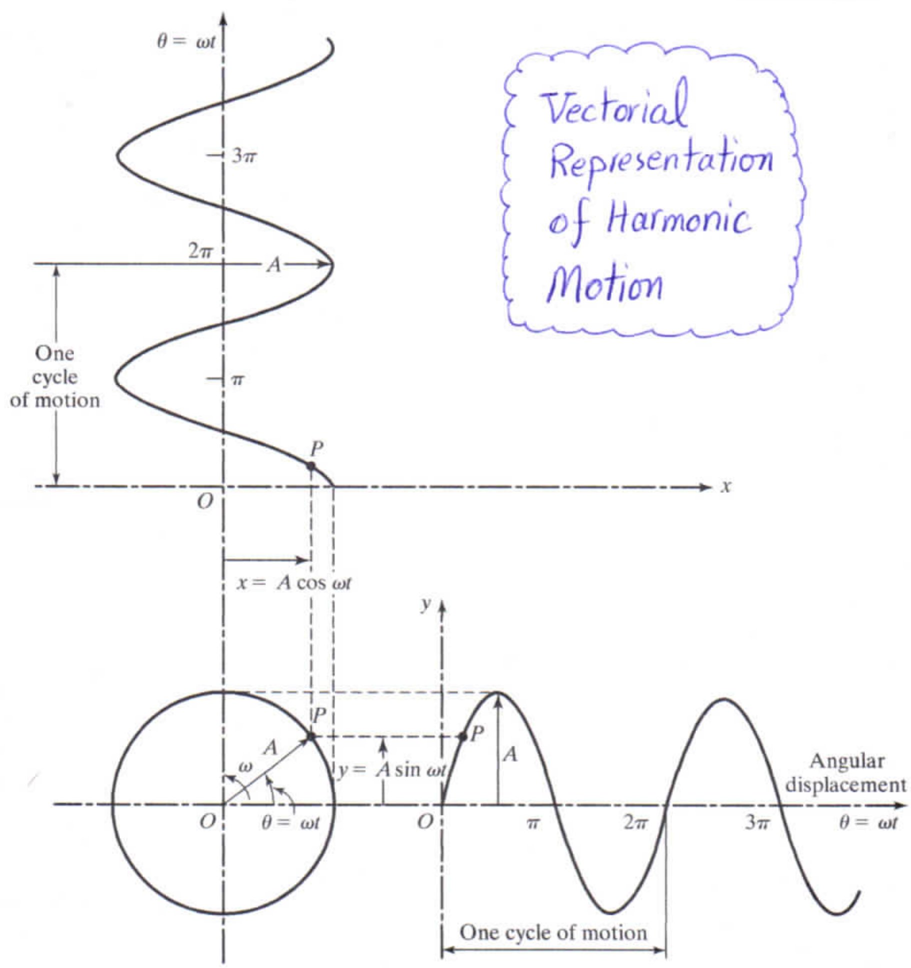
<u>Displacement:</u> $x = A\sin\theta = A\sin\omega t$ → <u>Velocity:</u> $\frac{dx}{dt} = A\omega\cos\omega t$
→ <u>Acceleration:</u> $\frac{d^2x}{dt^2} = -A\omega^2\sin\omega t = -\omega^2 x$

– Harmonic motion can be represented by means of a vector $\vec{OP}$ of magnitude 'A' rotating at a constant angular velocity 'ω'. In the figure below, the projection of the tip of the vector $\vec{OP}$ on the vertical axis is given by

$y = A \sin\omega t$

and its projection on the horizontal axis by

$x = A \cos\omega t.$

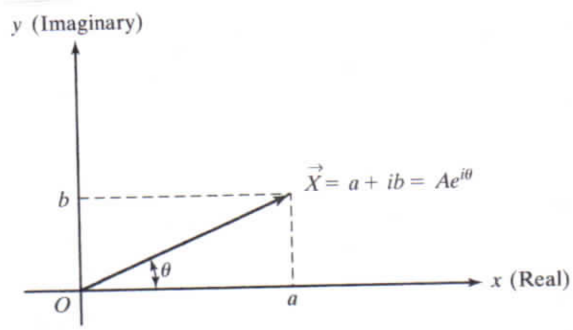


Vectorial Representation of Harmonic Motion

- Complex Number Representation of Harmonic Motion

Any vector $\vec{X}$ in x-y plane can be represented as a complex number

$$\vec{X} = a + ib \quad \text{where} \begin{cases} a = \text{real part} \\ b = \text{imaginary part} \end{cases} \quad \text{and } i^2 = -1$$



$\vec{X}$ can be also represented as $\vec{X} = A\cos\theta + iA\sin\theta$

with $A = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$.

We know that $i^2 = -1,\ i^3 = -i,\ i^4 = 1, \ldots$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$i\sin\theta = i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots$$

As a result $$(\cos\theta + i\sin\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = e^{i\theta}$$

$$(\cos\theta - i\sin\theta) = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \cdots = e^{-i\theta}$$

So, $\vec{X}$ can be expressed as

$$\vec{X} = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$

since $\theta = \omega t \rightarrow \vec{X} = Ae^{i\omega t}$ where ω = circular frequency of rotation of $\vec{X}$ (rad/sec)

$$\frac{d\vec{X}}{dt} = iA\omega e^{i\omega t} = i\omega\vec{X}$$

$$\frac{d^2\vec{X}}{dt^2} = -\omega^2 A e^{i\omega t} = -\omega^2\vec{X}$$

So, the displacement, velocity, and acceleration can be expressed as

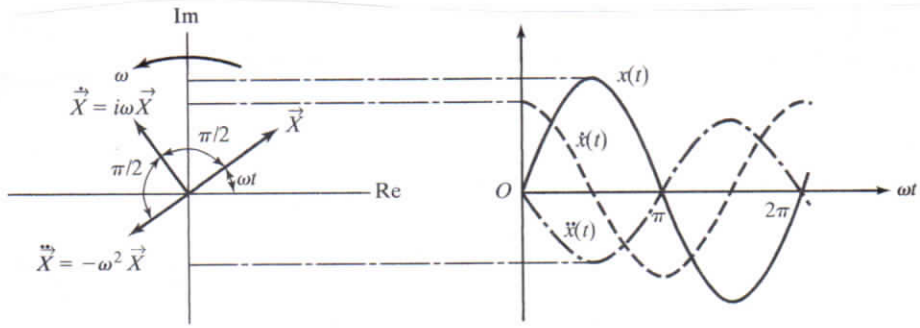
$$\begin{cases} \text{displacement} = Re\left[Ae^{i\omega t}\right] = A\cos\omega t \\ \text{velocity} = Re\left[iA\omega e^{i\omega t}\right] = -A\omega\sin\omega t = A\omega\cos(\omega t + 90^\circ) \\ \text{acceleration} = Re\left[-\omega^2 A e^{i\omega t}\right] = -A\omega^2\cos\omega t = A\omega^2\cos(\omega t + 180^\circ) \end{cases}$$

It is seen that the acceleration vector leads the velocity vector by 90°, and the latter leads the displacement vector by 90°.

Note that if the harmonic displacement is originally given as $x(t) = A\sin\omega t$, then we have

$$\begin{cases} \text{displacement} = Im\left[Ae^{i\omega t}\right] = A\sin\omega t \\ \text{velocity} = Im\left[iA\omega e^{i\omega t}\right] = A\omega\sin(\omega t + 90^\circ) \\ \text{acceleration} = Im\left[-\omega^2 A e^{i\omega t}\right] = A\omega^2\sin(\omega t + 180^\circ) \end{cases}$$

See the figure on the next page →



Displacement, velocity, and accelerations as rotating vectors.

– Definitions:

- Cycle: The movement of a vibrating body from its undisturbed or equilibrium position to its extreme position in one direction, then to the equilibrium position, then to its extreme position in the other direction, and back to equilibrium position is called a 'cycle' of vibration.

- Amplitude: The maximum displacement of a vibrating body from its equilibrium position is called the 'amplitude' of vibration.

- Period of Oscillation: The time taken to complete one cycle of motion is known as the 'period of oscillation' or 'time period'.

$$\tau = \frac{2\pi}{\underbrace{\omega}_{\text{circular frequency}}}$$

- Frequency of Oscillation: The number of cycles per unit time is called the 'frequency of oscillation' or simply the 'frequency'.

Note: $\omega \rightarrow$ circular frequency
$f \rightarrow$ linear frequency

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi} \quad ; \; f \rightarrow \left(\frac{\text{cycles}}{\text{second}}\right) \text{ or Hz}$$

- Phase Angle: Consider two vibratory motions

$$x_1 = A_1 \sin\omega t$$
$$x_2 = A_2 \sin(\omega t + \phi)$$

with the same frequency or angular velocity. The second motion, x_2, leads the first one, x_1, by an angle 'ϕ', known as the 'phase angle'. This means that the maximum of the second motion, x_2, would occur 'ϕ' radians earlier than that of the first motion, x_1.

- Natural Frequency: If a system, after an initial disturbance, is left to vibrate on its own, the frequency with which it oscillates without external forces is known as its 'natural frequency'.

- Example: Find the sum of the two harmonic motions

$x_1(t) = 10\cos\omega t$ and $x_2(t) = 15\cos(\omega t + 2)$.

Solution: Method 1

We know that $x(t) = \underbrace{A\cos(\omega t + \alpha)}_{\text{expand}} = x_1(t) + x_2(t)$

$A\cos(\omega t + \alpha) = A(\cos\omega t\cos\alpha - \sin\omega t\sin\alpha) =$

$10\cos\omega t + 15\underbrace{\cos(\omega t + 2)}_{\text{expand}} =$

$10\cos\omega t + 15(\cos\omega t\cos 2 - \sin\omega t\sin 2)$

$\Downarrow$

$\cos\omega t\,(A\cos\alpha) - \sin\omega t\,(A\sin\alpha) =$

$\cos\omega t(10 + 15\cos 2) - \sin\omega t(15\sin 2)$

So, $A\cos\alpha = 10 + 15\cos 2$

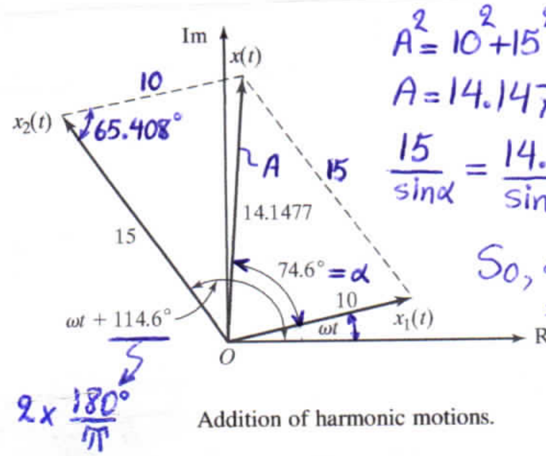
$A\sin\alpha = 15\sin 2 \quad \rightarrow A = \sqrt{(10 + 15\cos 2)^2 + (15\sin 2)^2}$

$\rightarrow A = 14.1477$

Also, $\alpha = \tan^{-1}\left(\frac{15\sin 2}{10 + 15\cos 2}\right) \rightarrow \theta = 74.5963^\circ$

Consequently: $x(t) = 14.1477\cos(\omega t + 74.5963^\circ)$

Method 2 For an arbitrary value of ωt, the harmonic motions $x_1(t)$ and $x_2(t)$ can be denoted graphically as shown. By adding them vectorially, the resultant vector $x(t)$ can be found.



Addition of harmonic motions.

$$A^2 = 10^2 + 15^2 - 2\times 10\times 15\times \cos 65.408^\circ$$
$$A = 14.1477$$
$$\frac{15}{\sin\alpha} = \frac{14.1477}{\sin 65.408^\circ} \rightarrow \alpha = 74.5963^\circ$$

So, $x(t) = 14.1477\cos(\omega t + 74.5963^\circ)$

* Harmonic Analysis

In many cases, the vibrations of systems are periodic. Any periodic function of time can be represented by Fourier series as an infinite sum of sine and cosine terms.

If $x(t)$ is a periodic function with period 'τ', its Fourier series representation is given by

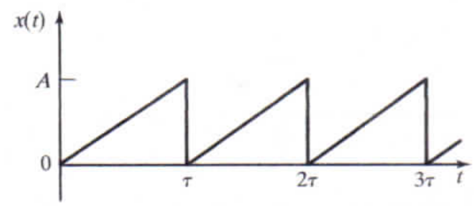
$$\boxed{\begin{aligned} x(t) &= \frac{a_0}{2} + a_1\cos\omega t + a_2\cos 2\omega t + \cdots + b_1\sin\omega t + b_2\sin 2\omega t + \cdots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t) \end{aligned}}$$

where: $\omega = \frac{2\pi}{\tau} \rightarrow$ fundamental frequency ; $a_0, a_1, \cdots, b_1, b_2, \cdots \rightarrow$ Constant Coefficients

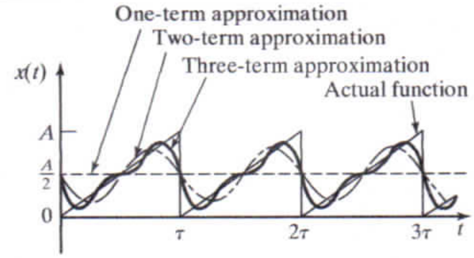
$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt$$

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt$$

$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt$$



Periodic Function



Representation Using Harmonic Functions

Fourier series can also be represented by the sum of sine terms only or cosine terms only. For example, the series using cosine terms only can be expressed as

$$\boxed{x(t) = d_0 + d_1 \cos(\omega t - \phi_1) + d_2 \cos(2\omega t - \phi_2) + \cdots}$$

where: $d_0 = \frac{a_0}{2}$; $d_n = \sqrt{a_n^2 + b_n^2}$; $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

Complex Fourier Series: The Fourier series can also be represented in terms of complex numbers. $x(t)$ can be expressed as

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t}}$$

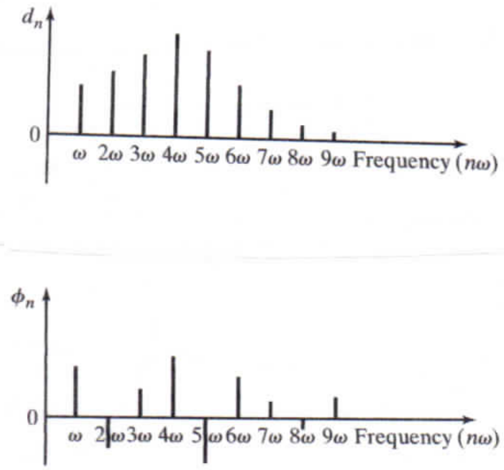
where: $c_n = \frac{a_n - ib_n}{2} = \frac{1}{\tau}\int_0^{\tau} x(t)\left[\cos n\omega t - i \sin n\omega t\right] dt$

$$= \frac{1}{\tau}\int_0^{\tau} x(t)\, e^{-in\omega t}\, dt$$

Note: Remember $e^{in\omega t} = \cos n\omega t + i \sin n\omega t$
$e^{-in\omega t} = \cos n\omega t - i \sin n\omega t$ → Euler's Formula!

Frequency Spectrum: Consider

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\underbrace{a_n \cos n\omega t + b_n \sin n\omega t}_{\text{harmonics of order 'n' of } x(t) \text{ with period } \frac{\tau}{n}} \right)$$

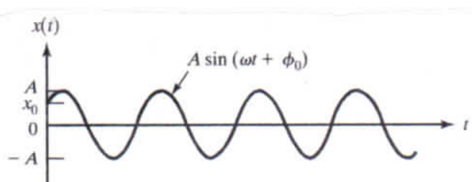


Frequency spectrum of a typical periodic function of time

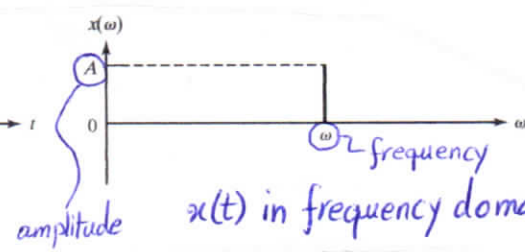
'$a_n \cos n\omega t$' and '$b_n \sin n\omega t$' can be plotted as vertical lines on a diagram of amplitude (a_n and b_n or d_n and ϕ_n) versus frequency ($n\omega$), called the 'frequency spectrum' or 'spectral diagram.'

Time- and Frequency-Domain Representations

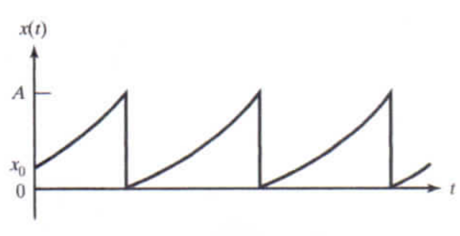
Fourier series expansion permits the description of any periodic function using either a time-domain or a frequency-domain representation.



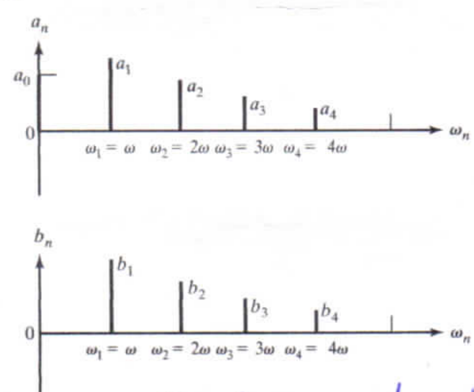
$x(t)$ in time domain



$x(t)$ in frequency domain



$x(t)$ in time domain



$x(t)$ in frequency domain

Even and Odd Functions

An even function satisfies the relation $x(-t) = x(t)$. In this case, the Fourier series expansion of $x(t)$ contains only cosine terms

$$\boxed{x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t}$$

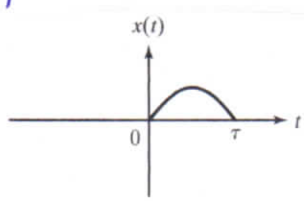
where a_0 and a_n were previously introduced.

An odd function satisfies the relation $x(-t) = -x(t)$. In this case, the Fourier series expansion of $x(t)$ contains only sine terms

$$\boxed{x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t}$$

where b_n was previously introduced.

Half-Range Expansions



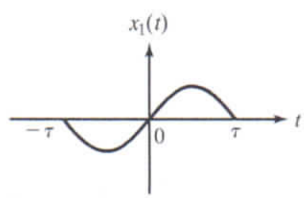
Original function

→ Sometimes $x(t)$ is defined in the interval 0 to τ with no periodicity. We can extend the function arbitrarily to include the interval $-\tau$ to 0.

odd

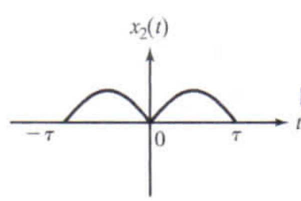
even

Note: Fourier series expansions of $x_1(t)$ and $x_2(t)$ are known as 'half-range' expansions!



Extension as an odd function

→ result: an odd function (Fourier series expansion yields only sine terms)

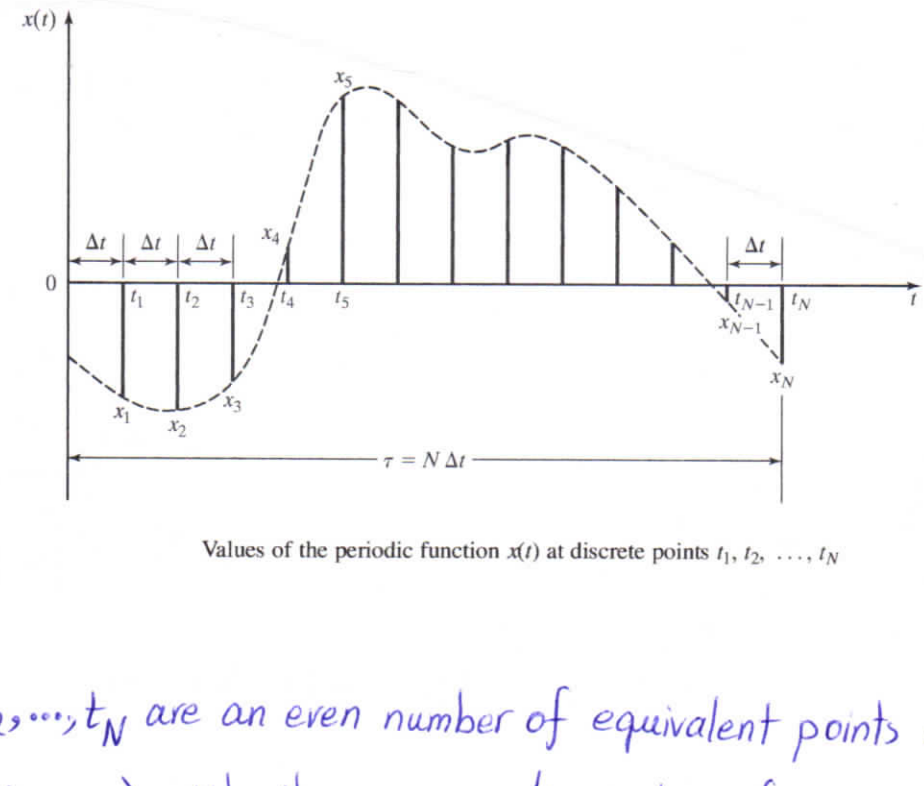


Extension as an even function

→ result: an even function (Fourier series expansion involves only cosine terms)

Numerical Computation of Coefficients

For simple forms of the function $x(t)$, the integrals of a_0, a_n, and b_n can be evaluated easily. However, the integration becomes involved if $x(t)$ does not have a simple form. In some practical applications, the function $x(t)$ is not available in the form of a mathematical expression; only the values of $x(t)$ at a number of points $t_1, t_2, \dots, t_N$ are available, as shown. In these cases, the coefficients can be evaluated by using a numerical integration procedure like the 'trapezoidal' or 'Simpson's' rule.

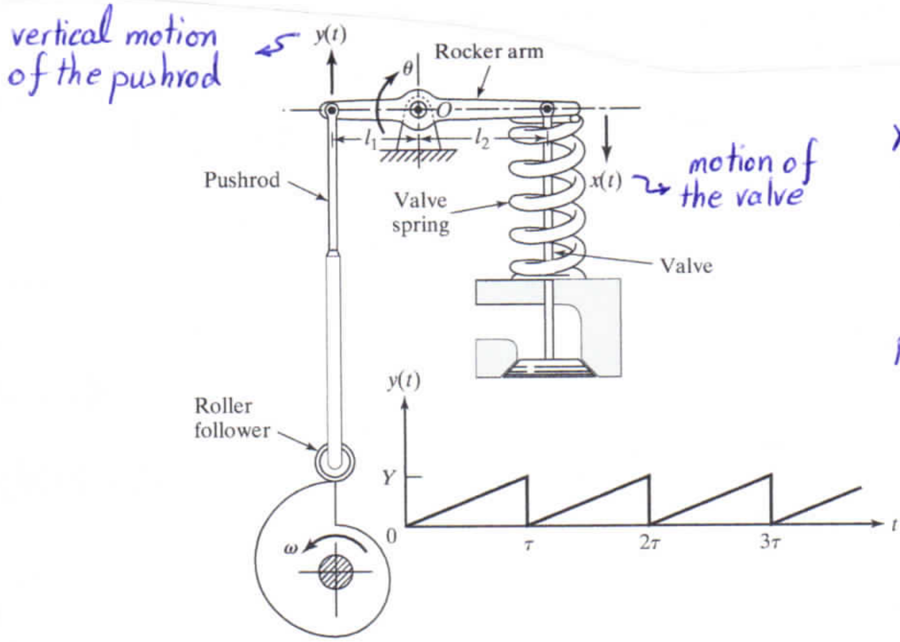


Values of the periodic function $x(t)$ at discrete points $t_1, t_2, \dots, t_N$

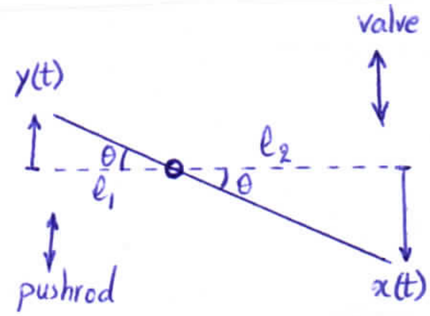
If $t_1, t_2, \dots, t_N$ are an even number of equivalent points over the period τ (N = even) with the corresponding values of $x(t)$ given by $x_1 = x(t_1), x_2 = x(t_2), \dots, x_N = x(t_N)$, respectively, then the application of the trapezoidal rule gives the coefficients as: $(\tau = N\Delta t)$

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} x_i \;;\quad a_n = \frac{2}{N}\sum_{i=1}^{N} x_i \cos\frac{2n\pi t_i}{\tau} \;;\quad b_n = \frac{2}{N}\sum_{i=1}^{N} x_i \sin\frac{2n\pi t_i}{\tau}$$

– Example: Determine the Fourier series expansion of the motion of the valve in the system shown.



Cam-follower system.



Solution: We can write $\tan\theta = \dfrac{y(t)}{\ell_1} = \dfrac{x(t)}{\ell_2} \rightarrow x(t) = \dfrac{\ell_2}{\ell_1}\, y(t)$

where $y(t) = Y\dfrac{t}{\tau} \quad 0 \leq t \leq \tau.$

By defining $A = \dfrac{Y\ell_2}{\ell_1} \rightarrow x(t) = A\dfrac{t}{\tau} \quad 0 \leq t \leq \tau.$

$$
\begin{cases}
a_0 = \dfrac{\omega}{\pi}\displaystyle\int_0^{2\pi/\omega} A\frac{t}{\tau}\,dt = A \quad (\text{verify!}) \\[2ex]
a_n = \dfrac{\omega}{\pi}\displaystyle\int_0^{2\pi/\omega} A\frac{t}{\tau}\cos n\omega t\,dt = 0 \quad n = 1, 2, \dots \quad (\text{verify!}) \\[2ex]
b_n = \dfrac{\omega}{\pi}\displaystyle\int_0^{2\pi/\omega} A\frac{t}{\tau}\sin n\omega t\,dt = -\dfrac{A}{n\pi} \quad n = 1, 2, \dots \quad (\text{verify!})
\end{cases}
$$

$$
\begin{aligned}
\text{So, } x(t) &= \frac{A}{2} - \frac{A}{\pi}\sin\omega t - \frac{A}{2\pi}\sin 2\omega t - \dots \\
&= \frac{A}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \dots\right\}\right]
\end{aligned}
$$

- Example: The pressure fluctuations of water in a pipe, measured at 0.01-second intervals, are given in the following Table. These fluctuations are repetitive in nature. Make a harmonic analysis of the pressure fluctuations and determine the first three harmonics of the Fourier series expansion.

TABLE

Time Station, i	Time (sec), t_i	Pressure (kN/m^2), p_i
0	0	0
1	0.01	20
2	0.02	34
3	0.03	42
4	0.04	49
5	0.05	53
6	0.06	70
7	0.07	60
8	0.08	36
9	0.09	22
10	0.10	16
11	0.11	7
12	0.12	0

Solution: $\tau = 0.12 \rightarrow \omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.12} = 52.36 \text{ rad/sec}$; $N = 12$

$$\begin{cases} a_0 = \frac{2}{12} \sum_{i=1}^{12} P_i = 68166.7 \\ a_n = \frac{2}{12} \sum_{i=1}^{12} P_i \cos \frac{2n\pi t_i}{0.12} \\ b_n = \frac{2}{12} \sum_{i=1}^{12} P_i \sin \frac{2n\pi t_i}{0.12} \end{cases} \Rightarrow$$

See the Table in the next page!

TABLE

i	t_i	p_i	$n = 1$: $p_i \cos \frac{2\pi t_i}{0.12}$	$n = 1$: $p_i \sin \frac{2\pi t_i}{0.12}$	$n = 2$: $p_i \cos \frac{4\pi t_i}{0.12}$	$n = 2$: $p_i \sin \frac{4\pi t_i}{0.12}$	$n = 3$: $p_i \cos \frac{6\pi t_i}{0.12}$	$n = 3$: $p_i \sin \frac{6\pi t_i}{0.12}$
1	0.01	20000	17320	10000	10000	17320	0	20000
2	0.02	34000	17000	29444	−17000	29444	−34000	0
3	0.03	42000	0	42000	−42000	0	0	−42000
4	0.04	49000	−24500	42434	−24500	−42434	49000	0
5	0.05	53000	−45898	26500	26500	−45898	0	53000
6	0.06	70000	−70000	0	70000	0	−70000	0
7	0.07	60000	−51960	−30000	30000	51960	0	−60000
8	0.08	36000	−18000	−31176	−18000	31176	36000	0
9	0.09	22000	0	−22000	−22000	0	0	22000
10	0.10	16000	8000	−13856	−8000	−13856	−16000	0
11	0.11	7000	6062	−3500	3500	−6062	0	−7000
12	0.12	0	0	0	0	0	0	0
$\sum_{i=1}^{12}()$		409000	−161976	49846	8500	21650	−35000	−14000
		a_0	a_1	b_1	a_2	b_2	a_3	b_3
$\frac{1}{6}\sum_{i=1}^{12}()$		68166.7	−26996.0	8307.7	1416.7	3608.3	−5833.3	−2333.3

Finally, the Fourier series expansion of the pressure fluctuations $p(t)$ can be obtained:

$$
\begin{aligned}
p(t) &= 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t \\
&\quad + 1416.7 \cos 104.72t + 3608.3 \sin 104.72t - 5833.3 \cos 157.08t \\
&\quad - 2333.3 \sin 157.08t + \cdots \quad \text{N/m}^2
\end{aligned}
$$

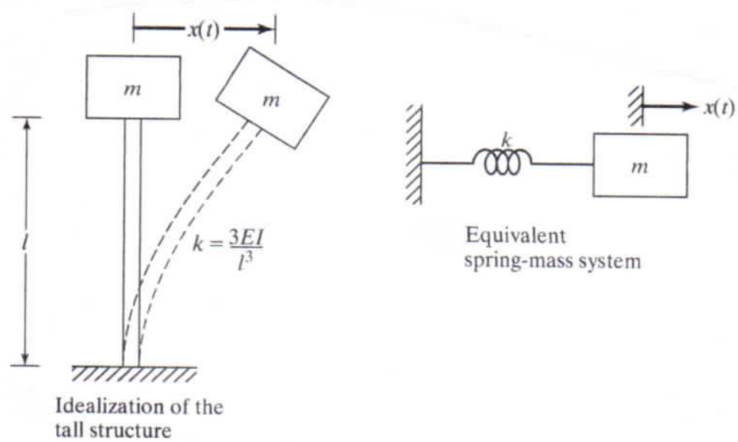
Chapter 2: Free Vibration of Single-Degree-of-Freedom Systems

* Introduction

Q: When does a system undergo 'free' vibration?

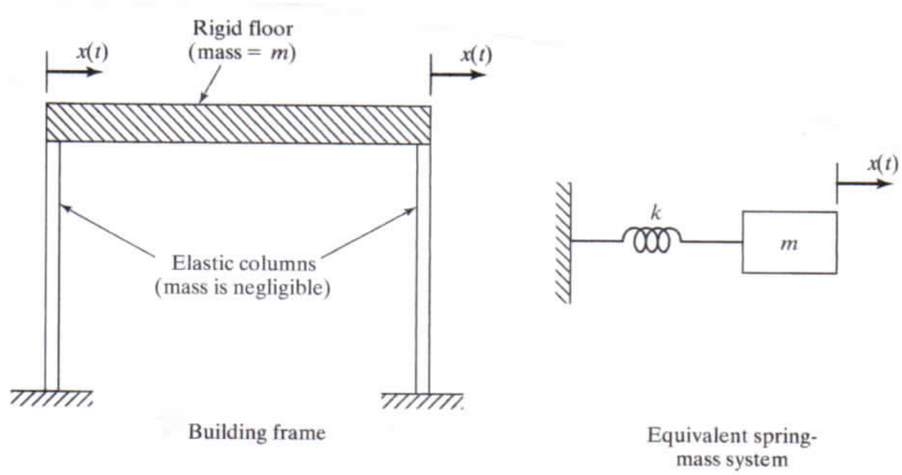
A: When it oscillates only under an initial disturbance with no external forces acting afterward.

For a simple analysis, systems/machines/structures can be idealized as a single-degree-of-freedom (SDOF) spring-mass system. Two examples are shown below.



Note: Both are undamped systems!

Modeling of tall structure as spring-mass system.



Idealization of a building frame.

* Free Vibration of an Undamped Translational System

① Derivation of equation of motion using 'Newton's second law of motion':

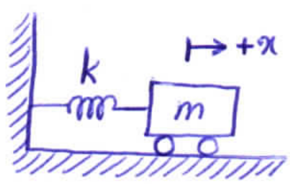
Rate of change of momentum of a mass is equal to the force acting on it. $\rightarrow \vec{F}(t) = \frac{d}{dt}\left(m\frac{d\vec{x}(t)}{dt}\right) \rightarrow$ if $m =$ constant $\rightarrow$

$$\boxed{\vec{F}(t) = m\frac{d^2\vec{x}(t)}{dt^2} = m\ddot{\vec{x}}(t)}$$ where $\ddot{\vec{x}}(t) = \frac{d^2\vec{x}(t)}{dt^2} =$ accel. of the mass

$\rightarrow$ In other words: Resultant force on the mass = mass x acceleration

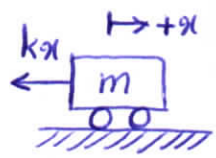
For a mass undergoing rotational motion: $\boxed{\underbrace{\vec{M}(t)}_{\text{resultant moment acting on the body}} = J\underbrace{\ddot{\vec{\theta}}}_{\text{angular acceleration}}}$

Consider,



undamped SDOF system

$\xrightarrow[\text{+x from its static equilibrium position}]{\text{If mass is displaced}}$



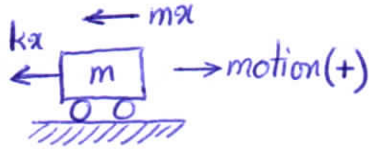
$$F(t) = m\ddot{x} \longrightarrow -kx = m\ddot{x} \longrightarrow \boxed{m\ddot{x} + kx = 0}$$

② Derivation of equation of motion using 'D'Alembert's Principle'

$\vec{F}(t) = m\ddot{\vec{x}} \xrightarrow[\text{as}]{\text{can be rewritten}} \vec{F}(t) - m\ddot{\vec{x}} = 0$

$\vec{M}(t) = J\ddot{\vec{\theta}} \xrightarrow[\text{as}]{\text{can be rewritten}} \vec{M}(t) - J\ddot{\vec{\theta}} = 0$

$\rightarrow$ These can be considered as equilibrium equations provided that $-m\ddot{\vec{x}}$ and $-J\ddot{\vec{\theta}}$ are treated as a force or moment. This fictitious force/moment is known as 'inertia' force/moment.

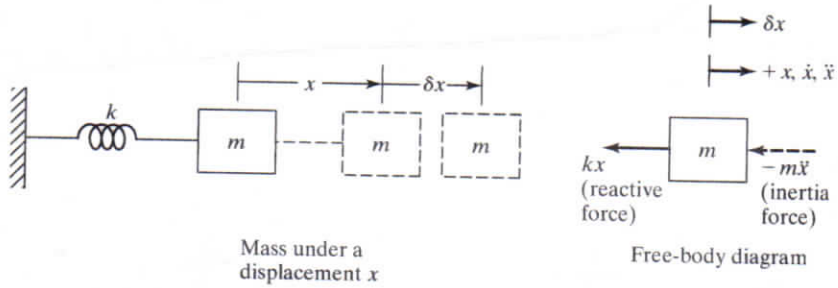
So, consider 

$\Rightarrow -kx - m\ddot{x} = 0 \rightarrow \boxed{m\ddot{x} + kx = 0}$

③ Derivation of equation of motion using 'principle of virtual displacements'

> If a system that is in equilibrium under the action of a set of forces is subjected to a virtual displacement, then the total virtual work done by the forces will be zero!

Consider the following mass under virtual displacement:



Mass under virtual displacement.

δx ↓ virtual displacement

negative sign! (why?)

Virtual work done by $\begin{cases} \text{the spring force: } \delta W_s = -(kx)\,\delta x \\ \text{the inertia force: } \delta W_i = -(m\ddot{x})\,\delta x \end{cases}$ →

Total virtual work: $-m\ddot{x}\,\delta x - kx\,\delta x = 0 \xrightarrow{\delta x \neq 0} \boxed{m\ddot{x} + kx = 0}$

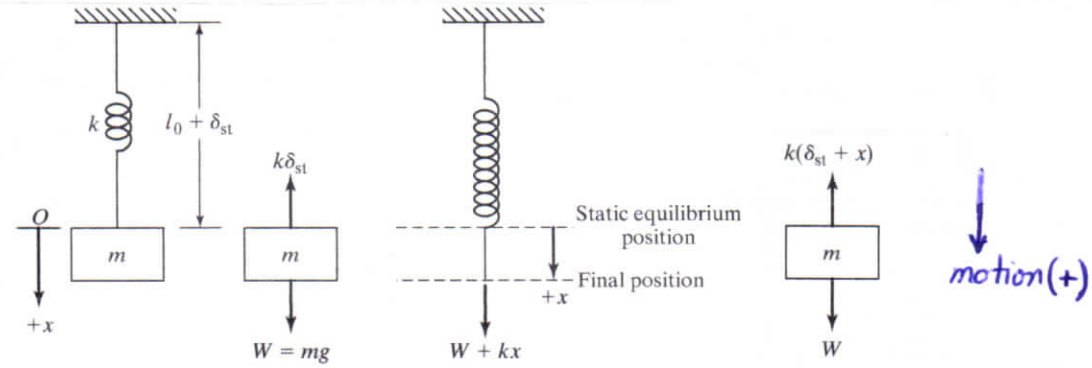
④ Derivation of equation of motion using 'principle of conservation of energy'

If no energy is lost, then $\underbrace{T}_{\text{kinetic energy}} + \underbrace{U}_{\text{potential energy}} = \text{constant} \rightarrow \frac{d}{dt}(T+U) = 0$

We know that $\begin{cases} T = \frac{1}{2} m\dot{x}^2 \\ U = \frac{1}{2} kx^2 \end{cases} \rightarrow \frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2\right) = 0 \rightarrow \boxed{m\ddot{x} + kx = 0}$

– Equation of motion of a spring-mass system in vertical position:

Consider the following system:



Static Equil. Position

Upward spring force balances downward gravitational force

δ_{st} = static deflection or elongation due to W

$$W = mg = k\delta_{st}$$

If mass is deflected $+x$ from static equil. position

• Newton's second law

$$-k(\delta_{st}+x)+W=m\ddot{x}$$

$$-\cancel{k\delta_{st}}-kx+\cancel{W}=m\ddot{x}$$

$$m\ddot{x}+kx=0$$

• Principle of conservation of energy

$$T = \text{kinetic} = \tfrac{1}{2}m\dot{x}^2$$

$$U = \text{potential (spring)} = mgx + \tfrac{1}{2}kx^2$$

$$U = \text{potential (mass)} = -mgx$$

$$\frac{d}{dt}(T+U)=0$$

$$\frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2+\cancel{mgx}+\tfrac{1}{2}kx^2-\cancel{mgx}\right)=0$$

$$m\ddot{x}+kx=0$$

– Solution of $m\ddot{x}+kx=0$

Assume $\underset{\text{constant}}{x(t)=Ce^{st}} \xrightarrow[\text{EOM}]{\text{substitute in}} C(ms^2+k)=0 \xrightarrow[C\neq 0]{\text{since}}$

$\underbrace{ms^2+k=0}_{\text{auxiliary or characteristic equation}} \longrightarrow \underbrace{s=\pm\left(-\frac{k}{m}\right)^{1/2}=\pm i\omega_n}_{\text{eigenvalues or characteristic values}} \longrightarrow$ Note: $i=\sqrt{-1} \longrightarrow$

$\underbrace{\omega_n=\sqrt{\frac{k}{m}}}_{\text{system's natural frequency of vibration}}$

General solution of EOM: $x(t)=C_1e^{i\omega_n t}+C_2e^{-i\omega_n t}$ where C_1 and C_2 = Consts.

$\xrightarrow[e^{\pm i\alpha t}=\cos\alpha t\pm i\sin\alpha t]{\text{using}}$ $\boxed{x(t)=A_1\cos\omega_n t+A_2\sin\omega_n t}$ where A_1 and A_2 = Consts.

Initial Conditions (I.C.-s)

$x(t=0)=x_0$

$\dot{x}(t=0)=\dot{x}_0$

$\xrightarrow{\text{apply}} \begin{cases} x(t=0)=A_1=x_0 \longrightarrow A_1=x_0 \\ \dot{x}(t=0)=\omega_n A_2=\dot{x}_0 \longrightarrow A_2=\dfrac{\dot{x}_0}{\omega_n} \end{cases}$

$\longrightarrow$ Finally, $\boxed{x(t)=x_0\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t}$

– Harmonic Motion

$x(t)=A_1\cos\omega_n t+A_2\sin\omega_n t$ can be expressed in two different forms as following:

① If $\begin{cases} A_1 = A\cos\phi \\ A_2 = A\sin\phi \end{cases}$; A and ϕ = consts.

then $\boxed{x(t) = A\cos(\omega_n t - \phi)}$

where

$$A = \text{amplitude} = \sqrt{A_1^2 + A_2^2} = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

$$\phi = \text{phase angle} = \tan^{-1}\left(\frac{A_2}{A_1}\right) = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right)$$

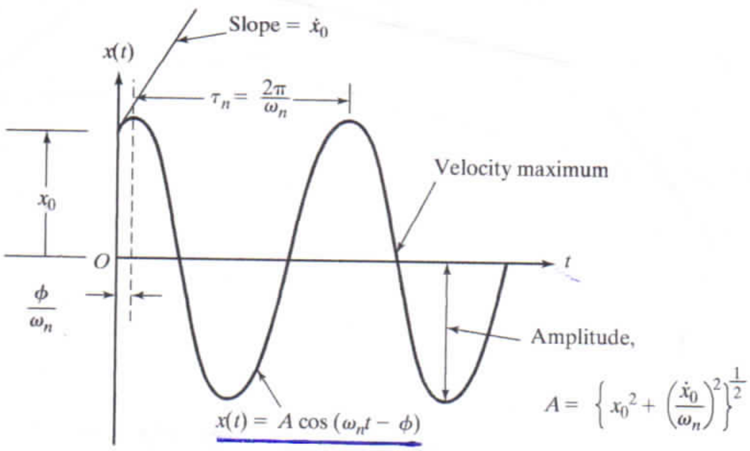
② If $\begin{cases} A_1 = A_0\sin\phi_0 \\ A_2 = A_0\cos\phi_0 \end{cases}$; A_0 and ϕ_0 = consts.

then $\boxed{x(t) = A_0\sin(\omega_n t + \phi_0)}$

where

$$A_0 = A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)$$

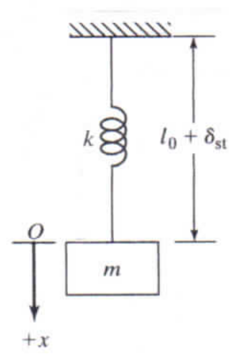


Graphical representation of the motion of a harmonic oscillator

– Some Points

① The circular natural frequency of a spring-mass system in a vertical position can be expressed as:

$$\omega_n = \sqrt{\frac{k}{m}}$$



It can be also shown that: $k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}}$

So, $\omega_n = \sqrt{\frac{g}{\delta_{st}}}$ $\rightarrow$ f = natural frequency $= \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$

$\rightarrow$ τ_n = natural period $= 2\pi\sqrt{\frac{\delta_{st}}{g}}$

② If $x(t) = A\cos(\omega_n t - \phi)$, then

$$\begin{cases} \dot{x}(t) = \text{velocity} = \frac{dx(t)}{dt} = -\omega_n A\sin(\omega_n t - \phi) = \omega_n A\cos\left(\omega_n t - \phi + \frac{\pi}{2}\right) \\ \ddot{x}(t) = \text{accel.} = \frac{d^2x(t)}{dt^2} = -\omega_n^2 A\cos(\omega t - \phi) = \omega_n^2 A\cos(\omega_n t - \phi + \pi) \end{cases}$$

③ Consider $\begin{cases} x(t) = A\cos(\omega_n t - \phi) \rightarrow \frac{x}{A} = \cos(\omega_n t - \phi) \\ \dot{x}(t) = -A\omega_n\sin(\omega_n t - \phi) \rightarrow -\frac{\dot{x}}{A\omega_n} = \sin(\omega_n t - \phi) = -\frac{y}{A} \end{cases}$ $y = \frac{\dot{x}}{\omega_n}$

So, by squaring and adding the two equations, we obtain

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = \cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1 \rightarrow \boxed{\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1}$$

graph

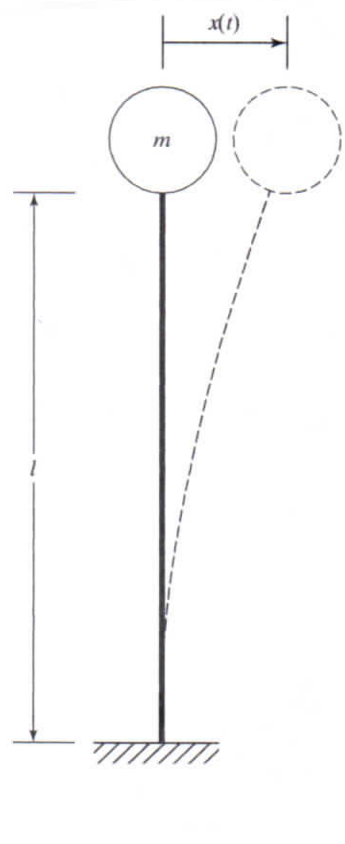
Phase-plane representation of an undamped system

– Example: The column of the water tank shown is 300 ft and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs 6×10^5 lbs when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 4×10^6 psi, determine the following:

(a) the natural frequency and the natural time period of transverse vibration of the water tank. ($\omega_n, \tau_n = ?$)

(b) the vibration response of the water tank due to an initial transverse displacement of 10 in. ($x_0 = 10$ in $\rightarrow x(t) = ?$)

(c) the maximum values of the velocity and acceleration experienced by the water tank. ($\dot{x}_{max}, \ddot{x}_{max} = ?$)



Solution: ⓐ we know that $\left.\begin{array}{l}\delta = \dfrac{P\ell^3}{3EI} \\ P = k\delta\end{array}\right\} \rightarrow k = \dfrac{3EI}{\ell^3}$

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(120^4 - 96^4) = 600.9554 \times 10^4 \text{ in}^4$$

$$k = \frac{3(4\times10^6)(600.9554\times10^4)}{(300\times12)^3} = 1545.6672 \text{ lb/in}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1545.6672}{\underbrace{(6\times10^5)}_{W(\text{lb})}/\underbrace{(386.4)}_{g\,(\text{in/s}^2)}}} = \underline{\underline{0.9977 \text{ rad/sec}}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{0.9977} = \underline{\underline{6.2977 \text{ sec}}}$$

ⓑ I.C.-s: $x_0 = 10$ in and $\dot{x}_0 = 0$

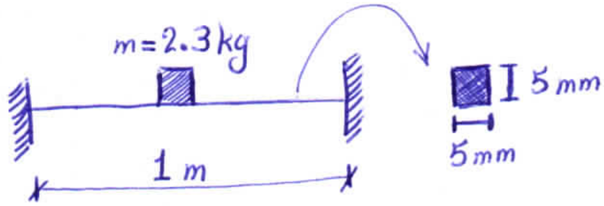
Harmonic response: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

$$\left\{\begin{array}{l} A_0 = 10 \text{ in} \\ \phi_0 = \frac{\pi}{2} \end{array}\right. \text{ (How?)} \rightarrow \boxed{\begin{array}{l} x(t) = 10\sin\left(0.9977t + \frac{\pi}{2}\right) \\ x(t) = 10\cos(0.9977t) \end{array}}$$

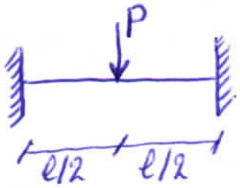
ⓒ $\dot{x}(t) = 10(0.9977)\cos\left(0.9977t + \frac{\pi}{2}\right) \rightarrow \underline{\dot{x}_{max} = 9.977 \text{ in/sec}}$

$\ddot{x}(t) = -10(0.9977)^2 \sin\left(0.9977t + \frac{\pi}{2}\right) \rightarrow \underline{\ddot{x}_{max} = 9.9540 \text{ in/sec}^2}$

- Example: A fixed-fixed beam of square cross section 5mm x 5mm and length 1 m, carrying a mass of 2.3 kg at the middle, is found to have a natural frequency of transverse vibration of 30 rad/s. Determine the Young's modulus of elasticity of the beam.



Solution: Consider



$$\delta_{max} = \frac{Pl^3}{192EI} \rightarrow k = \frac{P}{\delta_{max}} = \frac{192EI}{l^3}$$

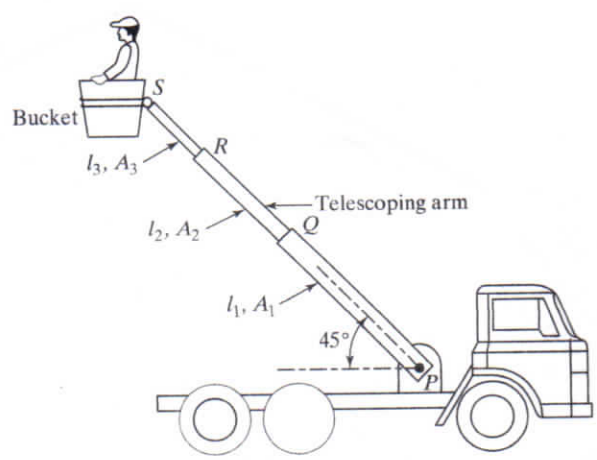
On the other hand, $k = m\omega_n^2$

$$\frac{192EI}{l^3} = m\omega_n^2 \rightarrow E = \frac{m\omega_n^2 l^3}{192 I}$$

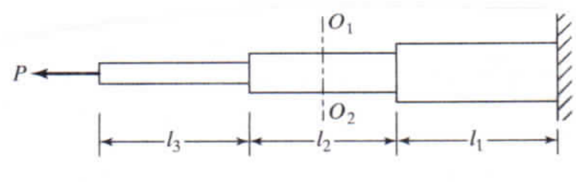
$$E = \frac{2.3 \times (30)^2 \times (1.0)^3}{192 \times \frac{1}{12}(0.005)^4} = \underline{207 \times 10^9 \ N/m^2}$$

– Example: The cockpit of a firetruck is located at the end of a telescoping boom, as shown. The cockpit, along with the fireman, weighs 2000 N. Find the cockpits natural frequency of vibration in the vertical direction.

$$\begin{cases} E = 2.1 \times 10^{11} \ N/m^2 \\ l_1 = l_2 = l_3 = 3 \ m \\ A_1 = 20 \ cm^2 \\ A_2 = 10 \ cm^2 \\ A_3 = 5 \ cm^2 \end{cases}$$



Solution: Since the force induced at any cross section O_1O_2 is equal to the axial Load applied at the end of the boom, as shown, the axial stiffness of the boom (k_b) can be expressed as following:



$$\frac{1}{k_b} = \frac{1}{k_{b_1}} + \frac{1}{k_{b_2}} + \frac{1}{k_{b_3}} \quad ; \quad k_{b_i} = \frac{E_i A_i}{\ell_i} \qquad i = 1,2,3$$

$$\left.\begin{aligned} k_{b_1} &= \frac{(2.1\times10^{11})(20\times10^{-4})}{3} = 14\times10^{7}\ N/m \\ k_{b_2} &= \frac{(2.1\times10^{11})(10\times10^{-4})}{3} = 7\times10^{7}\ N/m \\ k_{b_3} &= \frac{(2.1\times10^{11})(5\times10^{-4})}{3} = 3.5\times10^{7}\ N/m \end{aligned}\right\} \quad \begin{gathered} k_b = 2\times10^{7}\ N/m \\ \text{(verify!)} \end{gathered}$$

The stiffness of the telescoping boom in the vertical direction, k, is determined as

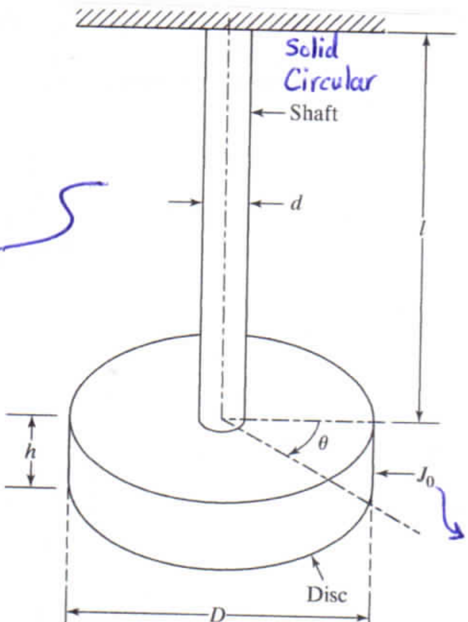
$$k = k_b \cos^2 45 = 2\times10^7 \cos^2 45 = 10^7\ N/m \quad \rightarrow \text{why?}$$

Consequently, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^7}{2000/9.81}} = \underline{221.4723\ \frac{rad}{sec}}$

* Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called 'torsional vibration'. In this case, the displacement of the body is measured in terms of an angular coordinate.

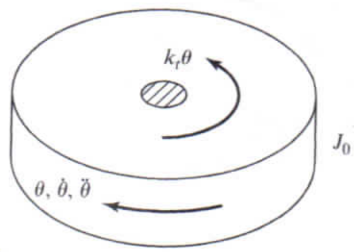
Also called 'torsional pendulum' or 'torsional spring-inertia system'



polar mass moment of inertia

Torsional vibration of a disc

θ: angular rotation of the disk about the axis of the shaft
or
shaft's angle of twist



FBD of the disk

SHAFT

From the theory of torsion of circular shafts: $\underset{\text{torque}}{M_t} = \overbrace{\frac{GI_0}{\ell}}^{k}\theta \quad ; \quad I_0 = \frac{\pi d^4}{32}$

shaft diameter

I_0: polar moment of inertia of the cross section of the shaft

Note that shaft acts as a torsional spring with spring constant: $k_t = \frac{M_t}{\theta} = \frac{GI_0}{\ell} = \frac{\pi G d^4}{32\ell}$

Equation of angular motion of the disk: $\boxed{J_0\ddot{\theta} + k_t\theta = 0}$

J_0: polar mass moment of inertia
k_t: torsional spring constant

Natural circular frequency of the torsional system: $\boxed{\omega_n = \sqrt{\dfrac{k_t}{J_0}}}$

Period: $\boxed{\tau_n = 2\pi\sqrt{\dfrac{J_0}{k_t}}}$

Frequency: $\boxed{f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_t}{J_0}}}$

Polar mass moment of inertia of the disk: $\boxed{J_0 = \dfrac{\rho h \pi D^4}{32} = \dfrac{WD^2}{8g}}$

$\begin{cases} W: \text{weight} \\ \rho: \text{density} \\ h: \text{thickness} \\ D: \text{diameter} \end{cases}$

Solution: The general solution of $J_0\ddot{\theta} + k_t\theta = 0$ can be obtained as:

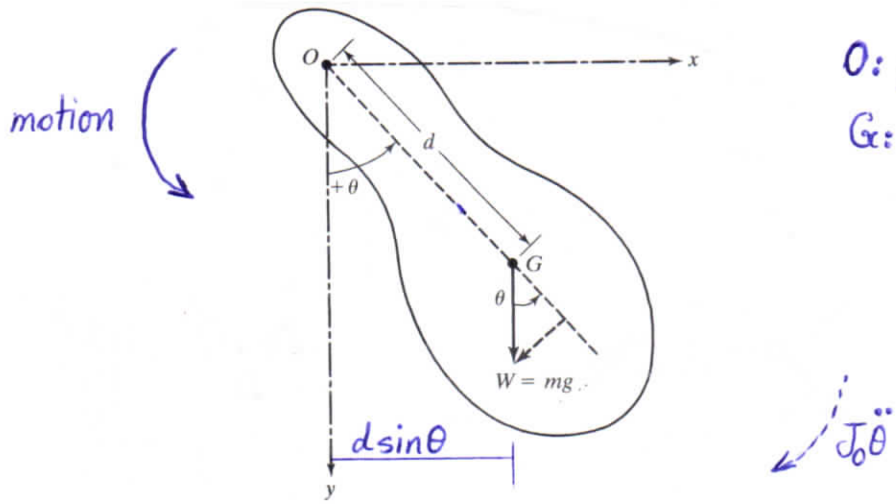
$\boxed{\theta(t) = A_1 \cos\omega_n t + A_2 \sin\omega_n t}$ which represents a simple harmonic motion. A_1 and A_2 can be determined from the initial conditions.

I.C.-s $\begin{cases} \theta(t=0) = \theta_0 \\ \dot{\theta}(t=0) = \dfrac{d\theta}{dt}\Big|_{t=0} = \dot{\theta}_0 \end{cases} \rightarrow$ Constants A_1 and A_2 can be found as $\begin{cases} A_1 = \theta_0 \\ A_2 = \dot{\theta}_0 / \omega_n \end{cases}$

(verify!)

So, $\boxed{\theta(t) = \theta_0 \cos\omega_n t + \dfrac{\dot{\theta}_0}{\omega_n} \sin\omega_n t}$

– Example: Any rigid body pivoted at a point other than its center of mass will oscillate about the pivot point under its own gravitational force. Such a system is known as a 'compound pendulum'. Find the natural frequency of such a system.



O: point of suspension
G: center of mass

Solution: Note that rigid body oscillates in $x-y$ plane. θ describes the motion. J_0 is the mass moment of inertia of the body about z-axis.

The restoring torque due to weight of the body is $Wd\sin\theta$.

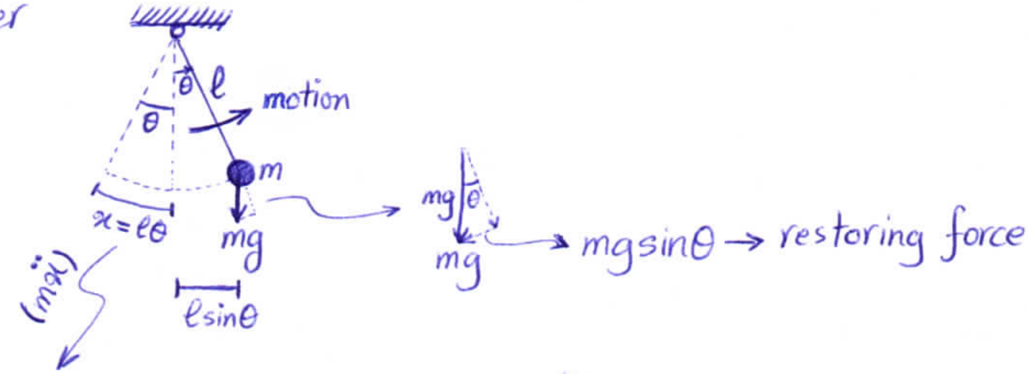
EOM: $J_0\ddot{\theta} + Wd\sin\theta = 0 \xrightarrow[\text{displacement } (\sin\theta \approx \theta)]{\text{assume small angular}}$

$$J_0\ddot{\theta} + Wd\theta = 0$$

Natural frequency of the system: $\omega_n = \sqrt{\dfrac{Wd}{J_0}} = \sqrt{\dfrac{mgd}{J_0}}$

– Question: Verify that the natural frequency of a 'simple' pendulum is $\omega_n = (g/\ell)^{1/2}$.

Answer: Consider



So, $m\ell\ddot{\theta} + mg\sin\theta = 0 \xrightarrow[\text{displ. } (\sin\theta \approx \theta)]{\text{assume small}} \cancel{m}\ell\ddot{\theta} + \cancel{m}g\theta = 0$

$\rightarrow$ EOM: $\ell\ddot{\theta} + g\theta = 0 \rightarrow \omega_n = \sqrt{\dfrac{g}{\ell}}$

* Rayleigh's Energy Method

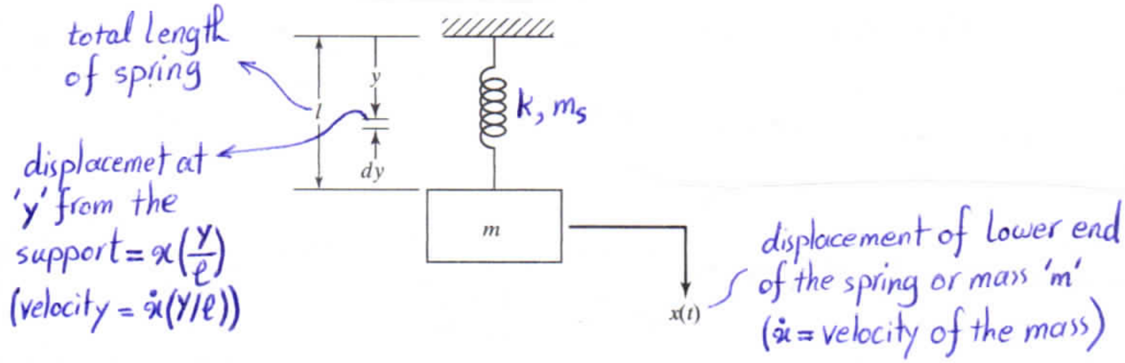
Energy method is used to find the natural frequencies of SDOF systems. The principle of conservation of energy, in the context of an undamped vibrating system, for two different instants of time, can be expressed as:

$$T_1 + U_1 = T_2 + U_2$$

- If ① is where the mass passes through the static equilibrium position, then $U_1 = 0$ and $T_1 = T_{max}$
- If ② is the time corresponding to maximum displacement of mass, then $T_2 = 0$ and $U_2 = U_{max}$

So $\rightarrow$ $T_{max} = U_{max}$

— Example: Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown below.



Solution: Total kinetic energy of the system is:

$$T = \frac{1}{2} m\dot{x}^2 + \underbrace{\int_{y=0}^{\ell} \frac{1}{2}\left(\frac{m_s}{\ell}dy\right)\left(\frac{y\dot{x}}{\ell}\right)^2}_{\text{kinetic energy of spring element of length 'dy'}}$$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2$$

Total potential energy of the system is:

$$U = \frac{1}{2} kx^2$$

If harmonic motion is assumed: $x(t) = X \cos\omega_n t$ → (max. displ. of mass → X)

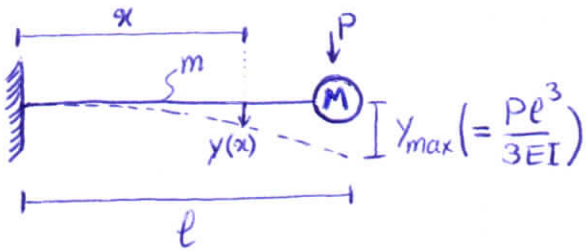
Max. kinetic energy: $T_{max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right) X^2 \omega_n^2$

Max. potential energy: $U_{max} = \frac{1}{2} k X^2$

$$T_{max} = U_{max} \rightarrow \boxed{\omega_n = \sqrt{\frac{k}{m + \frac{m_s}{3}}}}$$

So, effect of mass of spring can be accounted for by adding 1/3 of its mass to the main mass!

– Example: Find the natural frequency of transverse vibration of the following mass by including the mass of the beam.



Solution: To include the mass of the beam, we find the equivalent mass of the beam at the free end using the equivalence of kinetic energy. We know that

$$y(x)=\frac{Px^2}{6EI}(3\ell-x)=\frac{\cancel{3}Y_{max}}{\ell^3}\cdot\frac{x^2}{\cancel{6}_2}(3\ell-x)=\frac{Y_{max}x^2}{2\ell^3}(3\ell-x)$$

$$y(x)=\frac{Y_{max}}{2\ell^3}(3\ell x^2-x^3)\rightarrow \dot{y}(x)=\frac{\dot{Y}_{max}}{2\ell^3}(3\ell x^2-x^3)$$

$$T_{max}=\text{Max. kinetic energy of the beam itself}=\frac{1}{2}\int_0^\ell \frac{m}{\ell}[\dot{y}(x)]^2dx$$

$$T_{max}=\frac{1}{2}\int_0^\ell \frac{m}{\ell}\left[\frac{\dot{Y}_{max}}{2\ell^3}(3\ell x^2-x^3)\right]^2dx=\frac{m\dot{Y}_{max}^2}{8\ell^7}\int_0^\ell(3\ell x^2-x^3)^2dx$$

$$T_{max}=\frac{1}{2}\left(\frac{33}{140}m\right)\dot{Y}_{max}^2$$

If m_{eq} = equivalent mass of the cantilever at the free end, then

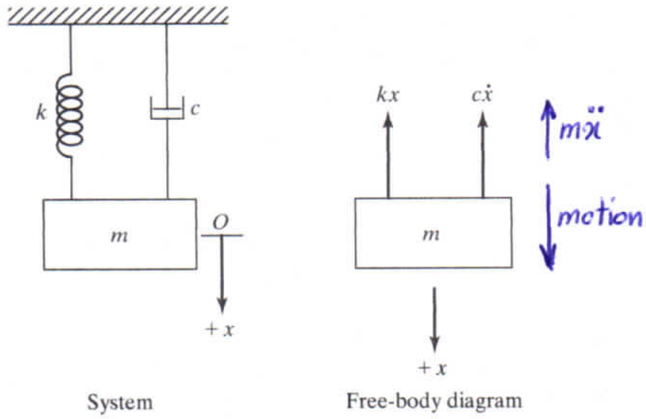
$$T_{max}=\frac{1}{2}m_{eq}\dot{Y}_{max}^2\rightarrow m_{eq}=\frac{33}{140}m$$

So, $M_{eff}=$ total effective mass acting at the end of the cantilever beam $=M+m_{eq}$

Finally, $$\omega_n=\sqrt{\frac{k}{M_{eff}}}=\sqrt{\frac{k}{M+\frac{33}{140}m}}$$

* Free Vibration with Viscous Damping – Translational Systems

A single-degree-of-freedom system with a viscous damper is shown below:



EOM: $m\ddot{x}+c\dot{x}+kx=0$ (why?)

Solution: Assume $x(t)=Ce^{st}$ where C and s = constants $\xrightarrow[\text{EOM}]{\text{insert in}}$

$\underbrace{ms^2+cs+k=0}_{\text{characteristic eq.}} \longrightarrow$ Roots: $s_{1,2}=\dfrac{-c\pm\sqrt{c^2-4mk}}{2m}$ or

$$s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}} \xrightarrow{\text{So}} \begin{cases} x_1(t)=C_1e^{s_1t} \\ x_2(t)=C_2e^{s_2t}\end{cases} \longrightarrow$$

General solution: $x(t)=C_1e^{s_1t}+C_2e^{s_2t}$ where C_1 and C_2 = Consts. determined from I.C.-s

Critical damping Constant and the damping ratio

If $s_{1,2}=-\frac{c}{2m}\pm\underbrace{\sqrt{\left(\frac{c}{2m}\right)^2-\frac{k}{m}}}_{=0} \rightarrow \left(\frac{c_c}{2m}\right)^2-\frac{k}{m}=0 \rightarrow c_c=2m\sqrt{\frac{k}{m}}=2\sqrt{km}=2m\omega_n$

c_c = critical damping

ξ = damping ratio = $\dfrac{c}{c_c}$ → damping constant / → critical damping constant

$$\frac{c}{2m}=\frac{c}{c_c}\cdot\frac{c_c}{2m}=\xi\omega_n$$

As a result, the solution of $m\ddot{x}+c\dot{x}+kx=0$, can be written as

$$\boxed{x(t)=C_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t}+C_2 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}}$$

Note: if $\xi=0 \rightarrow$ undamped system

Note: $s_{1,2}=\left(-\xi \pm \sqrt{\xi^2-1}\right)\omega_n$ (Why?)

There are three possible cases:
$$\begin{cases} ① \ \xi<1 \text{ or } c<c_c \rightarrow \text{underdamped system} \\ ② \ \xi=1 \text{ or } c=c_c \rightarrow \text{critically-damped system} \\ ③ \ \xi>1 \text{ or } c>c_c \rightarrow \text{overdamped system} \end{cases}$$

① Underdamped System ($\xi<1$ or $c<c_c$ or $\frac{c}{2m}<\sqrt{\frac{k}{m}}$)

Consider $s_{1,2}=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_n \xrightarrow{\xi^2-1<0} \begin{cases} s_1=\left(-\xi+i\sqrt{1-\xi^2}\right)\omega_n \\ s_2=\left(-\xi-i\sqrt{1-\xi^2}\right)\omega_n \end{cases}$

So, the solution can be written in different forms:

$$\boxed{\begin{aligned} x(t) &= C_1 e^{\left(-\xi+i\sqrt{1-\xi^2}\right)\omega_n t}+C_2 e^{\left(-\xi-i\sqrt{1-\xi^2}\right)\omega_n t} \\ &= e^{-\xi\omega_n t}\left\{C_1 e^{i\sqrt{1-\xi^2}\,\omega_n t}+C_2 e^{-i\sqrt{1-\xi^2}\,\omega_n t}\right\} \\ &= e^{-\xi\omega_n t}\left\{(C_1+C_2)\cos\sqrt{1-\xi^2}\,\omega_n t+i\,(C_1-C_2)\sin\sqrt{1-\xi^2}\,\omega_n t\right\} \\ &= e^{-\xi\omega_n t}\left\{C_1'\cos\sqrt{1-\xi^2}\,\omega_n t+C_2'\sin\sqrt{1-\xi^2}\,\omega_n t\right\} \\ &= X_0 e^{-\xi\omega_n t}\sin\left(\sqrt{1-\xi^2}\,\omega_n t+\phi_0\right) \\ &= X e^{-\xi\omega_n t}\cos\left(\sqrt{1-\xi^2}\,\omega_n t-\phi\right) \end{aligned}}$$

Note: $C_1', C_2', X_0, X, \phi_0, \phi \rightarrow$ Constants to be determined from I.C.-s

$$\text{I.C.-s:}\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = \dot{x}_0 \end{cases}$$

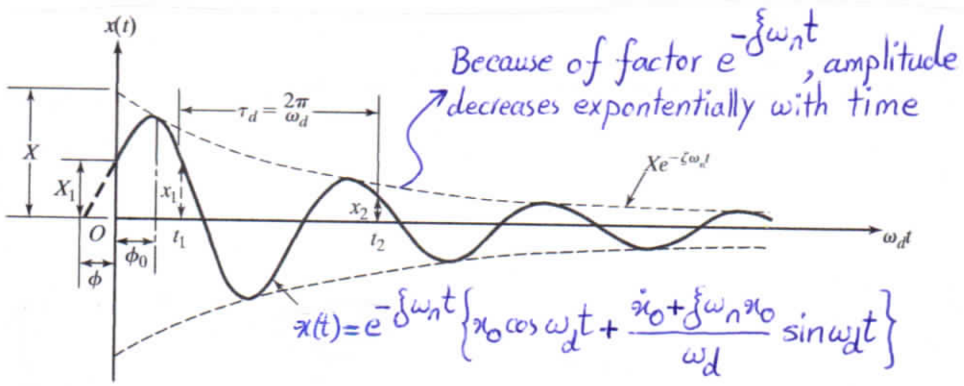
$$\begin{cases} C_1' = x_0 \\ C_2' = \dfrac{\dot{x}_0 + \xi\omega_n x_0}{\sqrt{1-\xi^2}\,\omega_n} \end{cases} \quad \text{(verify!)}$$

and

$$\begin{cases} X = X_0 = \sqrt{C_1'^2 + C_2'^2} = \dfrac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\xi\omega_n}}{\sqrt{1-\xi^2}\,\omega_n} \\ \phi_0 = \tan^{-1}\left(\dfrac{C_1'}{C_2'}\right) = \tan^{-1}\left(\dfrac{x_0\omega_n\sqrt{1-\xi^2}}{\dot{x}_0 + \xi\omega_n x_0}\right) \\ \phi = \tan^{-1}\left(\dfrac{C_2'}{C_1'}\right) = \tan^{-1}\left(\dfrac{\dot{x}_0 + \xi\omega_n x_0}{x_0\omega_n\sqrt{1-\xi^2}}\right) \end{cases} \quad \text{(verify!)}$$

Note: For damped harmonic motion:

$$\boxed{\omega_d = \text{frequency of damped vibration} = \sqrt{1-\xi^2}\,\omega_n}$$



Underdamped solution.

② Critically-Damped System $\left(\zeta = 1 \text{ or } c = c_c \text{ or } \frac{c}{2m} = \sqrt{\frac{k}{m}}\right)$

Consider $s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \xrightarrow{\zeta^2 - 1 = 0} s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$

Solution: $\boxed{x(t) = (C_1 + C_2 t)\, e^{-\omega_n t}}$

I.C.-s: $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = \dot{x}_0 \end{cases} \rightarrow \begin{cases} C_1 = x_0 \\ C_2 = \dot{x}_0 + \omega_n x_0 \end{cases} \rightarrow \boxed{x(t) = \left\{x_0 + (\dot{x}_0 + \omega_n x_0)\, t\right\} e^{-\omega_n t}}$

- nonperiodic motion!
- motion will eventually diminish to zero $\left(t \to \infty \Rightarrow e^{-\omega_n t} \to 0\right)$
- See figure below!

③ Overdamped System $\left(\zeta > 1 \text{ or } c > c_c \text{ or } \frac{c}{2m} > \sqrt{\frac{k}{m}}\right)$

Consider $s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \xrightarrow{\zeta^2 - 1 > 0} \begin{cases} s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0 \\ s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0 \end{cases}$

$(s_2 \ll s_1)$

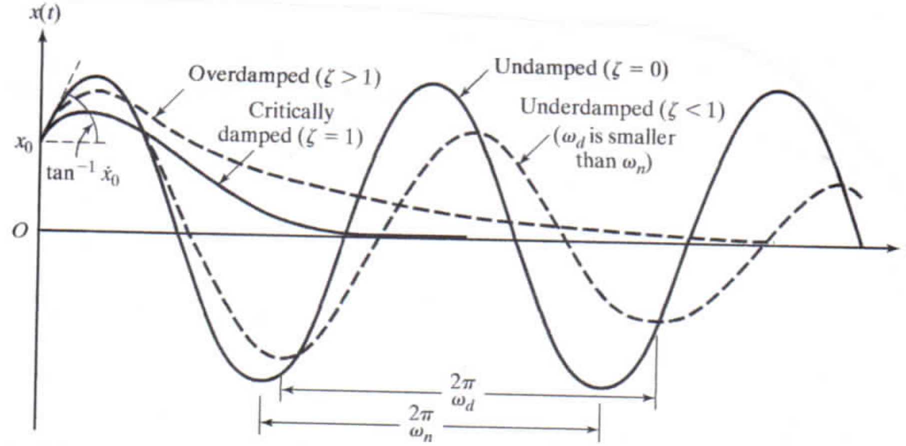
Solution: $\boxed{x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}}$

I.C.-s: $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = \dot{x}_0 \end{cases}$

$$C_1 = \frac{x_0 \omega_n \left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

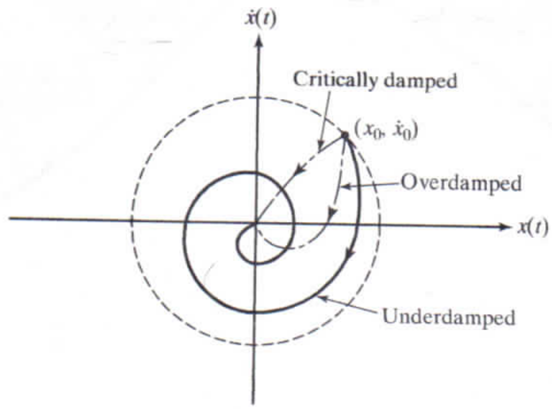
$$C_2 = \frac{-x_0 \omega_n \left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

- nonperiodic motion!
- motion diminishes exponentially with time
- See figure below!



Comparison of motions with different types of damping

The free damped response of a SDOF system can be represented in phase-plane or state space as indicated in the figure below:



Phase plane of a damped system

– Logarithmic Decrement

Q: What does logarithmic decrement do?

A: It represents the rate at which the amplitude of a free-damped vibration decreases.

Logarithmic decrement is defined as the natural logarithm of the ratio of any two successive amplitudes.

Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes (displacements), measured one cycle apart for an underdamped system, So

x_1 @ t_1
x_2 @ t_2 and $t_1 + \tau_d = t_2$ where $\tau_d = \frac{2\pi}{\omega_d}$

Since $x(t) = X_0 e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_0\right) \longrightarrow$

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi_0)} \quad \text{(ratio)}$$

$t_2 = t_1 + \tau_d$

We can write: $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$

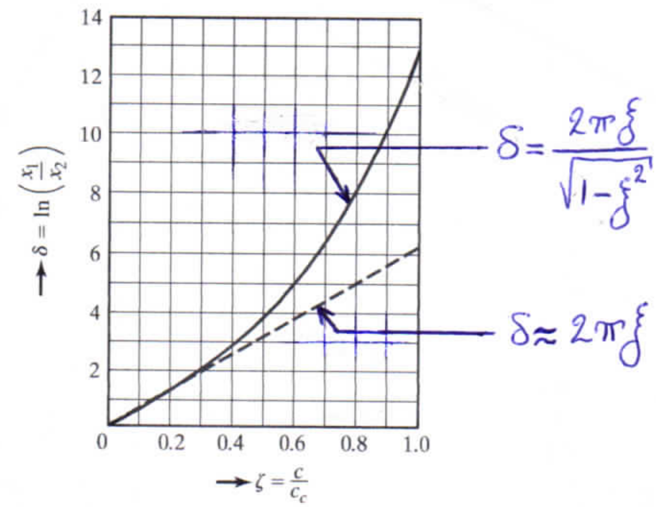
So, $\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d}$

Logarithmic Decrement $\boxed{\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \frac{2\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m}}$

For small damping, where $\zeta \ll 1 \rightarrow \boxed{\delta \approx 2\pi\zeta}$

So, $\begin{cases} \text{From } \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \rightarrow \zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \\ \text{From } \delta \approx 2\pi\zeta \rightarrow \zeta \approx \frac{\delta}{2\pi} \end{cases} \rightarrow$ Note: δ is indeed another form of ζ. Both are dimensionless.

The variation of δ with ξ is shown below:



Note: If damping is not known, it can be determined experimentally by measuring two consecutive displacements x_1 and x_2.

$\left\{\begin{array}{l}\text{If } x_1 \text{ and } x_{m+1} \text{ are separated} \\ \text{by some number of complete cycles}\end{array}\right. \rightarrow \dfrac{x_1}{x_{m+1}} = \dfrac{x_1}{x_2}\dfrac{x_2}{x_3}\dfrac{x_3}{x_4}\cdots\dfrac{x_m}{x_{m+1}}$

$\left\{\begin{array}{l}\text{For any two successive displacements} \\ \text{separated by one cycle}\end{array}\right. \rightarrow \dfrac{x_j}{x_{j+1}} = e^{\xi\omega_n\tau_d}$

$$\frac{x_1}{x_{m+1}} = \left(e^{\xi\omega_n\tau_d}\right)^m = e^{\xi\omega_n\tau_d m}$$

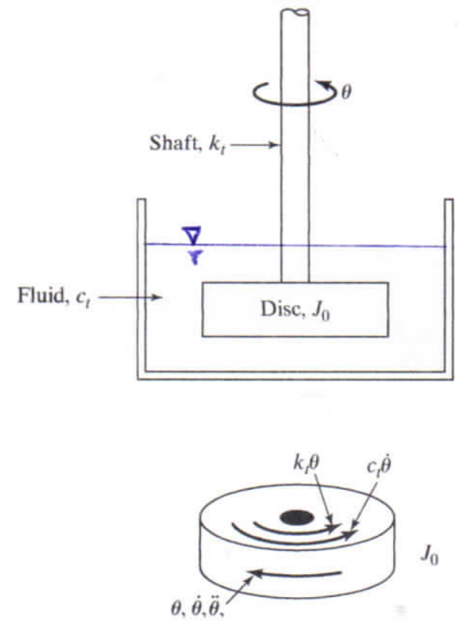
By considering $\delta = \xi\omega_n\tau_d \rightarrow \delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right)$

$$\downarrow$$

$$\xi = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}} \approx \frac{\delta}{2\pi}$$

* Free Vibration with Viscous Damping – Torsional Systems

Consider a SDOF torsional system with a viscous damper, as shown. The EOM can be derived as:



Torsional viscous damper

$$J_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$$

where

J_0 = mass moment of inertia of the disk

c_t = torsional viscous damping constant

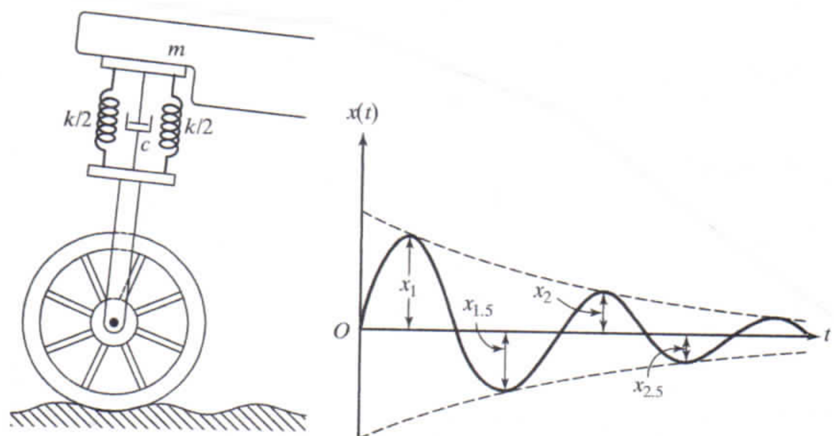
k_t = spring constant of the system

$\theta, \dot{\theta}, \ddot{\theta}$ = angular displacement, velocity, and acceleration of the disk

The solution of the EOM can be found exactly as in the case of linear vibrations. For example, in the underdamped case, the frequency of damped vibration is given by

$$\omega_d = \omega_n\sqrt{1-\xi^2} \quad \text{where} \quad \omega_n = \sqrt{\frac{k_t}{J_0}} \quad \text{and} \quad \xi = \frac{c_t}{\underbrace{c_{tc}}_{\text{critical torsional damping constant}}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_t J_0}}$$

– Example: An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg, as shown. When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displ.–time curve is to be as indicated in the figure below. Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 sec and the amplitude x_1 is to be reduced to one-fourth in one half cycle, i.e. $x_{1.5} = x_1/4$.



motorcycle and shock absorber resulting displ.–time curve

Solution: Since $x_{1.5} = \frac{x_1}{4}$, $x_2 = \frac{x_{1.5}}{4} = \frac{x_1}{16}$, then

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726 = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \rightarrow \xi = 0.4037$$

$$\tau_d = \frac{2\pi}{\omega_d} \rightarrow 2 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \rightarrow \omega_n = 3.4338 \, \frac{\text{rad}}{\text{sec}}$$

$$c_c = 2m\omega_n = 2 \times 200 \times 3.4338 = 1373.54 \ \text{N·sec/m}$$

$$c = \xi c_c = 0.4037 \times 1373.54 = \underline{554.4981 \ \text{N·sec/m}}$$

$$k = m\omega_n^2 = 200 \times (3.4338)^2 = \underline{2358.2652 \ \text{N/m}}$$

– Example: For a given vibrating system with viscous damping $W = 10$ lbs, $k = 30$ lb/in, and $c = 0.12$ lb·s/in. Determine the logarithmic decrement and the ratio of any two successive amplitudes.

Solution: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{10/386}} = 34.0$ rad/sec

$$c_c = 2m\omega_n = 2 \times \frac{10}{386} \times 34.0 = 1.76 \text{ lb·s/in}$$

$$\xi = \frac{c}{c_c} = \frac{0.12}{1.76} = 0.0681$$

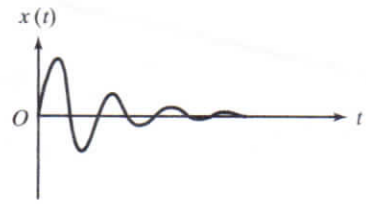
$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.0681}{\sqrt{1-0.0681^2}} = \underline{0.429}$$

$$\delta = \ln\left(\frac{x_1}{x_2}\right) \rightarrow \frac{x_1}{x_2} = e^{\delta} = e^{0.429} = \underline{1.54}$$

* Stability of Systems

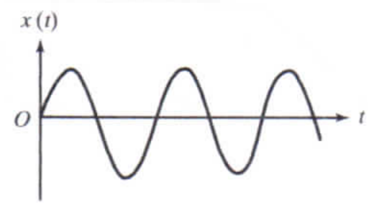
'Stability' is one of the most important characteristics for any vibrating system. A system can be either 'asymptotically stable' or 'stable' or 'unstable', as defined in the following.

- A system is defined to be 'asymptotically stable', if its free vibration response approaches zero as time approaches infinity.



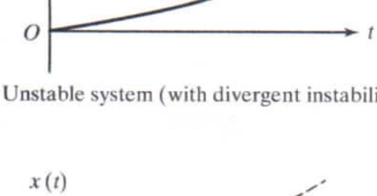
Asymptotically stable system

- A system is said to be 'stable', if its free vibration response neither decays nor grows, but remains constant or oscillates as time approaches infinity.



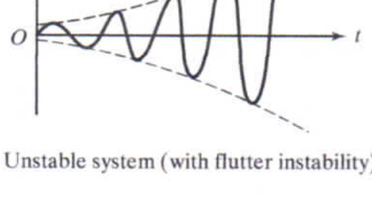
Stable system

- A system is considered to be 'unstable', if its free vibration response grows without bound (approaches infinity) as time approaches infinity.



→ no oscillation

Unstable system (with divergent instability)

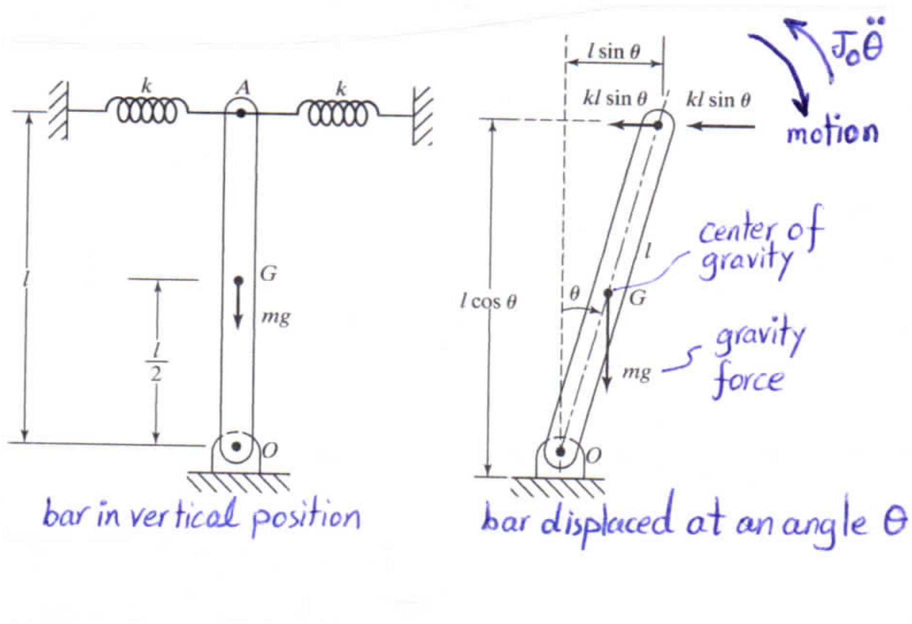


→ oscillation

Unstable system (with flutter instability)

Stability of a system can also be explained in terms of its energy. According to this scheme, a system is considered to be 'asymptotically stable', 'stable', or 'unstable' if its energy decreases, remains constant, or increases, respectively, with time.

- Example: Consider a uniform rigid bar, of mass 'm' and length 'ℓ', pivoted at one end and connected symmetrically by two springs at the other end, as shown. Assuming that the springs are unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability behavior of the system.



Solution: EOM of the bar is: $\left(\Sigma M_o = 0\right)$

$$\underbrace{\frac{m\ell^2}{3}}_{J_o}\ddot{\theta} + \underbrace{(2k\ell\sin\theta)\ell\cos\theta}_{\text{spring}} - \underbrace{W\frac{\ell}{2}\sin\theta}_{\text{weight}} = 0$$

For small oscillations $\begin{cases} \sin\theta \approx \theta \\ \cos\theta \approx 1 \end{cases} \rightarrow \frac{m\ell^2}{3}\ddot{\theta} + 2k\ell^2\theta - \frac{W\ell}{2}\theta = 0$

if $\frac{12k\ell^2 - 3W\ell}{2m\ell^2} = \alpha^2 \rightarrow$ EOM: $\ddot{\theta} + \alpha^2\theta = 0$

Characteristic equation: $s^2 + \alpha^2 = 0$

solution depends on the sign of α^2

① $\alpha^2 > 0 \rightarrow \boxed{\theta(t) = A_1\cos\omega_n t + A_2\sin\omega_n t}$ ($A_1, A_2 =$ consts.)

and $\omega_n = \sqrt{\frac{12k\ell^2 - 3W\ell}{2m\ell^2}}$

The solution represents a stable system with stable oscillations.

② $\alpha^2 = 0 \rightarrow$ EOM: $\ddot{\theta} = 0 \xrightarrow[\text{twice}]{\text{integrate}} \theta(t) = C_1 t + C_2$

($C_1, C_2 =$ Consts.)

I.C.'s $\begin{cases} \theta(t=0) = \theta_0 \\ \dot{\theta}(t=0) = \dot{\theta}_0 \end{cases} \rightarrow \boxed{\theta(t) = \dot{\theta}_0 t + \theta_0}$

The solution shows that the system is unstable with the angular displacement increasing linearly.

③ $\alpha^2 < 0 \rightarrow$ Solution: $\theta(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t}$ ($B_1, B_2 =$ Consts.)

I.C.-s $\begin{cases} \theta(t=0) = \theta_0 \\ \dot{\theta}(t=0) = \dot{\theta}_0 \end{cases} \rightarrow \boxed{\theta(t) = \frac{1}{2\alpha}\left[(\alpha\theta_0 + \dot{\theta}_0)e^{\alpha t} + (\alpha\theta_0 - \dot{\theta}_0)e^{-\alpha t}\right]}$

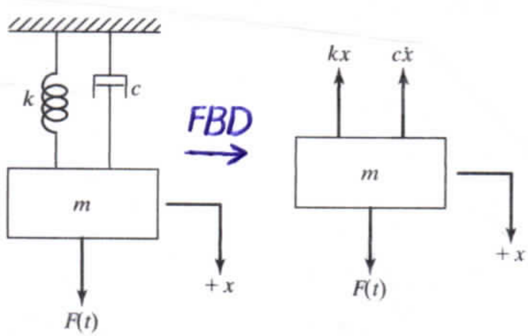
The solution shows that $\theta(t)$ increases exponentially with time; hence, the motion is unstable.

*Introduction

A mechanical or structural system is said to undergo forced vibration whenever external energy is supplied to the system during vibration. External energy can be supplied through either an applied force or an imposed displacement excitation. The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random in nature.

*Equation of Motion

If a force $F(t)$ acts on a viscously-damped spring-mass system, as shown, the following EOM can be obtained (how?).



EOM: $m\ddot{x} + c\dot{x} + kx = F(t)$

Since the EOM is nonhomogeneous, its general solution is given by:

$$x(t) = x_h(t) + x_p(t)$$

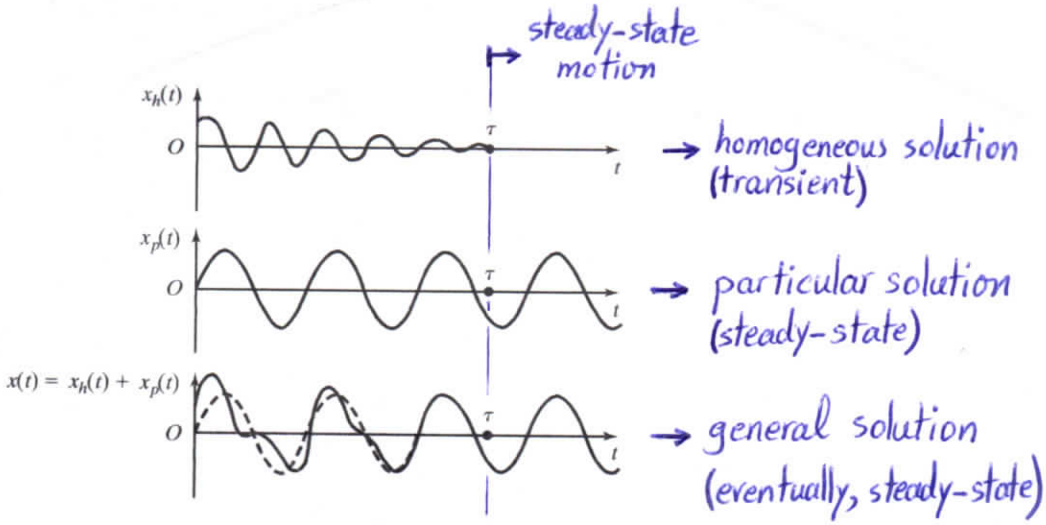
where

$x(t)$ = general solution

$x_h(t)$ = homogeneous solution, which is the solution of the homogeneous equation $m\ddot{x}+c\dot{x}+kx=0$ representing the free vibration of the system. Free vibration dies out with time under each of the three possible (underdamping, critical damping, and overdamping) conditions of damping. This part of the motion that dies out due to damping is called 'transient'.

$x_p(t)$ = particular solution. The general solution $x(t)$ eventually reduces to the particular solution, which represents the 'steady-state' vibration. The steady-state motion is present as long as the forcing function is present.

The vibrations of homogeneous, particular, and general solutions with time for a typical case are shown below. It is seen that $x_h(t)$ dies out and $x(t)$ becomes $x_p(t)$ after some time (τ).



* Response of an Undamped System Under Harmonic Force

Consider an undamped system subjected to a harmonic force, $F(t) = F_0 \cos\omega t$. The EOM of the system is:

$$\boxed{m\ddot{x} + kx = F_0 \cos\omega t} \quad \text{where} \begin{cases} F_0 : \text{amplitude of harmonic excitation} \\ \omega : \text{frequency of harmonic excitation} \end{cases}$$

The homogeneous solution is given by: $(m\ddot{x} + kx = 0)$

$$\boxed{x_h(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t} \quad \text{where} \quad \omega_n = \sqrt{\frac{k}{m}}$$

The particular solution is assumed to be:

$$\boxed{x_p(t) = X \cos\omega t} \quad \text{where } X: \text{maximum amplitude of } x_p(t)$$

By substituting $x_p(t)$ into the EOM and solving for X, we can write the total solution of the EOM as:

$$x(t) = \underbrace{C_1 \cos\omega_n t + C_2 \sin\omega_n t}_{\text{homogeneous sol.}} + \underbrace{\overbrace{\frac{F_0}{k - m\omega^2}}^{X} \cos\omega t}_{\text{particular sol.}}$$

$\xrightarrow{(why?)}$ $X = \dfrac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$

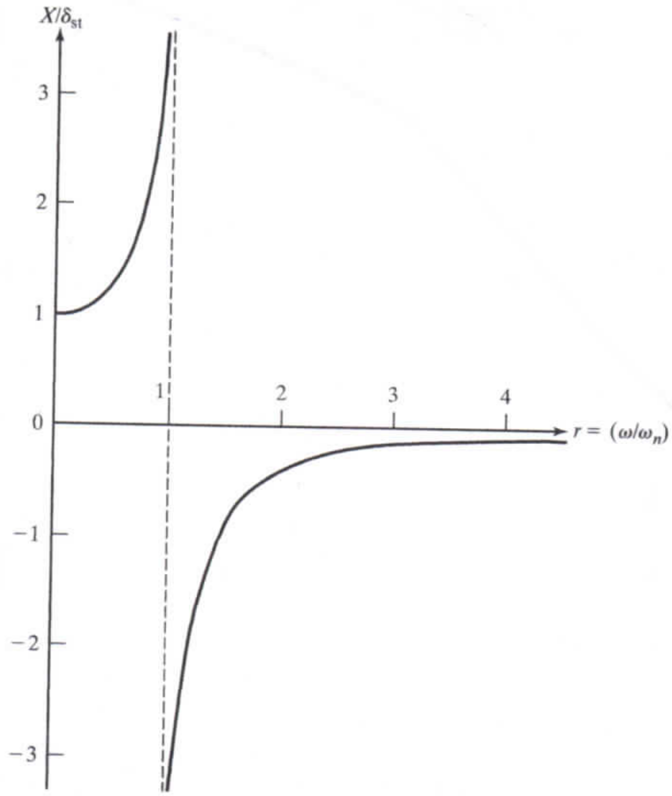
Note: $\underbrace{\delta_{st}}_{\text{static deflection}} = F_0 / k$

By using I.C.-s $\begin{cases} x(t=0) = x_0 \\ \dot{x}(t=0) = \dot{x}_0 \end{cases}$, C_1 and C_2 can be determined, and finally:

$$\boxed{x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t} \quad \text{(verify!)}$$

Since $X = \dfrac{F_0}{k - m\omega^2} = \dfrac{\delta_{st}}{1-(\omega/\omega_n)^2} \rightarrow \dfrac{X}{\delta_{st}} = \dfrac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$

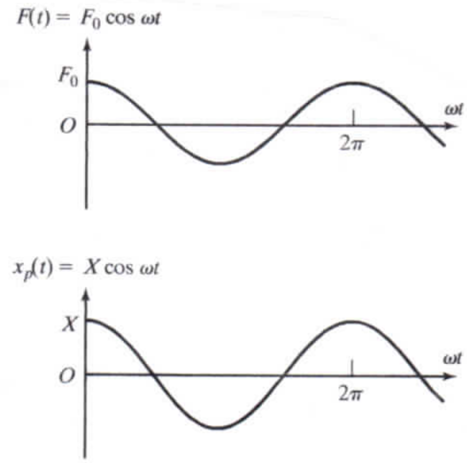
$\dfrac{X}{\delta_{st}}$ represents the ratio of the dynamic to the static amplitude of motion and is called the 'magnification factor', 'amplification factor', or 'amplitude ratio'. The variation of the amplitude ratio $\left(\dfrac{X}{\delta_{st}}\right)$ with the frequency ratio $\left(r = \dfrac{\omega}{\omega_n}\right)$ is shown below.



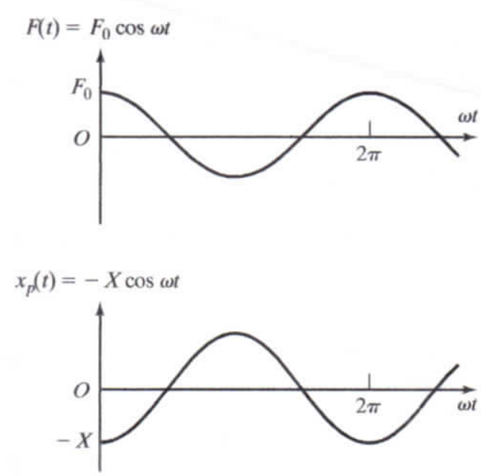
Magnification factor of an undamped system

From the figure, the response of the system can be identified to be of three types, as discussed in the following.

① When $0 < \frac{\omega}{\omega_n} < 1$, the denominator in $\frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$ is positive, and the response is given by $x_p(t) = X\cos\omega t$. $x_p(t)$ is said to be in phase with the external force as shown.



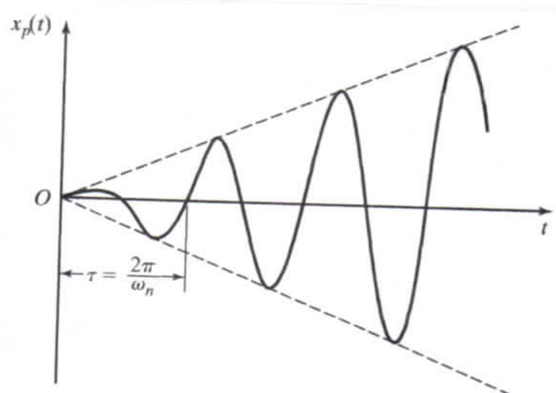
② When $\frac{\omega}{\omega_n} > 1$, the denominator in $\frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}$ is negative, and the steady-state solution can be expressed as $x_p(t) = -X\cos\omega t$, where $X = \frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$. The variations of $F(t)$ and $x_p(t)$ with time are shown below. Since $x_p(t)$ and $F(t)$ have opposite signs, the response is said to be $180°$ out of phase with the external force. Also note that as $\frac{\omega}{\omega_n} \to \infty$, $X \to 0$, which means that the response of the system to a harmonic force of very high frequency is close to zero.



③ When $\frac{\omega}{\omega_n}=1$, the amplitude X given by $X=\frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$ or $X=\frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2-1}$ becomes infinite. This condition, for which the forcing frequency (ω) is equal to the natural frequency of the system (ω_n), is called 'resonance'. Response of the system at resonance becomes

$$x(t)=x_0\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t+\frac{\delta_{st}\,\omega_n t}{2}\sin\omega_n t.$$

It is found that at resonance, $x(t)$ increases indefinitely. The response for $\frac{\omega}{\omega_n}=1$ is shown below.

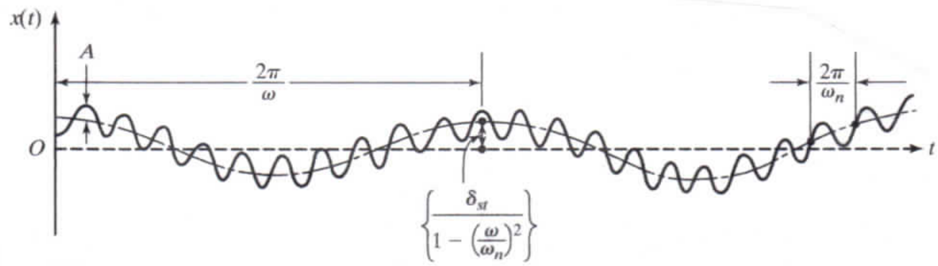


Total Response: The total response of the system,

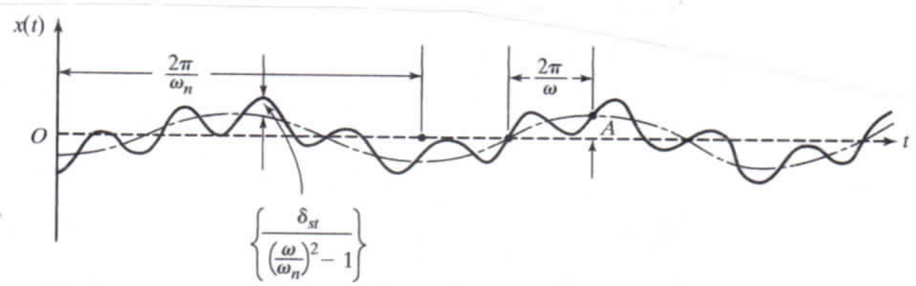
$$x(t)=C_1\cos\omega_n t+C_2\sin\omega_n t+\frac{F_0}{k-m\omega^2}\cos\omega t$$

can be expressed as the sum of two cosine curves of different frequencies, as shown in the following.

$$x(t) = A\cos(\omega_n t - \phi) + \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cos\omega t \qquad \text{for } \frac{\omega}{\omega_n} < 1$$



$$x(t) = A\cos(\omega_n t - \phi) - \frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}\cos\omega t \qquad \text{for } \frac{\omega}{\omega_n} > 1$$



where A and ϕ are constants. In fact, these equations are obtained by assuming $C_1 = A\cos\phi$ and $C_2 = A\sin\phi$ (verify!).

Beating Phenomenon: If the forcing frequency is close to, but not exactly equal to, the natural frequency of the system, a phenomenon known as 'beating' may occur.

The phenomenon of beating is explained by considering the solution

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t .$$ If

$x_0 = \dot{x}_0 = 0$, then

$$x(t) = \frac{(F_0/m)}{\omega_n^2 - \omega^2}\left(\cos\omega t - \cos\omega_n t\right) = \frac{(F_0/m)}{\omega_n^2 - \omega^2}\left[2\sin\frac{\omega_n + \omega}{2}t \cdot \sin\frac{\omega_n - \omega}{2}t\right]$$

Let ω be slightly less than $\omega_n \rightarrow \omega_n - \omega = 2\varepsilon$ → small positive quantity

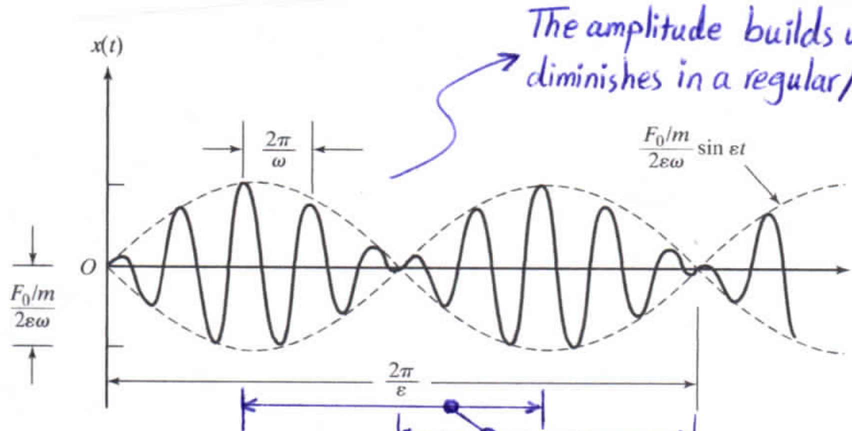
So, $\omega \approx \omega_n \rightarrow \omega + \omega_n \simeq 2\omega$

$$\omega_n^2 - \omega^2 \simeq 4\varepsilon\omega$$

Consequently, $x(t) = \underbrace{\overbrace{\left(\frac{F_0/m}{2\varepsilon\omega}\sin\varepsilon t\right)\sin\omega t}^{\text{motion with period } 2\pi/\omega}}$ (Plotted below)

$$x(t) = \underbrace{\left(\frac{F_0/m}{2\varepsilon\omega}\sin\varepsilon t\right)}_{\substack{\text{variable amplitude}\\ \text{with period } 2\pi/\varepsilon}}\sin\omega t$$

The amplitude builds up and then diminishes in a regular/continuous pattern.

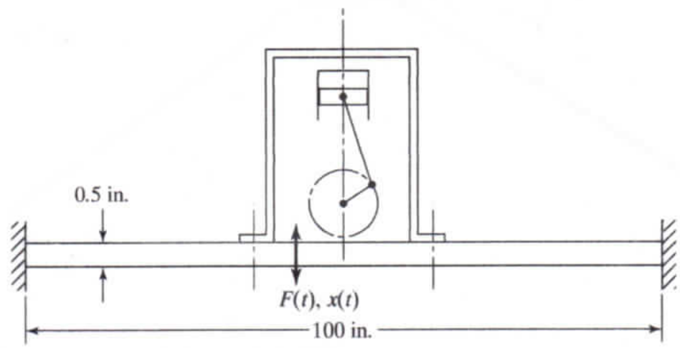


Phenomenon of beats.

→ period of beating: $\tau_b = \frac{2\pi}{2\varepsilon} = \frac{2\pi}{\omega_n - \omega}$

frequency of beating: $\omega_b = 2\varepsilon = \omega_n - \omega$

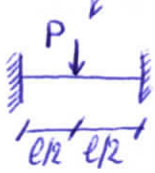
– Example: A reciprocating pump, weighing 150 lb, is mounted at the middle of a steel plate of thickness 0.5 in, width 20 in, and length 100 in, clamped along two edges as shown. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 50\cos 62.832t$ lb. Find the amplitude of vibration of the plate.



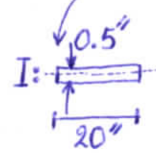
$$\left\{\begin{array}{l} W = 150 \text{ lb} \\ t = 0.5 \text{ in} \\ w = 20 \text{ in} \\ l = 100 \text{ in} \\ F(t) = 50\cos 62.832t \end{array}\right. \rightarrow X = ?$$

Solution:

$$X = \frac{F_0}{k - m\omega^2} = \frac{50}{\dfrac{192(30\times10^6)\left(\dfrac{20\times0.5^3}{12}\right)}{100^3} - \left(\dfrac{150}{\underbrace{386.4}_{g\ (\text{in/s}^2)}}\right)(62.832)^2}$$



$$k_{eq} = \frac{192EI}{l^3}$$



$$\underline{\underline{X = -0.1504 \text{ in}}}$$

means: response $x(t)$ of plate is out of phase with the excitation $F(t)$.

- Example: A spring-mass system, with a spring stiffness of 5000 N/m, is subjected to a harmonic force of magnitude 30 N and frequency 20 Hz. The mass is found to vibrate with an amplitude of 0.2 m. Assuming that vibration starts from rest ($x_0 = \dot{x}_0 = 0$), determine the mass of the system.

$$\left\{\begin{array}{l} k = 5000\ N/m \\ F_0 = 30N \\ f = 20\ Hz \\ \text{mass amplitude} = 0.2\ in \\ x_0 = \dot{x}_0 = 0 \end{array}\right. \rightarrow m = ?$$

Solution: Consider $x(t) = \left(\cancel{x_0} - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \cancel{\left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t} + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t$

$$\rightarrow x(t) = \frac{F_0}{k - m\omega^2}\left(\cos\omega t - \cos\omega_n t\right)$$

$$\rightarrow x(t) = \underline{\underline{\frac{2F_0}{k - m\omega^2}}} \sin\frac{\omega_n + \omega}{2}t \cdot \sin\frac{\omega_n - \omega}{2}t$$

$$\rightarrow \text{amplitude of vibration} = 0.2 = \frac{2F_0}{k - m\omega^2}$$

$$\rightarrow 0.2k - 0.2m\omega^2 = 2F_0 \rightarrow m = \frac{0.2k - 2F_0}{0.2\omega^2}$$

$$\rightarrow m = \frac{0.2 \times 5000 - 2 \times 30}{0.2 \times (2\pi \times 20)^2} \rightarrow \underline{\underline{m = 0.2976\ kg}}$$

* Response of a Damped System Under Harmonic Force

For a damped system subjected to $F(t) = F_0 \cos\omega t$, the EOM becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t$$

The particular solution is also harmonic and assumed to be

$$x_p(t) = X\cos(\omega t - \phi) \quad \text{where} \quad \begin{cases} X = \text{amplitude of response} \\ \phi = \text{phase angle of response} \end{cases}$$

By substituting $x_p(t)$ into the EOM, we arrive at

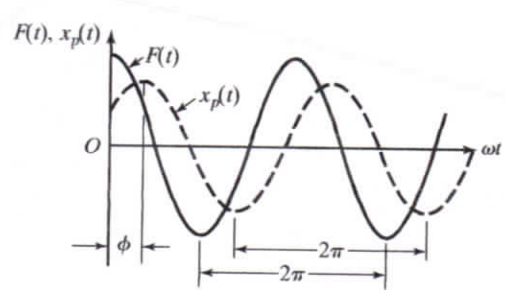
$$X\left[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)\right] = F_0 \cos\omega t .$$

By using the trigonometric relations $\begin{cases} \cos(\omega t - \phi) = \cos\omega t \cos\phi + \sin\omega t \sin\phi \\ \sin(\omega t - \phi) = \sin\omega t \cos\phi - \cos\omega t \sin\phi \end{cases}$

and equating the coefficients of $\cos\omega t$ and $\sin\omega t$ on both sides of the resulting equation, we obtain

$$\begin{cases} X\left[(k - m\omega^2)\cos\phi + c\omega \sin\phi\right] = F_0 \\ X\left[(k - m\omega^2)\sin\phi - c\omega \cos\phi\right] = 0 \end{cases} \rightarrow \quad X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

The following figure shows typical plots of the forcing function and (steady-state) response. →



Consider $X = \dfrac{F_0/k}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}/k}$ and also

$\omega_n = \sqrt{\dfrac{k}{m}}$

$\zeta = \dfrac{c}{c_c} = \dfrac{c}{2m\omega_n} = \dfrac{c}{2\sqrt{km}}$; $\dfrac{c}{m} = 2\zeta\omega_n$

$\delta_{st} = \dfrac{F_0}{k}$ (deflection under static force F_0)

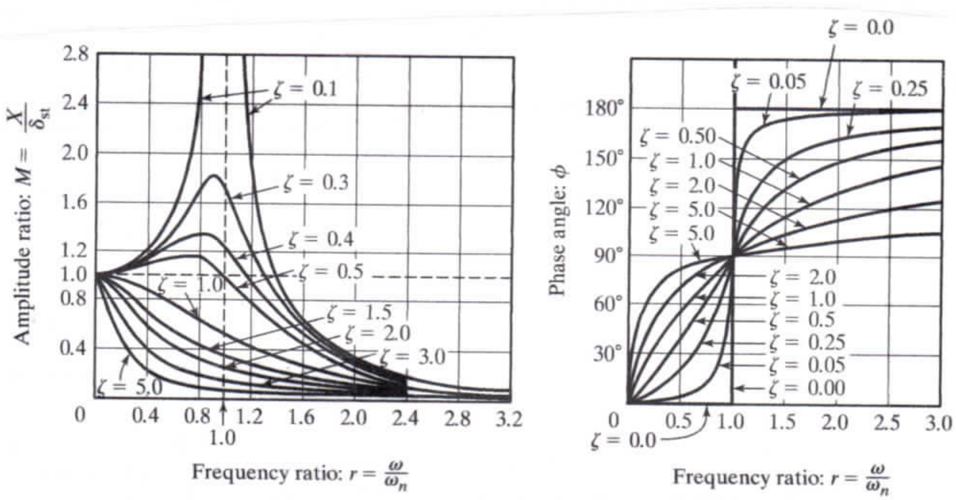
$r = \dfrac{\omega}{\omega_n}$ (frequency ratio)

we can obtain,

magnification factor, or
M = amplification factor, or $= \dfrac{X}{\delta_{st}} = \dfrac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} = \dfrac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$
amplitude ratio

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

The variations of $\dfrac{X}{\delta_{st}}$ and ϕ with the frequency ratio r and the damping ratio ζ are shown below.



Total Response: The complete solution of $m\ddot{x}+c\dot{x}+kx=F_0\cos\omega t$ is

$$x(t)=x_h(t)+x_p(t)$$

where

$$x_h(t)=X_0e^{-\xi\omega_n t}\cos\Big(\underbrace{\sqrt{1-\xi^2}\,\omega_n}_{\omega_d}t-\phi_0\Big)\rightarrow(m\ddot{x}+c\dot{x}+kx=0)$$

so,

$$\boxed{x(t)=X_0e^{-\xi\omega_n t}\cos(\omega_d t-\phi_0)+\frac{\delta_{st}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}\cos\left(\omega t-\tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]\right)}$$

X_0 and ϕ_0 are determined by applying the initial conditions:

$$\text{I.C.-s}\begin{cases}x(t=0)=x_0\\ \dot{x}(t=0)=\dot{x}_0\end{cases}\rightarrow \text{ we finally get}$$

$$X_0=\sqrt{(x_0-X\cos\phi)^2+\frac{1}{\omega_d^2}(\xi\omega_n x_0+\dot{x}_0-\xi\omega_n X\cos\phi-\omega X\sin\phi)^2}$$

$$\tan\phi_0=\frac{\xi\omega_n x_0+\dot{x}_0-\xi\omega_n X\cos\phi-\omega X\sin\phi}{\omega_d(x_0-X\cos\phi)}$$

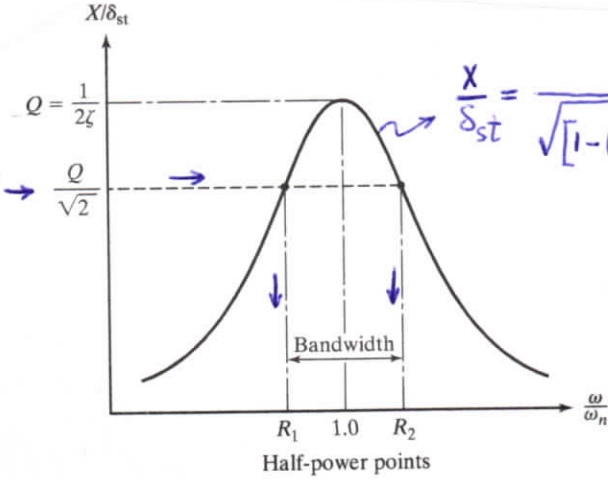
verify!

Quality Factor and Bandwidth: For small values of damping ($\xi<0.05$), we can take

$$\left(\frac{X}{\delta_{st}}\right)_{max}=\frac{1}{2\xi\sqrt{1-\xi^2}}\approx\frac{1}{2\xi}\approx Q$$

quality factor (Q factor)
value of amplitude ratio at resonance

Consider the following harmonic-response curve $\left(\frac{X}{\delta_{st}} \text{ vs. } \frac{\omega}{\omega_n}\right)$:



Consider → $\frac{Q}{\sqrt{2}}$

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

The points R_1 and R_2, where the amplification factor falls to $\frac{Q}{\sqrt{2}}$ are called 'half-power points'. The difference between the frequencies associated with the half-power points R_1 and R_2, i.e. ω_1 and ω_2, respectively, is called the 'bandwidth' of the system.

$$\left\{\begin{array}{l} R_1 \rightarrow \omega_1 \\ R_2 \rightarrow \omega_2 \end{array}\right. \rightarrow \text{bandwidth of the system} = \omega_2 - \omega_1$$

To find the values of R_1 and R_2 → $\frac{X}{\delta_{st}} = \frac{\overset{\frac{1}{2\zeta}}{Q}}{\sqrt{2}}$ ⟶

$$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{1}{2\sqrt{2}\zeta} \xrightarrow{\text{(verify!)}} \left\{\begin{array}{l} r_1^2 = 1-2\zeta^2-2\zeta\sqrt{1+\zeta^2} \\ r_2^2 = 1-2\zeta^2+2\zeta\sqrt{1+\zeta^2} \end{array}\right. \xrightarrow[\text{of } \zeta]{\text{for small values}}$$

$$\left\{\begin{array}{l} r_1^2 = R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1-2\zeta \\ r_2^2 = R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1+2\zeta \end{array}\right.$$

; Consider

$$\omega_2^2-\omega_1^2 = \overbrace{(\omega_2+\omega_1)}^{2\omega_n}(\omega_2-\omega_1) = (R_2^2-R_1^2)\omega_n^2 \approx 4\zeta\omega_n^2$$

$$\frac{1}{2}\left(\frac{\omega_1}{\omega_n}+\frac{\omega_2}{\omega_n}\right) = 1 \rightarrow \omega_2+\omega_1 = 2\omega_n$$

→ $\omega_2-\omega_1 \approx 2\zeta\omega_n$ → bandwidth!

$$\rightarrow \frac{\omega_2-\omega_1}{\omega_n} \approx 2\xi \rightarrow Q \approx \frac{1}{2\xi} \approx \frac{\omega_n}{\omega_2-\omega_1} \rightarrow Q/\text{quality factor!}$$

– Example: Find the total response of a single-degree-of-freedom system with $m = 10$ kg, $C = 20$ N·s/m, $k = 4000$ N/m, $x_0 = 0.01$ m, and $\dot{x}_0 = 0$ under the following conditions:

(a) An external force $F(t) = F_0 \cos\omega t$ acts on the system with $F_0 = 100$ N and $\omega = 10$ rad/s.

(b) Free vibration with $F(t) = 0$.

Solution:

(a) We know that $x(t) = X_0 e^{-\xi\omega_n t}\cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi)$

$$\left.\begin{aligned} \omega_n &= \sqrt{\frac{4000}{10}} = 20 \text{ rad/s} \\ \xi &= \frac{C}{2\sqrt{km}} = \frac{20}{2\sqrt{4000\times 10}} = 0.05 \end{aligned}\right\} \begin{aligned} \omega_d &= \omega_n\sqrt{1-\xi^2} = 20\sqrt{1-(0.05)^2} \\ \omega_d &= 19.975 \text{ rad/s} \end{aligned}$$

$$\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2+(2r\xi)^2}} = \frac{0.025}{\sqrt{(1-0.5^2)^2+(2\times 0.5\times 0.05)^2}} = 0.03326 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{2\times 0.05\times 0.5}{1-0.5^2}\right) = 3.8141^\circ$$

X_0 and ϕ_0 are found from initial conditions. $\xrightarrow[\text{(verify!)}]{}$ $\begin{cases} X_0 = 0.023297 \text{ m} \\ \phi_0 = 5.586765^\circ \end{cases}$

So, $x(t) = 0.023297 e^{-0.05(20)t} \cos(19.975t - 5.586765^\circ) + 0.03326 \cos(10t - 3.8141^\circ)$

ⓑ We know that $x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0)$

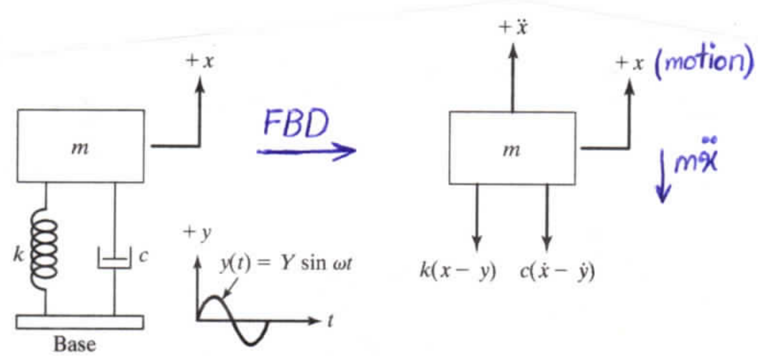
$\zeta = 0.05$
$\omega_n = 20$ rad/s
$\omega_d = 19.975$ rad/s

X_0 and ϕ_0 are found from initial conditions. $\xrightarrow[\text{(verify!)}]{}$ $\begin{cases} X_0 = 0.010012 \text{ m} \\ \phi_0 = -2.865984^\circ \end{cases}$

So, $x(t) = 0.010012 e^{-0.05(20)t} \cos(19.975t + 2.865984^\circ)$

* Response of a Damped System Under the Harmonic Motion of the Base

Assume the base or support of the spring-mass-damper system undergoes harmonic motion, as shown. Let $y(t)$ denote the displacement of the base and $x(t)$ the displacement of the mass from its static equilibrium position at time 't'. Then the net elongation of the spring is $x - y$ and the relative velocity between the two ends of the damper is $\dot{x} - \dot{y}$.



From FBD $\rightarrow$ $m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$ $\Rightarrow$ EOM

If $y(t) = Y\sin\omega t \rightarrow m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky = kY\sin\omega t + cY\omega\cos\omega t$

$$m\ddot{x} + c\dot{x} + kx = A\sin(\omega t - \alpha)$$

$$A = Y\sqrt{k^2 + (c\omega)^2} \qquad \alpha = \tan^{-1}\left(-\frac{c\omega}{k}\right)$$

This shows that giving excitation to the base is equivalent to applying a harmonic force of magnitude A to the mass!

Reminder: For $m\ddot{x} + c\dot{x} + kx = F_0\cos\omega t \rightarrow x_p(t) = X\cos(\omega t - \phi)$

where

$$X = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

similarly

For $m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2 + (c\omega)^2}\,\sin\left(\omega t - \tan^{-1}\left[-\frac{c\omega}{k}\right]\right)$

$$x_p(t) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}\sin(\omega t - \phi_1 - \alpha)$$

$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

So, we may write $\boxed{x_p(t) = X\sin(\omega t - \phi)}$, where

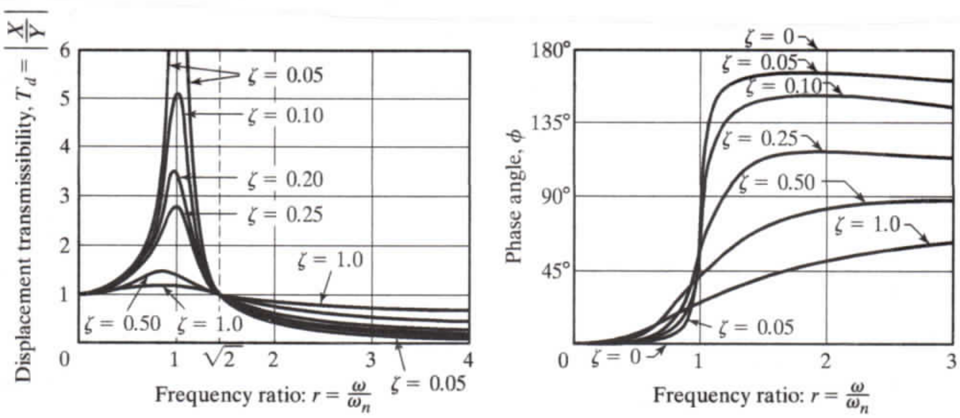
$$\boxed{\frac{X}{Y} = T_d \,\text{(displacement transmissibility)} = \sqrt{\frac{k^2+(c\omega)^2}{(k-m\omega^2)^2+(c\omega)^2}} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}}$$

and

$$\boxed{\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k-m\omega^2)+(c\omega)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right]}$$

Note: $r = \frac{\omega}{\omega_n}$

The variations of $\frac{X}{Y} = T_d$ and ϕ are shown below for different values of r and ζ.



Force Transmitted: In the spring-mass-damper system under the harmonic motion of the base, a force F is transmitted to the base or support due to the reactions from the spring and the dashpot. This force can be determined as:

$$F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}$$

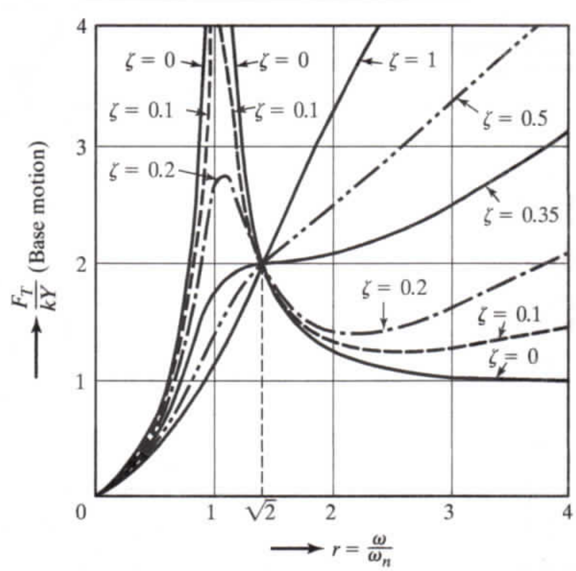
By considering $x_p(t) = X\sin(\omega t - \phi) \rightarrow \underbrace{F = m\omega^2 X\sin(\omega t-\phi)}_{(\text{how?})} = F_T \sin(\omega t - \phi)$

amplitude or maximum value of the force transmitted to the base

So, $F_T = m\omega^2 X$ and $X = Y\sqrt{\dfrac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}} \rightarrow \boxed{\underbrace{\dfrac{F_T}{kY}}_{\text{force transmissibility}} = r^2\sqrt{\dfrac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}}$

Note: $k = m\omega_n^2$ and $r = \dfrac{\omega}{\omega_n}$

The variation of the force transmitted to the base with the frequency ratio r is shown below for different values of ξ.



Relative Motion: If $z = x - y$ denotes the motion of the mass relative to the base, the EOM can be rewritten as:

$$\left.\begin{array}{l} m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0 \\ \text{where } y(t) = Y\sin\omega t \end{array}\right\} \rightarrow \underbrace{m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y\sin\omega t}_{\text{EOM}}$$

The steady-state solution of the EOM is given by:

$$z(t) = \frac{m\omega^2 Y\sin(\omega t - \phi_1)}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = Z\sin(\omega t - \phi_1)$$

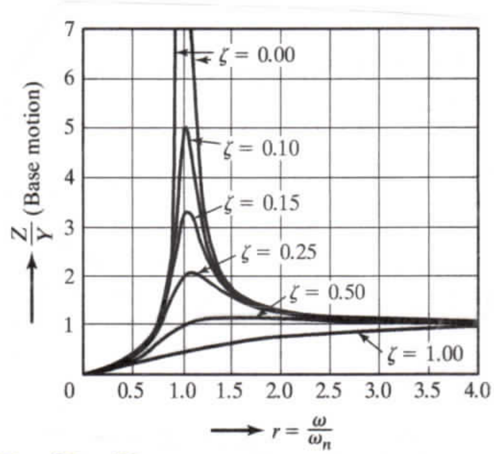
in which

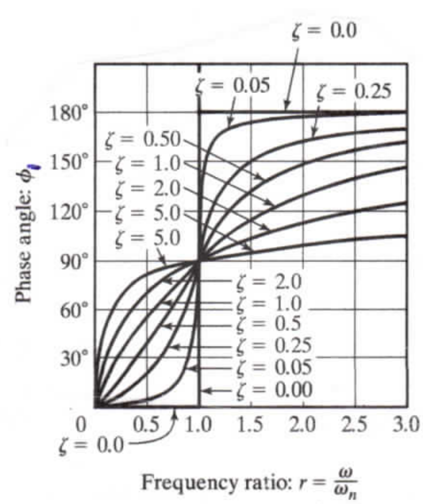
$$Z = \frac{m\omega^2 Y}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = Y\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \left(r = \frac{\omega}{\omega_n}\right)$$

and

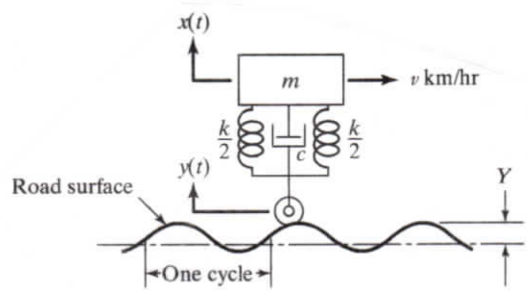
$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

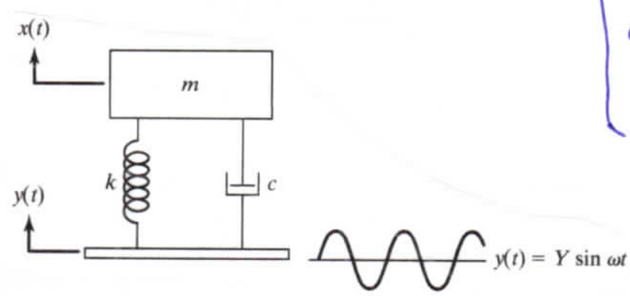
The ratio $\frac{Z}{Y}$ is shown graphically below. The variation of ϕ_1 with r and ζ is also shown below.





– Example: The following figure showes a simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road. The vehicle has a mass of 1200 kg. The suspension system has a spring constant of 400 kN/m and a damping ratio of $\xi = 0.5$. If the vehicle speed is 20 km/hr, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of $Y = 0.05$ m and a wavelength of 6 m.





$$\begin{cases} m = 1200 \text{ kg} \\ k = 400 \text{ kN/m} \\ \xi = 0.5 \\ v = 20 \text{ km/hr} \\ Y = 0.05 \text{ m} \\ \text{wavelength} = 6 \text{ m} \\ X = ? \end{cases}$$

$\frac{v \;\; (m/s)}{\text{length of one cycle of road roughness} \;\; (m)}$

Solution:

Frequency of base excitation: $\omega = 2\pi f = 2\pi \left(\frac{20 \times 1000}{3600}\right) \frac{1}{6}$

$\omega = 5.81778$ rad/sec

Natural frequency of vehicle: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400 \times 10^3}{1200}} = 18.2574$ rad/sec

Frequency ratio: $r = \frac{\omega}{\omega_n} = \frac{5.81778}{18.2574} = 0.318653$

Displacement amplitude of the vehicle: $X = Y\sqrt{\dfrac{1+(2\xi r)^2}{(1-r^2)^2+(2\xi r)^2}}$

This indicates that a 5-cm bump in the road is transmitted as a 5.5-cm bump to the chassis and the passengers of the car.

$\underline{\underline{X = 0.055048 \text{ m}}}$ (verify!)

– Example: A heavy machine, weighing 3000 N, is supported on a resilient foundation. The static deflection of the foundation due to the weight of the machine is found to be 7.5 cm. It is observed that the machine vibrates with an amplitude of 1 cm when the base of the foundation is subjected to harmonic oscillation at the undamped natural frequency of the system with an amplitude of 0.25 cm. Find

a) the damping constant of the foundation,

b) the dynamic force amplitude on the base, and

c) the amplitude of the displacement of the machine relative to the base.

$$\begin{cases} W = 3000 \text{N} \\ \delta_{st} = 7.5 \text{cm} \\ X = 1 \text{ cm} \\ Y = 0.25 \text{cm} \\ \omega = \omega_n \rightarrow r = 1 \end{cases} \rightarrow \begin{cases} c = ? \\ F_T = ? \\ Z = ? \end{cases}$$

Solution: next page

ⓐ $k = \frac{W}{\delta_{st}} = \frac{3000}{0.075} = 40000 \text{ N/m}$

For $r = 1 \rightarrow \frac{X}{Y} = \sqrt{\frac{1+(2\xi)^2}{(2\xi)^2}} \rightarrow \frac{0.01}{0.0025} = 4 = \sqrt{\frac{1+(2\xi)^2}{(2\xi)^2}} \rightarrow \xi = 0.1291$

$c = \xi c_c = \xi 2\sqrt{km} = 0.1291 \times 2 \times \sqrt{40000 \times \frac{3000}{9.81}} \rightarrow \underline{\underline{c = 903.0512 \ \frac{\text{N}\cdot\text{s}}{\text{m}}}}$

ⓑ For $r = 1 \rightarrow \underbrace{F_T = kY\sqrt{\frac{1+4\xi^2}{4\xi^2}} = kX}_{\text{(verify!)}} = 40000 \times 0.01 = \underline{\underline{400 \text{ N}}}$

ⓒ For $r = 1 \rightarrow Z = \frac{Y}{2\xi} = \frac{0.0025}{2 \times 0.1291} = \underline{\underline{0.00968 \text{ m}}}$

Note: $Z \neq X - Y \Rightarrow$ This is due to the phase differences between $x, y,$ and z.

* Forced Motion with Other Types of Damping

Viscous damping is the simplest form of damping to use in practice, since it leads to linear equations of motion. In the cases of other forms of damping, e.g. Coulomb, hysteretic, etc., equivalent viscous-damping coefficients are defined in order to simplify the analysis.

- **Energy Dissipated in Viscous Damping:** In a viscously-damped system the rate of change of energy with time (dW/dt) is given by:

viscous damping force

damping force is opposite to the direction of velocity

$F = -cv$

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c\left(\frac{dx}{dt}\right)^2$$

The negative sign denotes that energy dissipates with time.

Assume a simple harmonic motion as $x(t) = X \sin \omega_d t$. The energy dissipated in a complete cycle is given by:

$$\Delta W = \int_{t=0}^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi} c X^2 \omega_d \cos^2 \omega_d t \cdot d(\omega_d t) = \pi c \omega_d X^2$$

So, $\underline{\underline{\Delta W = \pi c \omega_d X^2}}$

- As an example, consider a 'quadratic' or 'velocity-squared' damping with a damping force:

$$F_d = \pm a(\dot{x})^2 \quad \text{where} \begin{cases} a: \text{a constant} \\ \dot{x}: \text{relative velocity across the damper} \\ \text{negative (positive) sign must be used when } \dot{x} \text{ is positive (negative)} \end{cases}$$

The energy dissipated per cycle during harmonic motion $x(t) = X \sin \omega t$ is given by:

$$\Delta W = 2\int_{-x}^{x} a(\dot{x})^2 dx = 2X^3 \int_{-\pi/2}^{\pi/2} a\omega^2 \cos^3 \omega t \, d(\omega t) = \frac{8}{3}\omega^2 a X^3$$

The equivalent viscous-damping coefficient (c_{eq}) corresponding to quadratic damping can be determined by equating $\frac{8}{3}\omega^2 a X^3$ to the energy dissipated in an equivalent viscous damper, i.e.

$$\frac{8}{3}\omega^2 a X^3 = \pi c_{eq}\omega X^2 \rightarrow \boxed{c_{eq} = \frac{8}{3\pi} a\omega X}$$

It is noted that c_{eq} is not a constant but varies with ω and X.

The amplitude of the steady-state response can be found from:

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\xi_{eq} r)^2}} \quad \text{where} \quad r = \frac{\omega}{\omega_n} \quad \text{and} \quad \xi_{eq} = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{2m\omega_n}$$

so, we can conclude that $X = \dfrac{3\pi m}{8ar^2}\sqrt{-\dfrac{(1-r^2)^2}{2} + \sqrt{\dfrac{(1-r^2)^4}{4} + \left(\dfrac{8ar^2\delta_{st}}{3\pi m}\right)^2}}$.

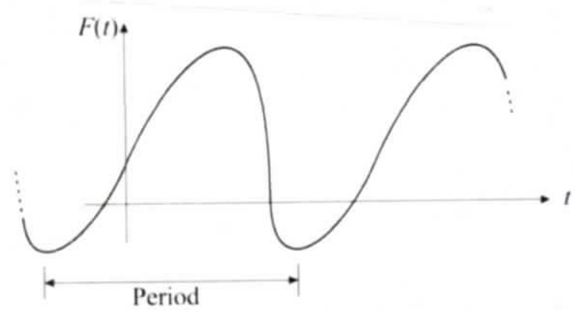
Please find the amplitude at resonance!

* Introduction

This chapter covers the vibration response of a single-degree-of-freedom system under arbitrary forcing conditions. Many practical systems are subjected to several types of forcing functions that are not harmonic. The general forcing functions may be periodic or nonperiodic. The nonperiodic forces include forces such as a suddenly-applied constant force (called a 'step force'), a linearly-increasing force (called a 'ramp force'), and an exponentially varying force. A nonperiodic forcing may be acting for a short, long, or infinite duration. An example of general forcing function is the ground vibration of a building frame during an earthquake.

It is noted that in forced vibration, the externally-applied force or displacement excitation may be harmonic (discussed in Chapter 3), nonharmonic but periodic, nonperiodic, or random in nature. Harmonic motion is a particular type of periodic motion. We may think of harmonic motion as a periodic motion that is represented by sine and cosine functions. However, the external loading may be periodic but not harmonic. In such a case, there exists a finite period of time after which the waveform is not representable by a single sine or cosine, a Fourier series can represent such periodic functions in terms of summation of an infinite number of sines and cosines.

In the following figure, a periodic but not harmonic function is shown. This case, i.e. periodic but not harmonic Loading, will be discussed in this chapter.



Periodic but not harmonic function

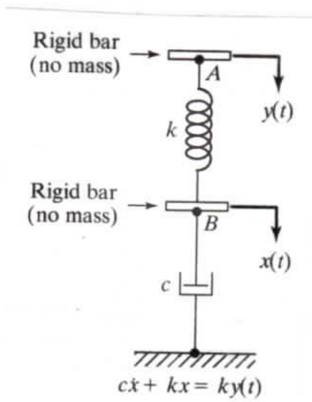
* Response Under a General Periodic Force

When an external force $F(t)$ is periodic with period $\tau = 2\pi/\omega$, it can be expanded in a Fourier series:

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \quad \text{where} \begin{cases} a_j = \frac{2}{\tau}\int_0^{\tau} F(t)\cos j\omega t\, dt \\ \qquad j = 0, 1, 2, \ldots \\ b_j = \frac{2}{\tau}\int_0^{\tau} F(t)\sin j\omega t\, dt \\ \qquad j = 1, 2, \ldots \end{cases}$$

In this chapter, we consider the responses of first- and second-order systems under general periodic forces. First-order systems are those for which the EOM is a first-order differential equation, and similarly, second-order systems are those for which the EOM is a second-order differential equation.

First-Order Systems: Consider the following spring-damper system subjected to a periodic excitation. The EOM of the system is given by:



$$c\dot{x} + k(x-y) = 0$$

where

$y(t)$: periodic motion (or excitation) imparted to the system at point A.

If the periodic displacement at point A, $y(t)$, is expressed in Fourier series, the EOM of the system can be expressed as:

$$\dot{x} + ax = ay = A_0 + \sum_{j=1}^{\infty} A_j \sin\omega_j t + \sum_{j=1}^{\infty} B_j \cos\omega_j t$$

where

$$a = \frac{k}{c} \;;\; A_0 = \frac{a a_0}{2} \;;\; A_j = a a_j \;;\; B_j = a b_j \;;\; \omega_j = j\omega$$

$j = 1, 2, 3, \ldots$

As seen, the right-hand side of the EOM is a constant plus a linear sum of harmonic (sine and cosine) functions. Using the principle of superposition, the steady-state solution of the EOM can be found by summing the steady-state solutions corresponding to the individual forcing terms on the right-hand side of the equation. So,

① $\dot{x}_0 + a x_0 = A_0$ (Note: x_0 is used for x) $\rightarrow$ Solution: $x_0(t) = \frac{A_0}{a}$

② $\dot{x}_j + a x_j = A_j \sin \omega_j t \longrightarrow$ Steady-state solution: $x_j(t) = X_j \sin(\omega_j t - \phi_j)$

where

$$X_j = \frac{A_j}{\sqrt{a^2 + \omega_j^2}} \; ; \; \phi_j = \tan^{-1}\left(\frac{\omega_j}{a}\right)$$

③ $\dot{x}_j + a x_j = B_j \cos \omega_j t \rightarrow$ Steady-state solution: $x_j(t) = Y_j \cos(\omega_j t - \phi_j)$

where

$$Y_j = \frac{B_j}{\sqrt{a^2 + \omega_j^2}} \; ; \; \phi_j = \tan^{-1}\left(\frac{\omega_j}{a}\right)$$

The complete steady-state (or particular) solution of the EOM can be expressed as:

$$x_p(t) = \frac{A_0}{a} + \sum_{j=1}^{\infty} \frac{A_j}{\sqrt{a^2+\omega_j^2}} \sin\left\{\omega_j t - \tan^{-1}\left(\frac{\omega_j}{a}\right)\right\} + \sum_{j=1}^{\infty} \frac{B_j}{\sqrt{a^2+\omega_j^2}} \cos\left\{\omega_j t - \tan^{-1}\left(\frac{\omega_j}{a}\right)\right\}$$

The total solution of the EOM is given by: $x(t) = x_h(t) + x_p(t)$.

The homogeneous solution can be expressed as: $x_h(t) = Ce^{-at}$ where C is an unknown constant to be determined using I.C.s.

So, $x(t) = Ce^{-at} + \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin(\omega_j t - \phi_j) + \sum_{j=1}^{\infty} Y_j \cos(\omega_j t - \phi_j)$

If $x(t=0) = x_0 \rightarrow C = x_0 - \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin\phi_j - \sum_{j=1}^{\infty} Y_j \cos\phi_j$

Finally, the total solution of the EOM of the first-order system is:

$$x(t) = \left[x_0 - \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin\phi_j - \sum_{j=1}^{\infty} Y_j \cos\phi_j\right] e^{-at} + \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin(\omega_j t - \phi_j) + \sum_{j=1}^{\infty} Y_j \cos(\omega_j t - \phi_j)$$

- Example: Determine the response of a spring-damper system with the EOM: $\dot{x} + 1.5x = 7.5 + 4.5\cos t + 3\sin 5t$. Assume $x(t=0) = 0$.

Solution: Consider the EOM: $\dot{x} + 1.5x = 7.5 + 4.5\cos t + 3\sin 5t$

we first find the solution of the differential equation by considering one forcing term at a time given on the right-hand side of the EOM, and then adding the solutions to find the total solution of the EOM.

① $\dot{x} + 1.5x = 7.5 \rightarrow$ Solution: $x(t) = \frac{7.5}{1.5} = 5.$

② $\dot{x} + 1.5x = 4.5\cos t \rightarrow$ solution: $x(t) = Y\cos(t - \phi)$

where

$$Y = \frac{4.5}{\sqrt{1.5^2 + 1.0^2}} = 2.4961$$

$$\phi = \tan^{-1}\left(\frac{1.0}{1.5}\right) = 0.5880 \text{ rad}$$

③ $\dot{x} + 1.5x = 3\sin 5t \rightarrow$ Solution: $x(t) = X\sin(5t - \phi)$

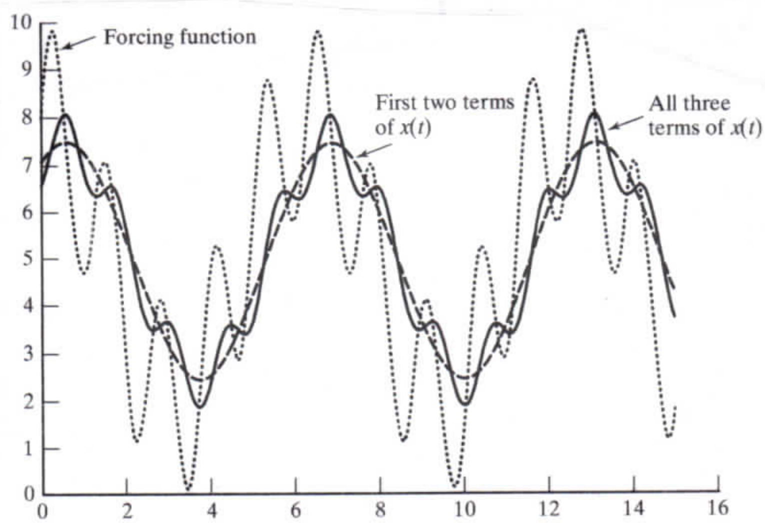
where

$$X = \frac{3.0}{\sqrt{1.5^2 + 5^2}} = 0.5747$$

$$\phi = \tan^{-1}(5/1.5) = 1.2793 \text{ rad}$$

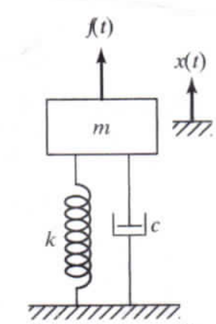
so, $\underline{\underline{x(t) = 5 + 2.4961\cos(t - 0.5880) + 0.5747\sin(5t - 1.2793)}}$

The forcing function given by the right-hand-side expression in EOM and the steady-state response of the system, $x(t)$, are shown graphically in the following figure. The first two terms of the response (given by the first two terms on the right-hand-side of $x(t)$) are also shown in the figure.



Second-Order Systems:

Consider the following spring-mass-damper system subjected to a periodic force. This is a second-order system because the governing equation is a second-order differential equation:



$m\ddot{x} + c\dot{x} + kx = f(t)$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

If the forcing function $f(t)$ is periodic, it can be expressed in Fourier series so that the equation of motion becomes:

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t}$$

The right-hand side of the EOM is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution is the sum of the steady-state solutions of the following equations:

① $m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \rightarrow$ solution: $x_p(t) = \frac{a_0}{2k}$

② $m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t \rightarrow$ solution: $x_p(t) = \frac{(a_j/k)}{\sqrt{(1-j^2r^2)^2 + (2\xi jr)^2}} \cos(j\omega t - \phi_j)$

③ $m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t \rightarrow$ solution: $x_p(t) = \frac{(b_j/k)}{\sqrt{(1-j^2r^2)^2 + (2\xi jr)^2}} \sin(j\omega t - \phi_j)$

where $\phi_j = \tan^{-1}\left(\frac{2\xi jr}{1-j^2r^2}\right)$ and $r = \frac{\omega}{\omega_n}$

The complete steady-state solution of the EOM is given by:

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1-j^2r^2)^2 + (2\xi jr)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1-j^2r^2)^2 + (2\xi jr)^2}} \cdot \sin(j\omega t - \phi_j)$$

The transient part of the total solution arising from the initial conditions can also be included to find the total solution. To find the total solution, we need to evaluate the arbitrary constants by setting the value of the total solution and its derivative equal to the specified values of initial displacement $x(t=0)=x_0$ and the initial velocity $\dot{x}(t=0)=\dot{x}_0$.

- Example: (Total Response Under Harmonic Base Excitation)

Find the total response of a viscously-damped SDOF system subjected to a harmonic base excitation for the following data: $m=10$ kg, $c=20$ N.s/m, $k=4000$ N/m, $y(t)=0.05\sin 5t$ m, $x_0=0.02$ m, and $\dot{x}_0=10$ m/s.

Solution: The EOM of the system is given by:

$$m\ddot{x}+c\dot{x}+kx=ky+c\dot{y}=kY\sin\omega t+c\omega Y\cos\omega t \quad \text{(why?)}$$

This equation is similar to the following equation:

$$m\ddot{x}+c\dot{x}+kx=\frac{a_0}{2}+\sum_{j=1}^{\infty}a_j\cos j\omega t+\sum_{j=1}^{\infty}b_j\sin j\omega t$$

with $a_0=0$, $a_1=c\omega Y$, $b_1=kY$, and $a_j=b_j=0$ for $j=2,3,\ldots$

so, the steady-state response of the system is:

$$x_p(t)=\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\left[\frac{a_1}{k}\cos(\omega t-\phi_1)+\frac{b_1}{k}\sin(\omega t-\phi_1)\right] \quad \text{(why?)}$$

We can find the following:

$$Y = 0.05\,m,\quad \omega = 5\ rad/s,\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\ rad/s$$

$$r = \frac{\omega}{\omega_n} = \frac{5}{20} = 0.25,\quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{4000\times 10}} = 0.05$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 19.975\ rad/s$$

$$a_1 = c\omega Y = 20\times 5\times 0.05 = 5,\quad b_1 = kY = 4000\times 0.05 = 200$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = 0.02666\ rad,\quad \sqrt{(1-r^2)^2+(2\zeta r)^2} = 0.937833$$

The homogeneous solution is:

$$x_h(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) = X_0 e^{-t}\cos(19.975t - \phi_0)$$

The total solution is: $x(t) = x_h(t) + x_p(t)$

$$x(t) = X_0 e^{-t}\cos(19.975t - \phi_0) + \frac{1}{0.937833}\left[\frac{5}{4000}\cos(5t-\phi_1) + \frac{200}{4000}\sin(5t-\phi_1)\right]$$

$$x(t) = X_0 e^{-t}\cos(19.975t - \phi_0) + 0.001333\cos(5t - 0.02666) + 0.053314\sin(5t - 0.02666)$$

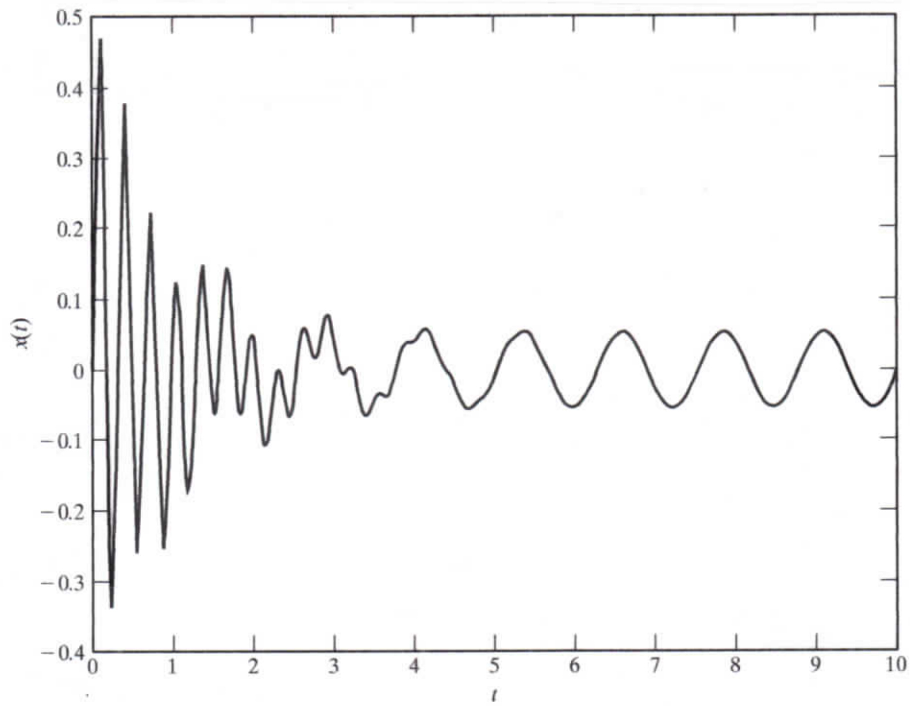
In this equation, X_0 and ϕ_0 can be found from the initial conditions:

$$I.C.\text{-s}\begin{cases} x_0 = 0.02\,m \\ \dot{x}_0 = 10\ m/s \end{cases} \rightarrow \begin{cases} X_0 = 0.488695 \\ \phi_0 = 1.529683 \end{cases}\ \text{(verify!)}$$

Thus, the total response of the mass under base excitation, in meters, is:

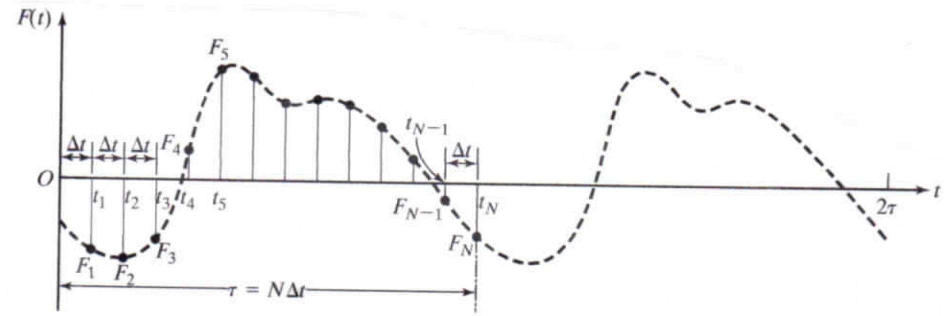
$$\underline{\underline{x(t) = 0.488695 e^{-t}\cos(19.975t - 1.529683) + 0.001333\cos(5t - 0.02666) + 0.053314\sin(5t - 0.02666)}}$$

The total response of the viscously-damped system subject to harmonic base excitation is plotted in the following.



* Response Under a Periodic Force of Irregular Form

In some cases, the force acting on a system may be quite irregular and may be determined only experimentally. Examples of such forces include wind and earthquake-induced forces. In such cases, the forces will be available in graphical form and no analytical expression can be found to describe $F(t)$. Sometimes, the value of $F(t)$ may be available only at a number of discrete points $t_1, t_2, \ldots, t_N$. In all these cases, it is possible to find the Fourier coefficients by using a numerical integration procedure. If $F_1, F_2, \dots, F_N$ denote the values of $F(t)$ at $t_1, t_2, \dots, t_N$, respectively, where N denotes an even number of equidistant points in one time period τ $(\tau = N\Delta t)$, as shown below, the application of trapezoidal rule gives:



$$a_0 = \frac{2}{N} \sum_{i=1}^{N} F_i$$

$$a_j = \frac{2}{N} \sum_{i=1}^{N} F_i \cos \frac{2j\pi t_i}{\tau} \qquad j = 1, 2, \dots$$

$$b_j = \frac{2}{N} \sum_{i=1}^{N} F_i \sin \frac{2j\pi t_i}{\tau} \qquad j = 1, 2, \dots$$

Once the Fourier coefficients a_0, a_j, b_j are known, the steady-state response of the system can be found using the following equation. (Note: $r = 2\pi / \tau\omega_n$)

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1-j^2r^2)^2 + (2\xi jr)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1-j^2r^2)^2 + (2\xi jr)^2}} \sin(j\omega t - \phi_j)$$

* Response Under a Nonperiodic Force

Periodic forces of any general waveform can be represented by Fourier series as a superposition of harmonic components of various frequencies. The response of a linear system is then found by superposing the harmonic response to each of the exciting forces. When the exciting force $F(t)$ is nonperiodic, such as that due to the blast from an explosion, a different method of calculating the response is required. Various methods can be used to find the response of the system to an arbitrary excitation.

Some of the methods are as follows:

1. Representing the excitation by a Fourier integral.
2. Using the method of convolution integral.
3. Using the method of Laplace transforms.
4. Numerically integrating the equations of motion (numerical solution of differential equations).

We will discuss method 2 in the following section.

* Convolution Integral

A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period and then stops. The simplest form is the 'impulsive force', a force that has a large magnitude F and acts for a very short time Δt. Impulse can be measured by finding the change it causes in momentum of the system. If $\dot{x}_1$ and $\dot{x}_2$ denote the velocities of the mass 'm' before and after the application of the impulse, we have

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1$$

By designating the magnitude of the impulse $F\Delta t$ by $\hat{F}$, we can write

$$\hat{F} = \int_t^{t+\Delta t} F\,dt$$ → Note: Impulse is the time integral of the force, and we designate it by the notation $\hat{F}$.

A unit impulse acting at $t=0$ ($\hat{f}$) is defined as

$$\hat{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\,dt = F\,dt = 1$$ → It should be noted that in order for $F\,dt$ to have a finite value, F tends to infinity (since dt tends to zero).

The unit impulse, $\hat{f}=1$, acting at $t=0$, is also denoted by the 'Dirac delta function' as:

$$\hat{f} = \hat{f}\delta(t) = \delta(t)$$

and the impulse of magnitude $\hat{F}$, acting at $t=0$, is denoted as

$$\hat{F} = \hat{F}\delta(t)$$

Additional Explanation

- We frequently encounter a force of very large magnitude that acts for a very short time but with a time integral that is finite. such forces are called 'impulsive'.
- Impulse is the time integral of the force: $\hat{F} = \int F(t)\,dt$
- When $\hat{F}$ is equal to unity, such a force in the limiting case $\Delta t \to 0$ is called the 'unit impulse' or the 'delta function'.
- The unit impulse, $\hat{f}$, acting at $t=0$, is also denoted by the 'Dirac delta function', $\delta(t)$. The Dirac delta function at time $t=\tau$, denoted as $\delta(t-\tau)$, has the following properties:

$$\delta(t-\tau) = 0 \text{ for } t \neq \tau; \rightarrow \delta(t-\tau) = \begin{cases} \infty & \text{if } t=\tau \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^\infty \delta(t-\tau)\,dt = 1.0$$

$$\int_0^\infty \delta(t-\tau)F(t)\,dt = F(\tau)$$

Further information/description is provided at the end of the lecture notes. (See Appendix 1)

where $0 < \tau < \infty$. Thus, an impulse of magnitude $\hat{F}$, acting at time $t=\tau$, can be denoted as $F(t) = \hat{F}\delta(t-\tau)$.

– Response to an Impulse:

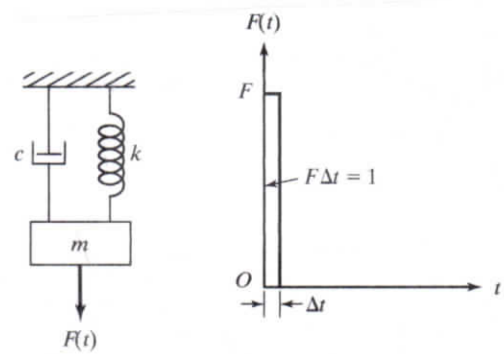
We consider the response of a SDOF system to an impulse excitation. Consider a viscously-damped spring mass system subjected to a unit impulse at $t=0$, as shown below:

For an underdamped system, the solution of the EOM:

$$m\ddot{x}+c\dot{x}+kx=0$$

is given by:

$$x(t)=e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t+\frac{\dot{x}_0+\zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right)$$



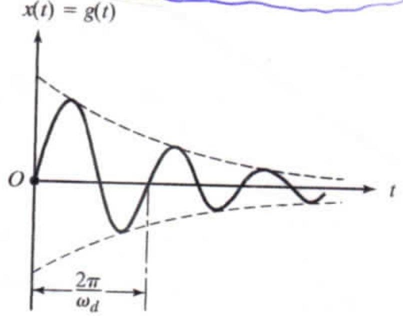
where: $\zeta=\frac{c}{2m\omega_n}$ $\quad \omega_d=\omega_n\sqrt{1-\zeta^2}=\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}$ $\quad \omega_n=\sqrt{\frac{k}{m}}$

If the mass is at rest before the unit impulse is applied ($x=\dot{x}=0$ for $t<0$ or at $t=0^-$), we can write:

$$\text{Impulse}=\hat{f}=1=m\dot{x}(t=0)-m\dot{x}(t=0^-)=m\dot{x}_0 \longrightarrow \text{so, I.C.s}\begin{cases}x(t=0)=x_0=0\\ \dot{x}(t=0)=\dot{x}_0=\frac{1}{m}\end{cases}$$

$$x(t)=g(t)=\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

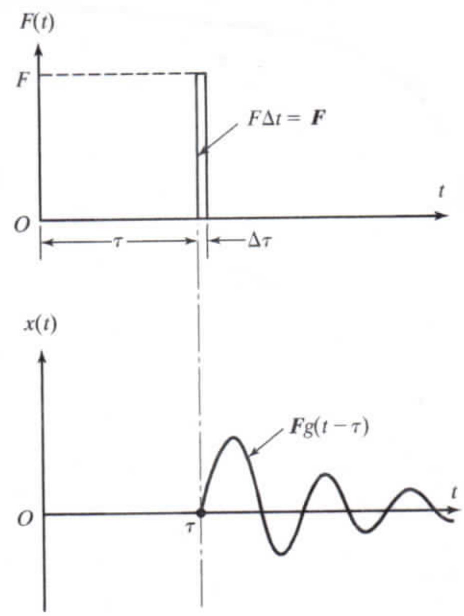
$\Rightarrow$ response of a SDOF system to a unit impulse, which is also known as 'impulse response function', denoted by $g(t)$. The function $g(t)$ is shown here.



If the magnitude of the impulse is $\hat{F}$ instead of unity, then initial velocity $\dot{x}_0 = \hat{F}/m$ and the response of the system becomes:

$$x(t) = \frac{\hat{F} e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t = \hat{F} g(t) \qquad \text{(why?)}$$

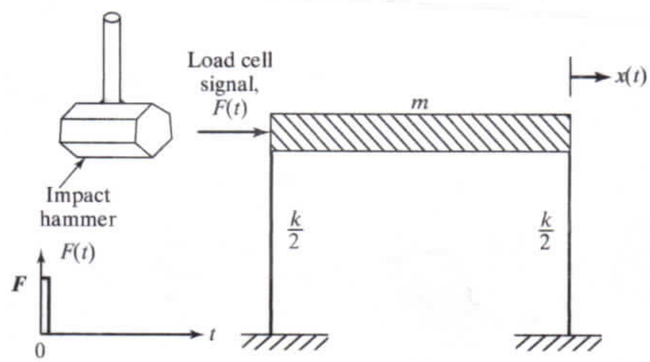
If the impulse $\hat{F}$ is applied at an arbitrary time $t=\tau$, as shown below, it will change the velocity at $t=\tau$ by an amount $\hat{F}/m$. Assuming that $x=0$ until the impulse is applied, the displacement 'x' at any subsequent time 't', caused by a change in the velocity at time τ, is give by the above-mentioned equation with 't' replaced by the time elapsed after the application of the impulse, i.e. $t-\tau$. So, we get:



shown

$$x(t) = \frac{\hat{F} e^{-\zeta\omega_n (t-\tau)}}{m\omega_d} \sin\omega_d (t-\tau) = \hat{F} g(t-\tau)$$

- Example: In the vibration testing of a structure, an impact hammer with a load cell to measure the impact force is used to cause excitation, as shown. Assuming $m = 5\,kg$, $k = 2000\,N/m$, $c = 10\,N.s/m$, and $\hat{F} = 20\,N.s$, find the response of the system.



Solution: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20\ rad/s$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{10}{2(5)(20)} = 0.05$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 19.975\ rad/s$$

Assuming that the impact is given at $t = 0$, we find the response of the system as:

$$x_1(t) = \hat{F}\,\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$x_1(t) = \frac{20}{5 \times 19.975}\,e^{-0.05(20)t}\sin 19.975t$$

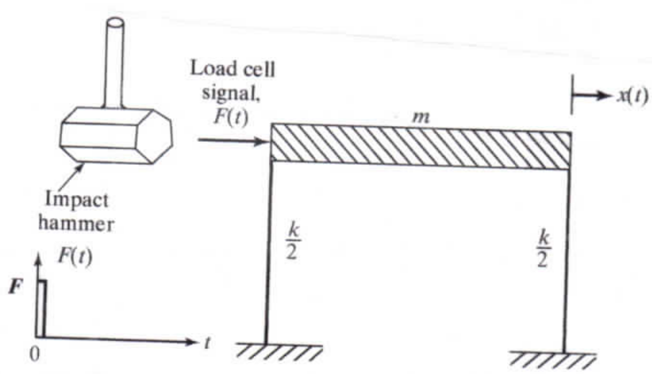
$$\underline{\underline{x_1(t) = 0.20025\,e^{-t}\sin 19.975t\ \ m}}$$

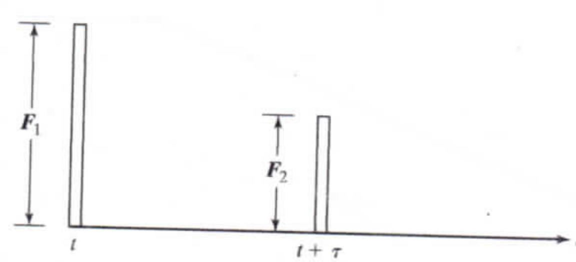
- Example: In many cases, providing only one impact to the structure using an impact hammer is difficult. Sometimes a second impact takes place after the first, as shown below, and the applied force, $F(t)$, can be expressed as:

$$F(t) = \hat{F}_1 \delta(t) + \hat{F}_2 \delta(t-\tau)$$

where $\delta(t)$ is the Dirac delta function and τ indicates the time between the two impacts of magnitudes $\hat{F}_1$ and $\hat{F}_2$.

For a structure with $m = 5\,kg$, $k = 2000\ N/m$, $c = 10\ N{\cdot}s/m$, and $F(t) = 20\delta(t) + 10\delta(t-0.2)\ N$, find the response of the structure.





Solution:
$$\left.\begin{aligned} \omega_n &= 20\ rad/s \\ \zeta &= 0.05 \\ \omega_d &= 19.975\ rad/s \end{aligned}\right\} \text{previous example!}$$

The response due to the impluse $\hat{F}_1\delta(t)$ is given by:

$$x_1(t) = 0.20025\, e^{-t} \sin 19.975t \quad m \quad \rightarrow \text{previous example!}$$

The response due to the impulse $\hat{F}_2\delta(t-0.2)$ can be found from:

$$x_2(t) = \frac{\hat{F}_2\, e^{-\zeta\omega_n(t-\tau)}}{m\omega_d}\sin\omega_d(t-\tau)$$

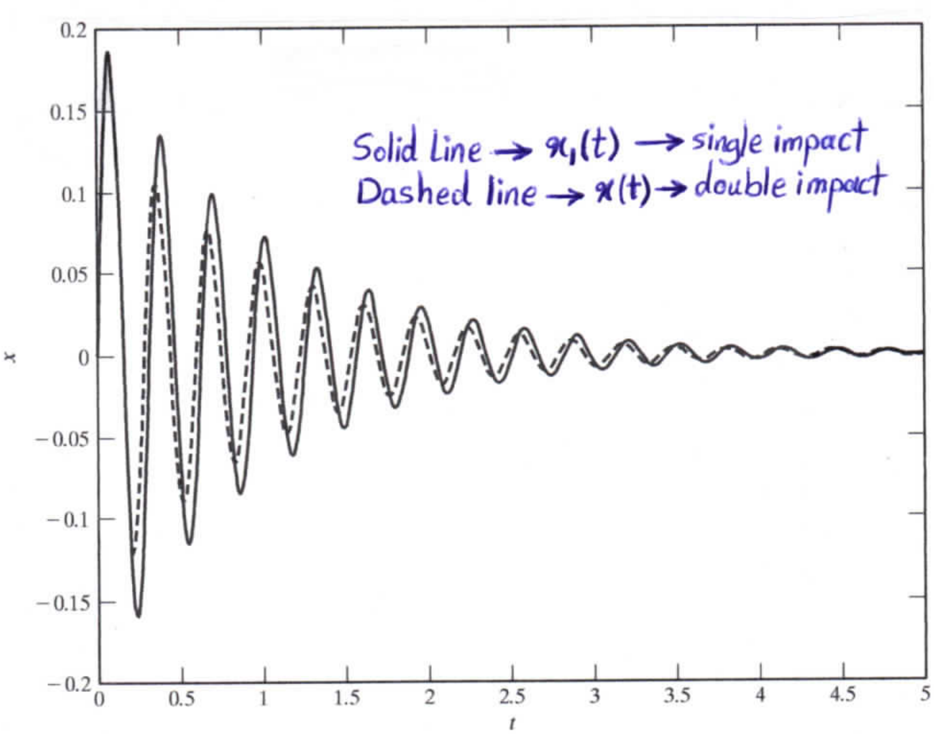
$$x_2(t) = \frac{10\times e^{-0.05(20)(t-0.2)}}{5\times 19.975}\sin 19.975(t-0.2)$$

$$x_2(t) = 0.100125\, e^{-(t-0.2)}\sin 19.975(t-0.2) \quad t>0.2$$

Using the superposition of the two responses $x_1(t)$ and $x_2(t)$, the response due to two impacts, in meters can be expressed as:

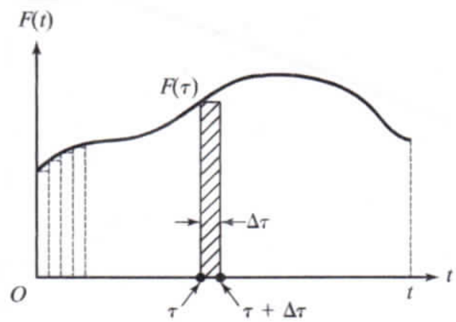
$$\boxed{x(t) = \begin{cases} 0.20025\, e^{-t}\sin 19.975t & 0\le t\le 0.2 \\ 0.20025\, e^{-t}\sin 19.975t + 0.100125\, e^{-(t-0.2)}\sin 19.975(t-0.2) & t>0.2 \end{cases}}$$

The graphs of these equations is shown below:



– Response to a General Forcing Condition:

Here we consider the response of the system under an arbitrary external force $F(t)$, shown below. This force may be assumed to be made up of a series of impulses of varying magnitude.



Assuming that at time 'τ', the fore '$F(\tau)$' acts on the system for a short period of time '$\Delta\tau$', the impulse acting at '$t=\tau$' is given by '$F(\tau)\Delta\tau$'. At any time 't', the elapsed time since the impulse is '$t-\tau$', so the response of the system at 't' due to this impulse alone is given by

$$\Delta x(t) = F(\tau)\Delta\tau\, g(t-\tau) \qquad (\text{Note: } \hat{F} = F(\tau)\Delta\tau)$$

The total response at time 't' can be found by summing all the responses due to the elementary impulses acting at all times 'τ':

$$x(t) \simeq \sum F(\tau)\, g(t-\tau)\, \Delta\tau$$

Letting $\Delta\tau \to 0$ and replacing the summation by integration, we get

$$\boxed{x(t) = \int_0^t F(\tau) g(t-\tau)\, d\tau = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau}$$

which represents the response of an underdamped SDOF system to the arbitrary excitation F(t). Note that this equation does not consider the effect of initial conditions of the system, because the mass is assumed to be at rest before the application of the impulse, as implied by the following equations:

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t \quad \text{and} \quad \Delta x(t) = F(\tau)\Delta\tau\, g(t-\tau) .$$

The integral in the above equation is called the 'convolution or 'Duhamel integral'. In many cases, the function F(t) has a form that permits an explicit integration of the equation. If such integration is not possible, we can evaluate numerically without much difficulty.

– Response to Base Excitation:

If a spring-mass-damper system is subjected to an arbitrary base excitation described by its displacement, velocity, or acceleration, the EOM can be expressed in terms of the relative displacement of the mass $z = x - y$ as follows:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

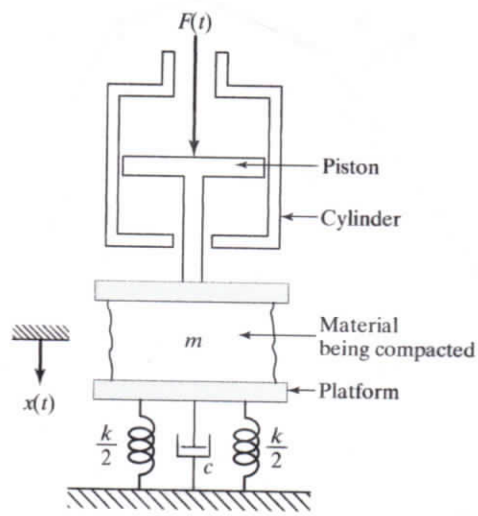
This is similar to the equation:

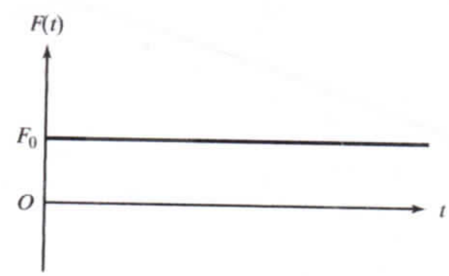
$$m\ddot{x} + c\dot{x} + kx = F$$

with the variable 'z' replacing 'x' and the term '$-m\ddot{y}$' replacing the forcing function 'F'. Hence, all of the results derived for the force-excited system are applicable to the base-excited system also for 'z' when the term 'F' is replaced by '$-m\ddot{y}$'. For an underdamped system subjected to base excitation, the relative displacement can be found from:

$$\boxed{z(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau}$$

- Example: (Step Force) A compacting machine, modeled as a SDOF system, is show below. The force acting on the mass 'm' ('m' includes the masses of the piston, the platform, and the material being compacted) due to a sudden application of the pressure can be idealized as a step force, as shown. Determine the response of the system.





Solution: $F(t) = F_0$

$$x(t) = \frac{F_0}{m\omega_d}\int_0^t e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\,d\tau$$

$$x(t) = \frac{F_0}{m\omega_d}\left[e^{-\zeta\omega_n(t-\tau)} \left\{ \frac{\zeta\omega_n \sin\omega_d(t-\tau) + \omega_d\cos\omega_d(t-\tau)}{(\zeta\omega_n)^2 + (\omega_d)^2} \right\} \right]_{\tau=0}^{t}$$

$$\underline{x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right]}, \text{ where}$$

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}} \right)$$

This response is shown in Fig. E1. If the system is undamped ($\xi = 0$ and $\omega_n = \omega_d$), then we get:

$$x(t) = \frac{F_0}{k}\left[1 - \cos\omega_n t\right]$$

This equation is shown graphically in Fig. E2. It is seen that if the load is instantaneously applied to an undamped system, a maximum displacement of twice the static displacement will be attained, i.e. $x_{max} = 2F_0/k$.

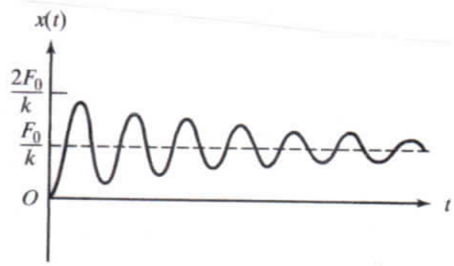


Fig. E1: Damped

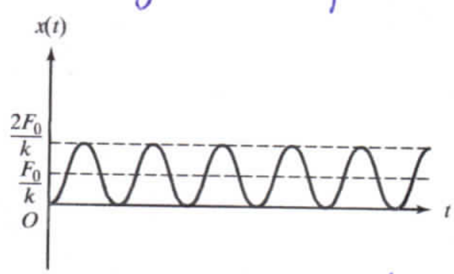
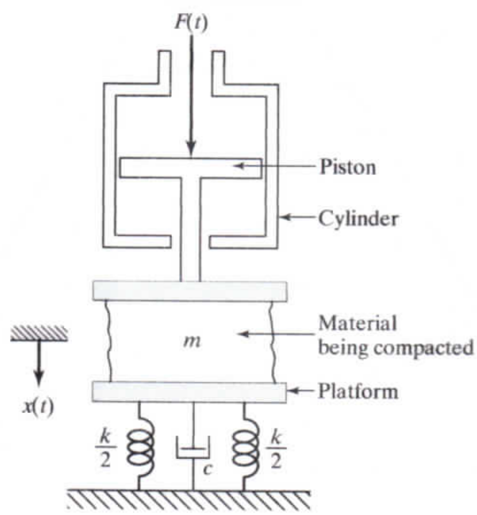
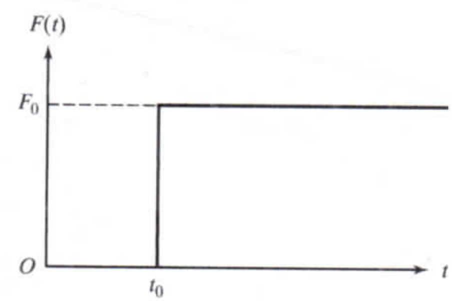


Fig. E2: Undamped

– Example: (Time-Delayed Step Force) Find the response of the compacting machine shown below, when it is subjected to the force shown below.





Solution: Forcing function starts at $t=t_0$ instead of at $t=0$, so the response can be obtained from the equation obtained in previous example, that is

$$x(t)=\frac{F_0}{k}\left[1-\frac{1}{\sqrt{1-\zeta^2}}\cdot e^{-\zeta\omega_n t}\cos\left(\omega_d t-\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right)\right]$$

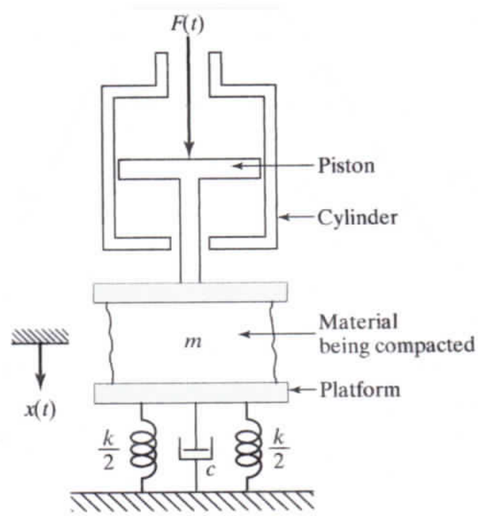
by replacing 't' by '$t-t_0$'. This gives

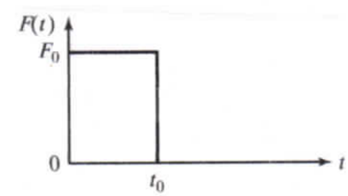
$$x(t)=\frac{F_0}{k\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}-e^{-\zeta\omega_n(t-t_0)}\cos\{\omega_d(t-t_0)-\phi\}\right]$$

If the system is undamped, we get:

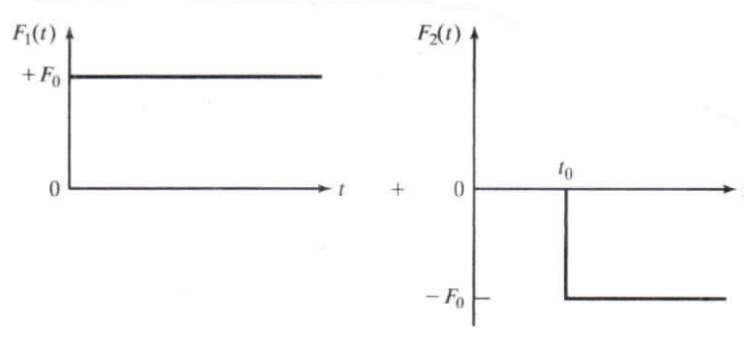
$$x(t)=\frac{F_0}{k}\left[1-\cos\omega_n(t-t_0)\right]$$

- Example: (Rectangular Pulse Load) If the compacting machine shown below is subjected to a constant force only during the time $0 \leq t \leq t_0$ (shown below), determine the response of the machine.





Solution: The given forcing function, $F(t)$, can be considered as the sum of a step function $F_1(t)$ of magnitude '$+F_0$' beginning at $t=0$ and a second step function $F_2(t)$ of magnitude '$-F_0$' starting at time $t=t_0$, as shown below:



Thus, the response of the system can be obtained by:

$$\underbrace{\frac{F_0}{k}\left[1-\frac{1}{\sqrt{1-\zeta^2}}\cdot e^{-\zeta\omega_n t}\cos(\omega_d t-\phi)\right]}_{\text{step force}} - \underbrace{\frac{F_0}{k\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}-e^{-\zeta\omega_n(t-t_0)}\cos\left(\omega_d(t-t_0)-\phi\right)\right]}_{\text{time-delayed step force}}$$

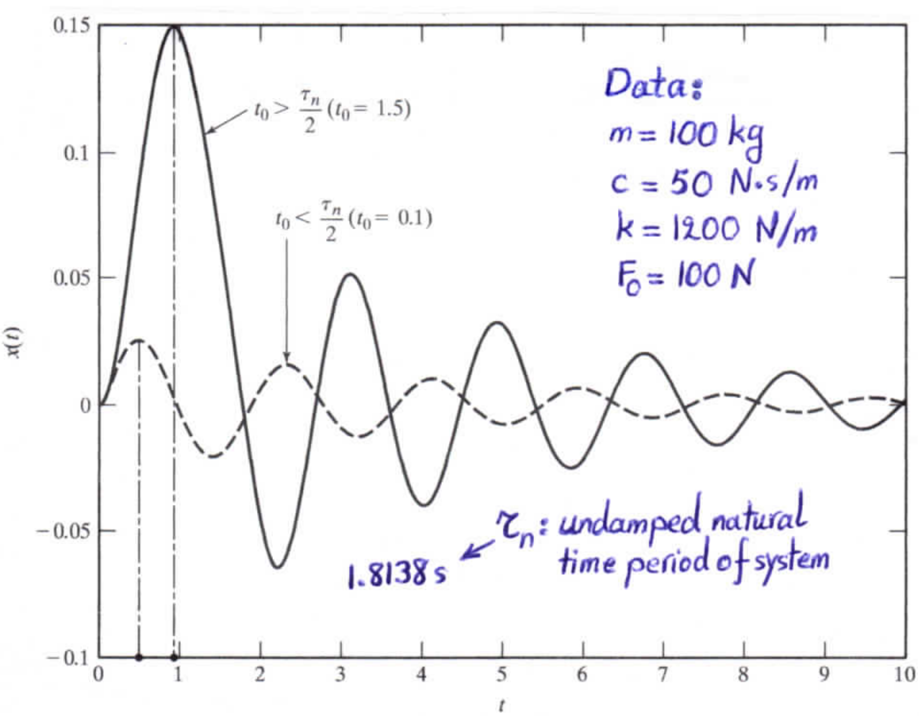
which results in

$$\underline{x(t)=\frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left[-\cos(\omega_d t-\phi)+e^{\zeta\omega_n t_0}\cos\left(\omega_d(t-t_0)-\phi\right)\right]} \quad \text{where } \phi=\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

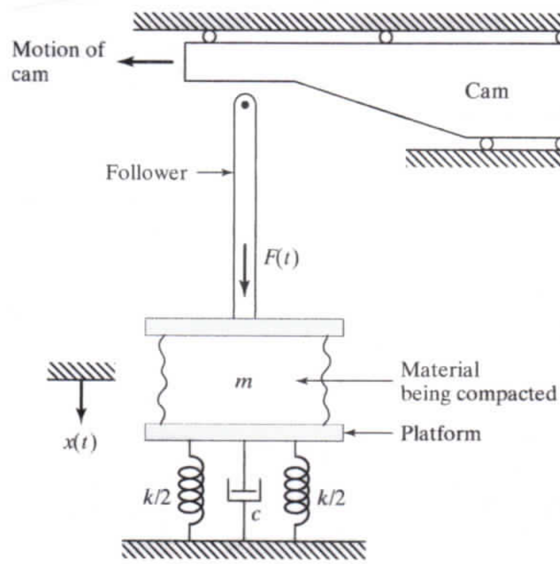
If the system is undamped, we get:

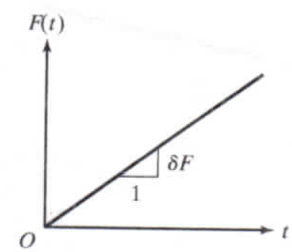
$$\underline{x(t)=\frac{F_0}{k}\left[\cos\omega_n(t-t_0)-\cos\omega_n t\right]}$$

The response is shown in the following for two different pulse widths of t_0.



– Example: (Linear Force) Determine the response of the compacting machine, shown below, when a linearly-varying force (shown below) is applied due to the motion of the cam.





Solution: The linearly-varying force is known as the ramp function. This forcing function can be represented as $F(\tau) = \delta F \cdot \tau$, where '$\delta F$' denotes the rate of increase of the force 'F' per unit time. So, we may write:

$$x(t) = \frac{1}{m\omega_d} \int_0^t \underbrace{\delta F . \tau}_{F(\tau)} . e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

$$x(t) = \frac{\delta F}{m\omega_d} \int_0^t \tau e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

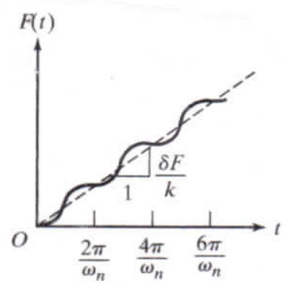
$$x(t) = \frac{\delta F}{m\omega_d} \int_0^t (t-\tau) e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)(-d\tau) - \frac{\delta F . t}{m\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)(-d\tau)$$

$$x(t) = \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t - \left\{\frac{\omega_d^2 - \zeta^2\omega_n^2}{\omega_n^2\omega_d}\right\}\sin\omega_d t\right)\right]$$

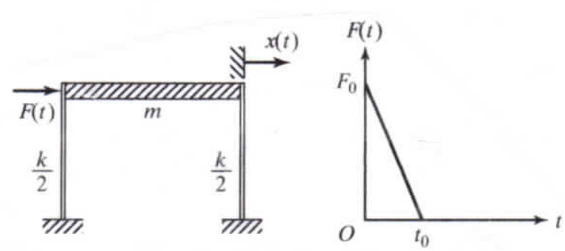
For an undamped system, we get:

$$x(t) = \frac{\delta F}{k\omega_n}\left[\omega_n t - \sin\omega_n t\right]$$

Following figure shows the response given by the above (undamped) equation:



– Example: (Blast Load) A building frame is modeled as an undamped SDOF system (shown below). Find the response of the frame if it is subjected to a blast loading represented by the triangular pulse shown below.



Solution: The forcing function is given by:

$$\begin{cases} F(\tau) = F_0\left(1 - \frac{\tau}{t_0}\right) & 0 \le \tau \le t_0 \\ F(\tau) = 0 & \tau > t_0 \end{cases}$$

Note: For an undamped system $\rightarrow x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$

① Response during $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\omega_n^2}\int_0^t \left(1 - \frac{\tau}{t_0}\right)[\sin\omega_n t\cos\omega_n\tau - \cos\omega_n t\sin\omega_n\tau]\,d(\omega_n\tau)$$

$$x(t) = \frac{F_0}{k}\sin\omega_n t\int_0^t \left(1 - \frac{\tau}{t_0}\right)\cos\omega_n\tau \cdot d(\omega_n\tau) - \frac{F_0}{k}\cos\omega_n t \int_0^t \left(1 - \frac{\tau}{t_0}\right)\sin\omega_n\tau \cdot d(\omega_n\tau)$$

By noting that integration by parts gives

$$\int \tau \cdot \cos\omega_n\tau \cdot d(\omega_n\tau) = \tau \cdot \sin\omega_n\tau + \frac{1}{\omega_n}\cos\omega_n\tau$$

$$\int \tau \cdot \sin\omega_n\tau \cdot d(\omega_n\tau) = -\tau \cdot \cos\omega_n\tau + \frac{1}{\omega_n}\sin\omega_n\tau$$

We get:

$$\text{Eq. } * \Leftarrow \begin{cases} x(t) = \frac{F_0}{k}\Big\{\sin\omega_n t\left[\sin\omega_n t - \frac{t}{t_0}\sin\omega_n t - \frac{1}{\omega_n t_0}\cos\omega_n t + \frac{1}{\omega_n t_0}\right] \\ \quad -\cos\omega_n t\left[-\cos\omega_n t + 1 + \frac{t}{t_0}\cos\omega_n t - \frac{1}{\omega_n t_0}\sin\omega_n t\right]\Big\} \end{cases}$$

By simplifying this expression, we get:

$$x(t) = \frac{F_0}{k}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right]$$

② Response during $t > t_0$:

Here we also use the forcing function for $F(\tau)$, but the upper limit of the following integration will be 't_0', since $F(\tau) = 0$ for $\tau > t_0$.

$$x(t) = \frac{1}{m\omega_n}\int_0^{t \rightarrow t_0} F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

Thus, the response can be found from Eq. * (previous page) by setting '$t = t_0$' within the square brackets.

This results in:

$$x(t) = \frac{F_0}{k\omega_n t_0}\left[(1 - \cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right]$$

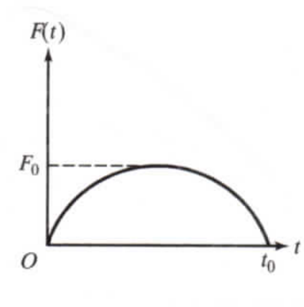
* Response Spectrum

The graph showing the variation of the maximum response (maximum displacement, velocity, acceleration, or any other quantity) with the natural frequency (or natural period) of a SDOF system to a specified forcing function is known as the 'response spectrum'. Since the maximum response is plotted against the natural frequency (or natural period), the response spectrum gives the maximum response of all possible SDOF systems. The response spectrum is widely used in earthquake engineering design.

Once the response spectrum corresponding to a specified forcing function is available, we need to know just the natural frequency of the system to find its maximum response.

– Example: (Construction of Response Spectrum of Sinusoidal Pulse)

Find the undamped response spectrum for a sinusoidal pulse force shown below using the initial conditions $x(0) = \dot{x}(0) = 0$.



Solution: The EOM of an undamped system can be expressed as:

$$m\ddot{x} + kx = F(t) = \begin{cases} F_0 \sin\omega t & 0 \le t \le t_0 \\ 0 & t > t_0 \end{cases} \qquad \text{where} \quad \omega = \frac{\pi}{t_0}$$

The solution of the EOM is:

$$x(t) = x_c(t) + x_p(t) \qquad \text{where} \begin{cases} x_c(t): \text{homogeneous solution} \\ x_p(t): \text{particular solution} \end{cases}$$

$$x(t) = \underbrace{A \cos\omega_n t + B \sin\omega_n t}_{\text{constants}} + \left(\frac{F_0}{k - m\omega^2}\right) \sin\omega t$$

natural frequency of system $= \omega_n = \dfrac{2\pi}{\tau_n} = \sqrt{\dfrac{k}{m}}$

Using the initial conditions $x(0) = \dot{x}(0) = 0 \rightarrow \begin{cases} A = 0 \\ B = -\dfrac{F_0 \omega}{\omega_n (k - m\omega^2)} \end{cases}$

(verify!)

$$\text{So,} \quad x(t) = \frac{F_0/k}{1 - (\omega/\omega_n)^2} \left\{ \sin\omega t - \frac{\omega}{\omega_n} \sin\omega_n t \right\} \qquad 0 \le t \le t_0$$

$$\text{Rewrite } x(t) \rightarrow \frac{x(t)}{\delta_{st}} = \underbrace{\frac{1}{1 - \left(\frac{\tau_n}{2t_0}\right)^2} \left\{ \sin\frac{\pi t}{t_0} - \frac{\tau_n}{2t_0} \sin\frac{2\pi t}{\tau_n} \right\}}_{\text{Eq. } *} \qquad 0 \le t \le t_0$$

$\delta_{st} = F_0/k$

Since there is no force applied for $t > t_0$, the solution can be expressed as a free-vibration solution:

$$x(t) = A' \cos\omega_n t + B' \sin\omega_n t \qquad t > t_0$$

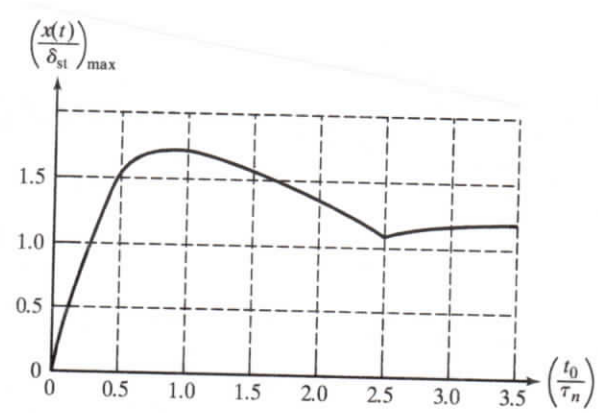
In previous equation, A' and B' can be found by using the values of $x(t=t_0)$ and $\dot{x}(t=t_0)$, given by Eq. $*$, as initial conditions for the duration $t>t_0$. This gives:

$$\begin{cases} x(t=t_0) = \alpha\left[-\dfrac{\tau_n}{2t_0}\sin\dfrac{2\pi t_0}{\tau_n}\right] = A'\cos\omega_n t_0 + B'\sin\omega_n t_0 \\[2ex] \dot{x}(t=t_0) = \alpha\left\{\dfrac{\pi}{t_0} - \dfrac{\pi}{t_0}\cos\dfrac{2\pi t_0}{\tau_n}\right\} = -\omega_n A'\sin\omega_n t_0 + \omega_n B'\cos\omega_n t_0 \end{cases} \quad \text{where } \alpha = \frac{\delta_{st}}{1-\left(\frac{\tau_n}{2t_0}\right)^2}$$

Finally $\longrightarrow$ $A' = \dfrac{\alpha\pi}{\omega_n t_0}\sin\omega_n t_0$ and $B' = -\dfrac{\alpha\pi}{\omega_n t_0}\left[1+\cos\omega_n t_0\right]$ (verify!)

$$\underbrace{\text{So, } \frac{x(t)}{\delta_{st}} = \frac{(\tau_n/t_0)}{2\left\{1-(\tau_n/2t_0)^2\right\}}\left[\sin 2\pi\left(\frac{t_0}{\tau_n}-\frac{t}{\tau_n}\right) - \sin 2\pi\frac{t}{\tau_n}\right]}_{\text{Eq. } **} \qquad t \geq t_0$$

Eq. $*$ and Eq. $**$ give the response of the system in nondimensional form, i.e. x/δ_{st} expressed in terms of t/τ_n. So, for any specified value of t_0/τ_n, the maximum value of x/δ_{st} can be found. This maximum value of x/δ_{st} when plotted against t_0/τ_n gives the response spectrum shown here. $\longrightarrow$



Note: In previous example, the input force is simple and hence a closed-form solution has been obtained for the response spectrum. However, if the input force is arbitrary, we can find the response spectrum only numerically. In such a case, the following equation can be used to express the peak response of an undamped SDOF system due to an arbitrary input force $F(t)$.

$$x(t)\Big|_{max} = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau\Big|_{max}$$

– Response Spectrum for Base Excitation:

In the design of machinery and structures subjected to a ground shock, such as that caused by an earthquake, the response spectrum corresponding to the base excitation is useful. If the base of a damped SDOF system is subjected to an acceleration '$\ddot{y}(t)$', the EOM, in terms of the relative displacement '$z=x-y$', is given by '$m\ddot{z}+c\dot{z}+kz=-m\ddot{y}$' and the response '$z(t)$' by '$z(t)=-\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$'. In case of a ground shock, the velocity response spectrum is generally used. The displacement and acceleration spectra are then expressed in terms of the velocity spectrum. For a harmonic oscillator (an undamped system under free vibration), we notice that

$$\begin{cases} \ddot{x}_{max} = -\omega_n^2 x_{max} \\ \dot{x}_{max} = \omega_n x_{max} \end{cases}$$

So, the acceleration and displacement spectra 'S_a' and 'S_d' can be obtained in terms of the velocity spectrum 'S_v':

$$\begin{cases} S_a = \omega_n S_v \\ S_d = \dfrac{S_v}{\omega_n} \end{cases}$$

To consider damping in the system, if we assume that the maximum relative displacement occurs after shock pulse has passed, the subsequent motion must be harmonic. In such a case, we can use '$S_a = \omega_n S_v$' and '$S_d = S_v/\omega_n$'. The fictitious velocity associated with this apparent harmonic motion is called the 'pseudo velocity' and its response spectrum, 'S_v', is called the 'pseudo spectrum'. The velocity spectra of damped systems are used extensively in earthquake analysis.

To find the relative velocity spectrum, consider

$$z(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)\, e^{-\xi\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

⇓ *differentiate*

$$\dot{z}(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)\, e^{-\xi\omega_n(t-\tau)} \left[-\xi\omega_n \sin\omega_d(t-\tau) + \omega_d \cos\omega_d(t-\tau)\right] d\tau$$

⇓ *rewrite*

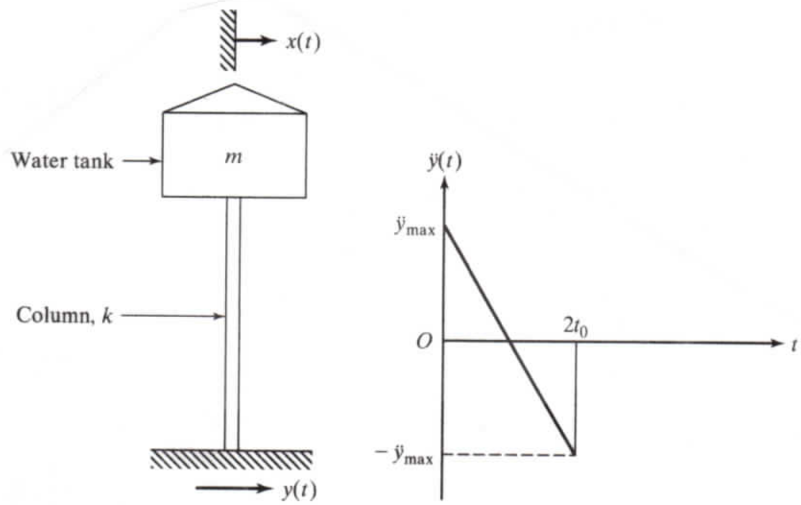
$$\dot{z}(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sqrt{P^2+Q^2}\,\sin(\omega_d t - \phi) \xrightarrow{\text{where}} \begin{cases} P = \int_0^t \ddot{y}(\tau)\, e^{\xi\omega_n t} \cos\omega_d\tau\, d\tau \\ Q = \int_0^t \ddot{y}(\tau)\, e^{\xi\omega_n t} \sin\omega_d\tau\, d\tau \\ \phi = \tan^{-1}\left\{\frac{-(P\sqrt{1-\xi^2}+Q\xi)}{(P\xi - Q\sqrt{1-\xi^2})}\right\} \end{cases}$$

So,

$$\text{Velocity response spectrum} = S_v = |\dot{z}(t)|_{max} = \left|\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sqrt{P^2+Q^2}\right|_{max}$$

$$\begin{cases} S_d = |z|_{max} = S_v/\omega_n \\ S_v = |\dot{z}|_{max} \\ S_a = |\ddot{z}|_{max} = \omega_n S_v \end{cases} \Rightarrow \text{Pseudo response spectra}$$

- Example: The water tank, shown below, is subjected to a linearly-varying ground acceleration, as shown, due to an earthquake. The mass of the tank is 'm', the stiffness of the column is 'k', and damping is negligible. Find the response spectrum for the relative displacement, $z = x - y$, of the water tank.



Solution: We model the water tank as an undamped SDOF system.

$$\text{Base acceleration} \begin{cases} \ddot{y}(t) = \ddot{y}_{max}\left(1 - \frac{t}{t_0}\right) & 0 \leq t \leq 2t_0 \\ \ddot{y}(t) = 0 & t > 2t_0 \end{cases}$$

* Response during $0 \leq t \leq 2t_0$:

$$z(t) = -\frac{1}{\omega_n}\ddot{y}_{max}\left[\int_0^t \left(1 - \frac{\tau}{t_0}\right)(\sin\omega_n t \cos\omega_n \tau - \cos\omega_n t \sin\omega_n \tau)\,d\tau\right]$$

⇓ rewrite

$$z(t) = -\frac{\ddot{y}_{max}}{\omega_n^2}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right]$$

⇓ Find z_{max}

$$\dot{z}(t) = -\frac{\ddot{y}_{max}}{t_o\omega_n^2}\left[-1+\omega_n t_o \sin\omega_n t + \cos\omega_n t\right] = 0 \rightarrow t_m = \frac{2}{\omega_n}\tan^{-1}(\omega_n t_o) \rightarrow$$

$$z_{max} = -\frac{\ddot{y}_{max}}{\omega_n^2}\left[1 - \frac{t_m}{t_o} - \cos\omega_n t_m + \frac{1}{\omega_n t_o}\sin\omega_n t_m\right]$$

* Response during $t > 2t_o$: Since $\ddot{y}(t) = 0$, we can use the solution of the free vibration $m\ddot{x} + kx = 0$, as

$$z(t) = z_o \cos\omega_n t + \left(\frac{\dot{z}_o}{\omega_n}\right)\sin\omega_n t \qquad \text{(verify!)}$$

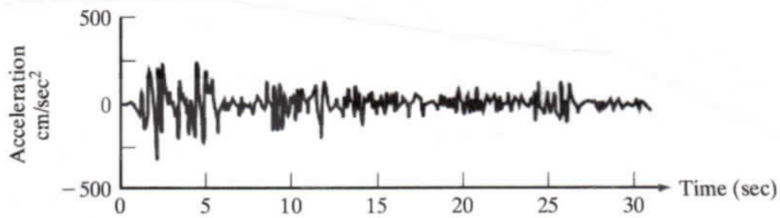
by considering the I.C.s as $\begin{cases} z(t=2t_o) = z_o \\ \dot{z}(t=2t_o) = \dot{z}_o \end{cases}$.

The maximum of $z(t)$ can be identified as:

$$z_{max} = \sqrt{z_o^2 + \left(\frac{\dot{z}_o}{\omega_n}\right)^2}$$

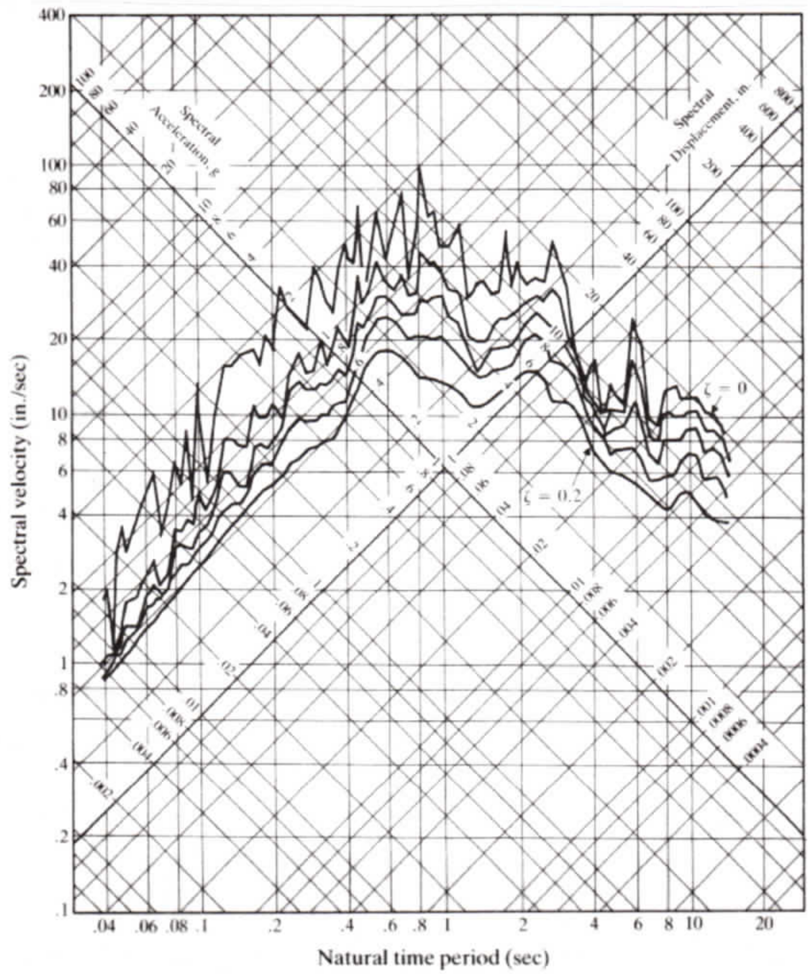
– Earthquake Response Spectra:

The most direct description of an earthquake motion in time domain is provided by accelerograms that are recorded by instruments called 'strong motion accelerographs'. They record three orthogonal components of ground acceleration at a certain location. A typical accelerogram is shown in the following. An accelerogram can be integrated to obtain the time variations of the ground velocity and ground displacement.



A typical accelerogram

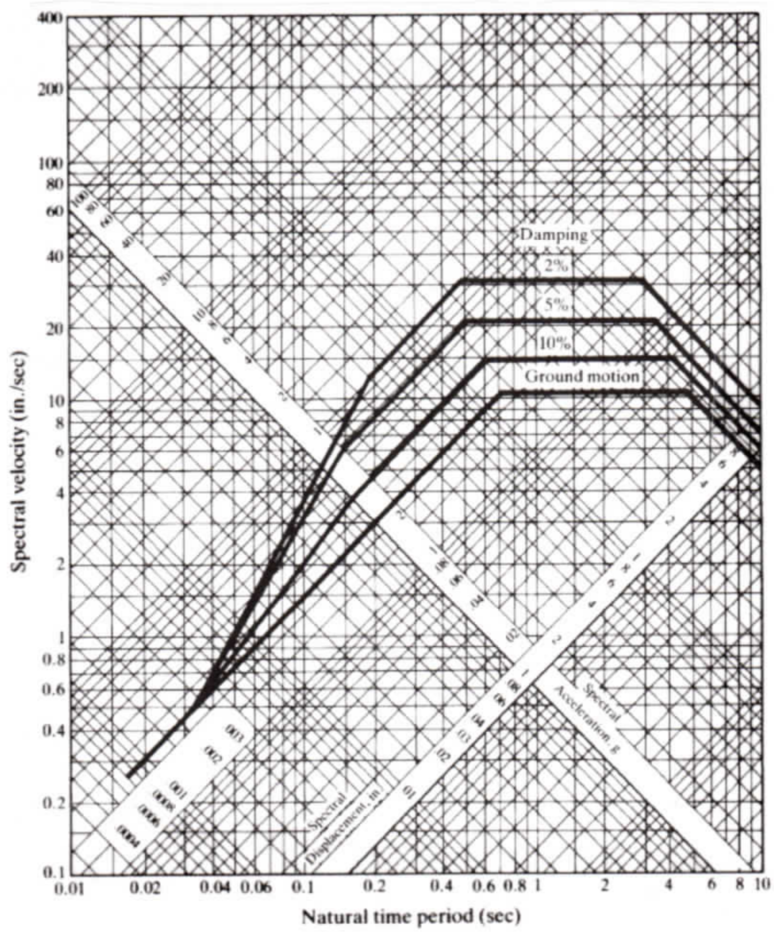
A response spectrum is used to provide the most descriptive representation of the influence of a given earthquake on a structure or machine. It is possible to plot the maximum response of a SDOF system in terms of the acceleration, relative pseudo velocity, and relative displacement using logarithmic scales. A typical response spectrum, plotted on a four-way logarithmic paper, is shown in the following. In the figure, the vertical axis denotes the spectral velocity, the horizontal axis represents the natural time period, the 45° inclined axis indicates the spectral displacement, and the 135° inclined axis shows the spectral acceleration. This figure indeed shows the response spectrum of the Imperial Valley Earthquake (May 18, 1940), for $\xi = 0, 0.02, 0.05, 0.10$, and 0.20 damping ratios.



Response Spectrum

As it is seen in the above figure, the response spectrum of a particular earthquake exhibits considerable irregularities in the frequency domain. Nevertheless, spectra corresponding to an ensemble of accelerograms produced by ground shakings of sites with similar geological and seismological features are smooth functions of time and provide statistical trends that characterize them collectively. This idea has led to the development of the concept of a design spectrum for use in earthquake-resistant design of structures and machines.

A typical design spectrum is shown below.



Design Spectrum

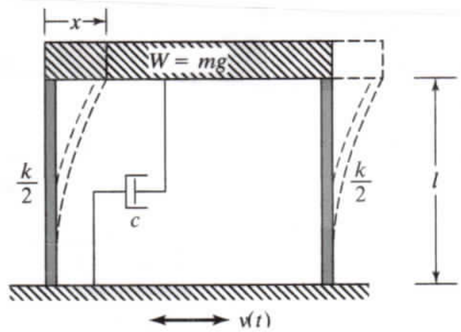
In the following examples, the use of the response and design spectra of earthquakes is demonstrated.

— Example: A building frame has a weight of 15,000 lb and two columns of total stiffness k, as indicated in the following figure. It has a damping ratio of 0.05 and a natural time period of 1.0 sec. For the earthquake characterized before (Imperial Valley Earthquake, May 18, 1940), determine the following:

ⓐ Maximum relative displacement of the mass, x_{max}.

ⓑ Maximum shear force in the columns.

ⓒ Maximum bending stress in the columns.



Solution: For $\tau_n = 1.0$ sec and $\xi = 0.05 \xrightarrow[\text{Spectrum}]{\text{Response}} \begin{cases} S_v = 25 \text{ in/s} \\ S_d = 4.2 \text{ in} \\ S_a = 0.42g \\ \quad = 162.288 \text{ in/s}^2 \end{cases}$

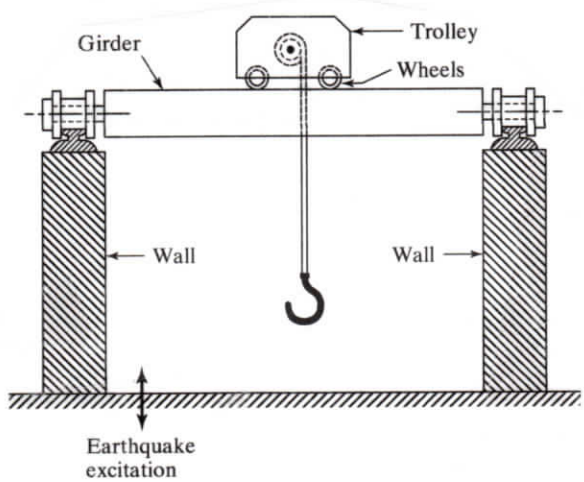
ⓐ $\underline{x_{max} = S_d = 4.2 \text{ in}}$

ⓑ $|kx_{max}| = m\ddot{x}_{max} = \frac{W}{g} S_a = \left(\frac{15000}{386.4}\right)(162.288) = 6300 \text{ lb}$

$\underline{F_{max/each column} = \frac{6300}{2} = 3150 \text{ lb}}$

ⓒ $M_{max} = F_{max} \cdot l \longrightarrow \underline{\sigma_{max} = \frac{M_{max}c}{I}}$

– Example: The trolley of an electric overhead traveling (EOT) crane travels horizontally on the girder as shown. Assuming the trolley as a point mass, the crane can be modeled as a SDOF system with a period of 2 sec and a damping ratio of 2%. Determine whether the trolley derails under a vertical earthquake excitation whose design spectrum was given before.



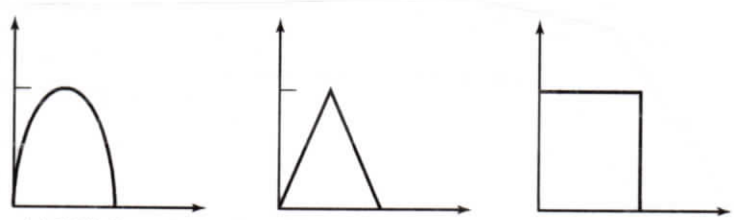
Solution: For $\begin{cases} \tau_n = 2 \text{ sec} \\ \xi = 0.02 \end{cases} \xrightarrow[\text{Spectrum}]{\text{Design}} \underline{S_a = 0.25g}$

The trolley will not derail, since the spectral acceleration of the trolley (mass) does not exceed a value of 1.0g.

– Design Under a Shock Environment:

When a force is applied for a short duration, usually for a period of less than one natural time period, it is called a 'shock Load'. A shock causes a significant increase in the displacement, velocity, acceleration, or stress in a mechanical system. Although fatigue is a major cause of failure under harmonic forces, usually it is not very important under shock loads. A shock may be described by a pulse shock, velocity shock, or a shock response spectrum. The pulse shocks are introduced by suddenly-applied forces or displacements in the form of a square, half-sine, triangular, or similar shape (shown below).

Typical shock pulses

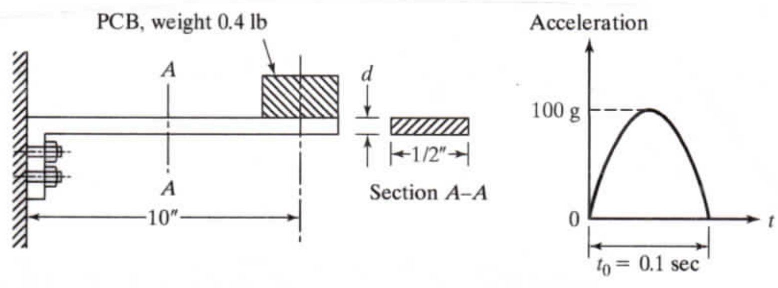


(a) Half-sine pulse (b) Triangular pulse (c) Rectangular pulse

A velocity shock is caused by sudden changes in the velocity such as those caused when packages are dropped from a height. The shock response spectrum describes the way in which a machine or structure responds to a specific shock instead of describing the shock itself.

The following example illustrates the method of limiting dynamic stresses in mechanical systems under a shock environment.

– Example: A printed circuit board (PCB) is mounted on a cantilevered aluminum bracket, as shown below. The bracket is placed in a container that is expected to be dropped from a low-flying helicopter. The resulting shock can be approximated as a half-sine-wave pulse, as shown below. Design the bracket to withstand an acceleration level of 100 g under the half-sine-wave pulse shown below. Assume a specific weight of 0.1 lb/in^3, a Young's modulus of 10^7 psi, and a permissible stress of 26,000 psi for aluminum.



Solution: self weight of the beam: $W = 10 \times \frac{1}{2} \times d \times 0.1 = 0.5d$

total weight: W = weight of beam + weight of PCB $= 0.5d + 0.4$

moment of inertia of cross section: $I = \frac{1}{12} \times \frac{1}{2} \times d^3 = 0.04167\, d^3$

static deflection of beam under the end Load W: $\delta_{st} = \frac{W\ell^3}{3EI} = \frac{(0.5d+0.4)(10^3)}{3 \times 10^7 \times 0.04167 d^3}$

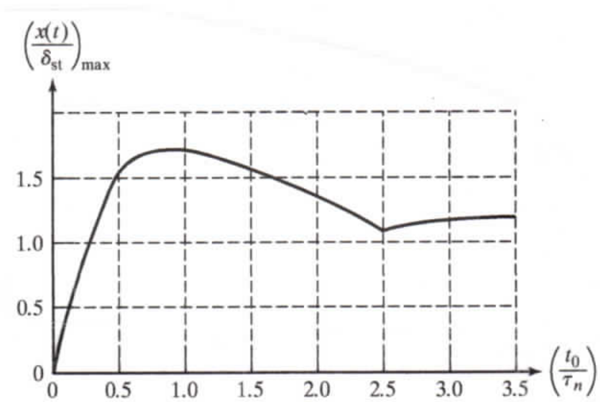
$$= \frac{0.5d+0.4}{d^3} \times 7.9994 \times 10^{-4}$$

we need to find τ_n, so that the shock amplification factor can be determined.

assume: $d = \frac{1}{2}$ in $\rightarrow \delta_{st} = 41.5969 \times 10^{-4}$ in $\rightarrow$

$$\tau_n = 2\pi \sqrt{\frac{\delta_{st}}{\underbrace{g}_{386.4}}} = 0.020615 \text{ sec} \rightarrow \frac{t_0}{\tau_n} = \frac{0.1}{0.020615} = 4.8508$$

the shock amplification factor (A_a) can be found from the following figure obtained for response spectrum of sinusoidal pulse:



the dynamic load acting on cantilever : $P_d = A_a \overbrace{\underbrace{M}_{\text{mass at the end of beam}}\underbrace{a_s}_{\text{acceleration corresponding to the shock}}}^{\text{inertia force on the beam}} = 1.1 \times \left(\frac{0.65}{g}\right) \times (100g) = 71.5 \text{ lb}$

note: $I = 0.04167 d^3 = 0.005209 \text{ in}^4$

maximum bending stress at the root of the cantilever bracket: $\sigma_{max} = \frac{M_b \cdot c}{I} = \frac{(71.5 \times 10)\left(\frac{0.5}{2}\right)}{0.005209} = 34315.6076 \text{ lb/in}^2$

since $\sigma_{max} > \sigma_{permissible} \longrightarrow$ assume: $d = 0.6$ in $\rightarrow$

$$\delta_{st} = \left(\frac{0.5 \times 0.6 + 0.4}{0.6^3}\right) \times 7.9994 \times 10^{-4} = 25.9240 \times 10^{-4} \text{ in}$$

$$\tau_n = 2\pi \sqrt{\frac{\delta_{st}}{g}} = 0.01627 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.01627} = 6.1445 \xrightarrow[\text{figure}]{\text{using the}} A_a \approx 1.1 \rightarrow$$

$$P_d = (1.1)\left(\frac{0.7}{g}\right)(100g) = 77.0 \text{ lb}$$

$$I = 0.04167 d^3 = 0.009001 \text{ in}^4$$

$$\sigma_{max} = \frac{M_b \cdot c}{I} = \frac{(77.0 \times 10)\left(\frac{0.6}{2}\right)}{0.009001} = 25663.8151 \text{ lb/in}^2 < \sigma_{permissible}$$

$\rightarrow$ <u>the thickness of the bracket can be taken as $d = 0.6$ in.</u>

* Introduction

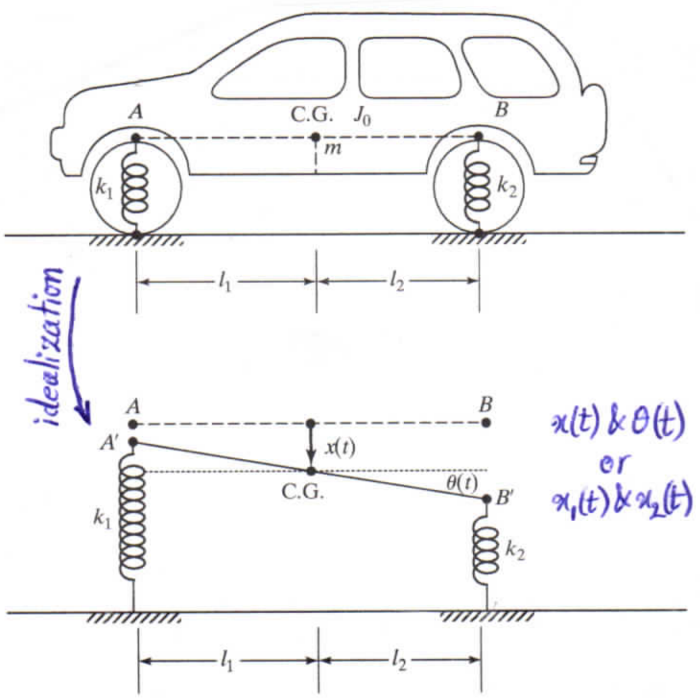
This chapter deals with multi-degree-of-freedom (MDOF) systems, which require two or more independent coordinates to describe their motion. The coupled equations of motion of the system are derived and expressed in matrix form. By expressing the EOMs in matrix form, the mass, damping, and stiffness matrices of the system are identified. For example, for a two-degree-of-freedom system, we get

EOM: $M\ddot{X} + C\dot{X} + KX = F \text{ or } 0$

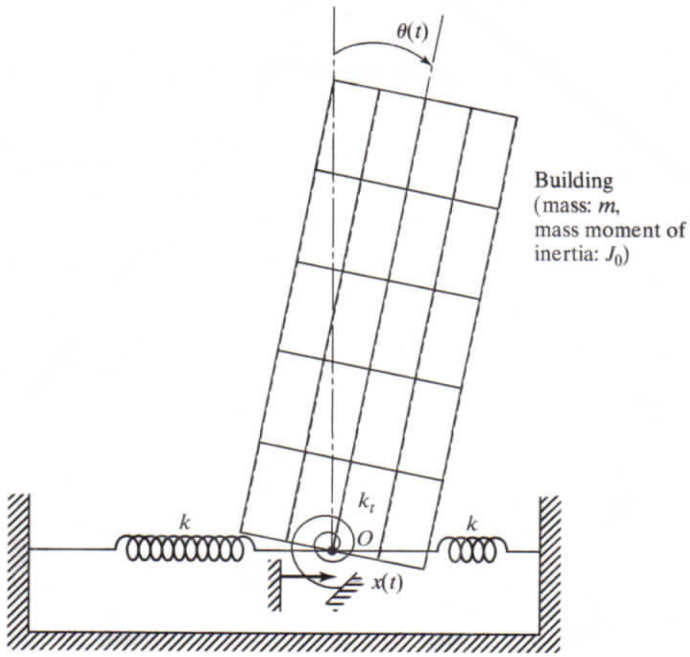
where

M, C, K are 2×2 matrices, and
$\ddot{X}, \dot{X}, X$ (F and 0) are 2×1 vectors.

In the following, consider the automobile shown. For the vibration of the automobile in the vertical plane, a two-degree-of-freedom model, as shown, can be used. The body is idealized as a bar of mass 'm' and a mass moment of inertia 'J_0', supported on the rear and front wheels (suspensions) of stiffness 'k_1' and 'k_2'. The displacement of the automobile at any time can be specified by the linear coordinate '$x(t)$' denoting the vertical displacement of the C.G. of the body and the angular coordinate '$\theta(t)$' indicating the rotation of the body about its C.G. Alternatively, the motion of the automobile can be specified using the independent coordinates, '$x_1(t)$' and '$x_2(t)$', of points 'A' and 'B'.



As another example, consider the motion of a multistory building under an earthquake. For simplicity, a two-degree-of-freedom model can be used as shown. The building is modeled as a rigid bar having a mass 'm' and mass moment of inertia 'J_0'. The resistance offered to the motion of the building by the foundation and surrounding soil is approximated by a linear spring of stiffness 'k' and a torsional spring of stiffness 'k_t'. The displacement of the building at any time can be specified by the horizontal motion of the base '$x(t)$' and the angular motion '$\theta(t)$' about the point O.



The general rule for the computation of the number of degrees of freedom can be stated as follows:

$$\boxed{\begin{array}{c}\text{Number of} \\ \text{degrees of freedom} \\ \text{of the system}\end{array} = \begin{array}{c}\text{Number of masses} \\ \text{in the system}\end{array} \times \begin{array}{c}\text{Number of possible types} \\ \text{of motion of each mass}\end{array}}$$

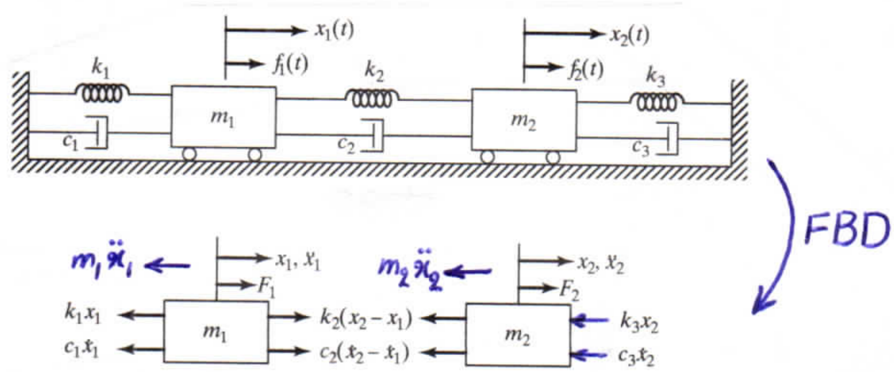
For example, in a two-degree-of-freedom system, we may have one of the following two possible cases:

① 1 mass and 2 coordinates,
② 2 masses and 1 coordinate for each mass.

The MDOF system differs from that of the SDOF system in that it has multi or 'N' natural frequencies, and for each of the natural frequencies, there corresponds a natural state of vibration with a displacement configuration known as the 'normal mode'. Mathematical terms related to these quantities are known as 'eigenvalues' and 'eigenvectors', respectively. They are established form the 'N' simultaneous EOMs of the system and possess certain dynamic properties associated with the system.

* Equations of Motion for Forced Vibration

Consider a viscously-damped two-degree-of-freedom spring-mass system, as shown. The motion of the system is completely described by the coordinates '$x_1(t)$' and '$x_2(t)$', which define the positions of the masses 'm_1' and 'm_2' at any time 't' from the respective equilibrium positions. The external forces '$F_1(t)$' and '$F_2(t)$' act on the masses 'm_1' and 'm_2', respectively. The free-body diagrams of the masses 'm_1' and 'm_2' are shown as well. The equations of motions of the masses are determined as follows:



$$m_1:\ \overset{+}{\rightarrow}\ \sum F_x = 0 \Rightarrow -k_1x_1 - c_1\dot{x}_1 - m_1\ddot{x}_1 + F_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = 0$$

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_2:\ \overset{+}{\rightarrow}\ \sum F_x = 0 \Rightarrow -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - m_2\ddot{x}_2 - k_3x_2 - c_3\dot{x}_2 = 0$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

Both involve x_1 and x_2. (Coupled 2nd-order DEs)

The two DEs can be written in matrix form:

$$[m]\ddot{\vec{x}}(t) + [c]\dot{\vec{x}}(t) + [k]\vec{x}(t) = \vec{F}(t)$$

where

$$[m]_{2\times2} = \text{mass matrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[c]_{2\times2} = \text{damping matrix} = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}$$

$$[k]_{2\times2} = \text{stiffness matrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

Note: 2x2 and symmetric!

$$\begin{cases} [m] = [m]^T \\ [c] = [c]^T \\ [k] = [k]^T \end{cases}$$

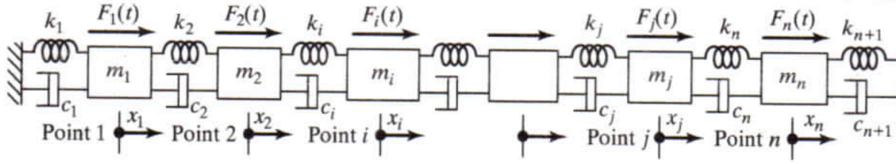
transpose!

$$\ddot{\vec{x}}(t) = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix}_{2\times1}, \quad \dot{\vec{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix}_{2\times1}, \quad \text{and} \quad \vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}_{2\times1}$$

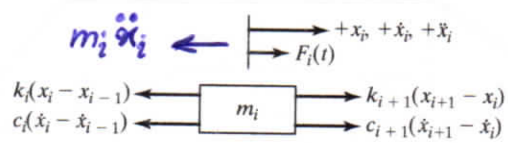
The I.C.s of the two masses can be specified as:

$$m_1: \begin{cases} x_1(t=0) = x_1(0) \\ \dot{x}_1(t=0) = \dot{x}_1(0) \end{cases} \qquad m_2 = \begin{cases} x_2(t=0) = x_2(0) \\ \dot{x}_2(t=0) = \dot{x}_2(0) \end{cases}$$

Now, we derive the equations of motion of the following multi-degree-of-freedom spring-mass-damper system:



The FBD of a typical interior mass 'm_i' is shown below:



The EOM of mass 'm_i' is found to be:

$$\boxed{m_i\ddot{x}_i - c_i\dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1} + (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = F_i \quad i = 2, 3, \ldots, n-1}$$

m_i

The equations of motion of the masses 'm_1' and 'm_n' can be derived from the above equation by:

m_1: $i = 1$ along with $x_0 = 0$

$$\boxed{m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2 x_2 = F_1}$$

m_1

m_n: $i = n$ along with $x_{n+1} = 0$

$$\boxed{m_n\ddot{x}_n - c_n\dot{x}_{n-1} + (c_n + c_{n+1})\dot{x}_n - k_n x_{n-1} + (k_n + k_{n+1})x_n = F_n}$$

m_n

The three EOMs can be expressed in matrix form as:

$$\boxed{[m]\ddot{\vec{x}}(t) + [c]\dot{\vec{x}}(t) + [k]\vec{x}(t) = \vec{F}(t)}$$

where $[m]$, $[c]$, and $[k]$ are called the mass, damping, and stiffness, matrices, respectively, and are given by:

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m_3 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & m_n \end{bmatrix} \qquad [c] = \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & (c_3 + c_4) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -c_n & (c_n + c_{n+1}) \end{bmatrix}$$

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & (k_3 + k_4) & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & -k_n & (k_n + k_{n+1}) \end{bmatrix}$$

and $\vec{x}(t)$, $\dot{\vec{x}}(t)$, $\ddot{\vec{x}}(t)$, and $\vec{F}(t)$ are the displacement, velocity, acceleration, and force vectors, given by:

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{Bmatrix}, \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{Bmatrix}, \quad \ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n(t) \end{Bmatrix}, \quad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(t) \end{Bmatrix}$$

The spring-mass-damper system considered above is a particular case of a general N-degree-of-freedom spring-mass-damper system. In their most general form, the mass, damping, and stiffness matrices are given by:

$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{12} & m_{22} & m_{23} & \cdots & m_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ m_{1n} & m_{2n} & m_{3n} & \cdots & m_{nn} \end{bmatrix}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{12} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix}$$

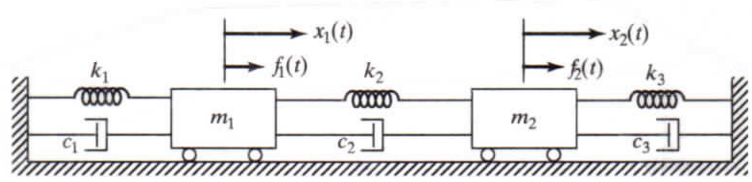
$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ k_{1n} & k_{2n} & k_{3n} & \cdots & k_{nn} \end{bmatrix}$$

If the mass matrix is not diagonal, the system is said to have mass or inertia coupling. If the damping matrix is not diagonal, the system is said to have damping or velocity coupling. Finally, if the stiffness matrix is not diagonal, the system is said to have elastic or static coupling. Both mass and damping coupling are also known as dynamic coupling.

The differential equations of the considered multi-degree-of-freedom spring-mass-damper system can be seen to be coupled; each equation involves more than one coordinate. This means that the equations can not be solved individually one at a time; they can only be solved simultaneously. In addition, the system can be seen to be statically coupled, since stiffnesses are coupled, i.e. the stiffness matrix has at least one nonzero off-diagonal term. On the other hand, if the mass matrix has at least one off-diagonal term nonzero, the system is said to be dynamically coupled. Further, if both the stiffness and mass matrices have nonzero off-diagonal terms, the system is said to be coupled both statically and dynamically.

* Free-Vibration Analysis of an Undamped System

For the free-vibration analysis of the system shown, we set $f_1(t) = f_2(t) = 0$.



Further, if damping is disregarded, $c_1 = c_2 = c_3 = 0$, and the EOMs are:

$$\begin{cases} m_1 \ddot{x}_1(t) + (k_1+k_2)x_1(t) - k_2 x_2(t) = 0 \\ m_2 \ddot{x}_2(t) - k_2 x_1(t) + (k_2+k_3)x_2(t) = 0 \end{cases}$$

If m_1 and m_2 oscillate harmonically with the same frequency (ω) and phase angle (ϕ) but with different amplitudes, then

$$\begin{cases} x_1(t) = X_1 \cos(\omega t + \phi) \rightarrow m_1 \\ x_2(t) = X_2 \cos(\omega t + \phi) \rightarrow m_2 \end{cases} \quad ; X_1 \text{ and } X_2 = \text{max. amplitudes of } x_1(t) \text{ and } x_2(t)$$

By substituting the above equations into the two abovementioned EOMs, we get:

$$\begin{cases} [\{-m_1\omega^2 + (k_1+k_2)\}X_1 - k_2 X_2]\cos(\omega t + \phi) = 0 \\ [-k_2 X_1 + \{-m_2\omega^2 + (k_2+k_3)\}X_2]\cos(\omega t + \phi) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \{-m_1\omega^2 + (k_1+k_2)\}X_1 - k_2 X_2 = 0 \\ -k_2 X_1 + \{-m_2\omega^2 + (k_2+k_3)\}X_2 = 0 \end{cases}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \{-m_1\omega^2 + (k_1+k_2)\} & -k_2 \\ -k_2 & \{-m_2\omega^2 + (k_2+k_3)\} \end{bmatrix}}_{\text{For nontrivial solution: det} = 0} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Downarrow$$

frequency or characteristic equation : $(m_1 m_2)\omega^4 - \{(k_1+k_2)m_2 + (k_2+k_3)m_1\}\omega^2 + \{(k_1+k_2)(k_2+k_3) - k_2^2\} = 0$

The solution of the frequency or characteristic equation will yield the frequencies or the characteristic values of the system. The roots are:

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1 m_2}\right\} \mp \frac{1}{2}\sqrt{\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1 m_2}\right\}}$$

ω_1 and ω_2 are the natural frequencies of the system.

We need to determine the values of X_1 and X_2. These values depend on the natural frequencies ω_1 and ω_2. We shall denote the values of X_1 and X_2 corresponding to ω_1 as $X_1^{(1)}$ and $X_2^{(1)}$ and those corresponding to ω_2 as $X_1^{(2)}$ and $X_2^{(2)}$. So,

$$\left.\begin{array}{l} \omega^2=\omega_1^2 \rightarrow X_1^{(1)} \text{ and } X_2^{(1)} \\ \omega^2=\omega_2^2 \rightarrow X_1^{(2)} \text{ and } X_2^{(2)} \end{array}\right\} \rightarrow \underbrace{\begin{array}{l} r_1 = \dfrac{X_2^{(1)}}{X_1^{(1)}} = \dfrac{-m_1\omega_1^2+(k_1+k_2)}{k_2} = \dfrac{k_2}{-m_2\omega_1^2+(k_2+k_3)} \\ r_2 = \dfrac{X_2^{(2)}}{X_1^{(2)}} = \dfrac{-m_1\omega_2^2+(k_1+k_2)}{k_2} = \dfrac{k_2}{-m_2\omega_2^2+(k_2+k_3)} \end{array}}$$

Note: The ratios r_1 and r_2 are found from:

$$\begin{cases} \{-m_1\omega^2+(k_1+k_2)\}X_1 - k_2 X_2 = 0 \\ -k_2 X_1 + \{-m_2\omega^2+(k_2+k_3)\}X_2 = 0 \end{cases}$$

The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed as:

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix}$$

These vectors are known as the modal vectors of the system.

The free-vibration solution or the motion in time can be expressed as:

$$\left.\begin{aligned} x_1(t) &= X_1 \cos(\omega t + \phi) \\ x_2(t) &= X_2 \cos(\omega t + \phi) \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \vec{x}^{(1)}(t) &= \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode} \\ \vec{x}^{(2)}(t) &= \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode} \end{aligned}\right.$$

In these equations, $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 are determined by the I.C.s.

I.C.s

It is noted that the system can be made to vibrate in its i-th normal mode ($i = 1, 2$) by subjecting it to the following specific initial conditions:

$$\begin{cases} x_1(t=0) = X_1^{(i)} = \text{some constant}, & \dot{x}_1(t=0) = 0 \\ x_2(t=0) = r_i X_1^{(i)}, & \dot{x}_2(t=0) = 0 \end{cases}$$

For any other general initial conditions, both modes will be excited. The resulting motion can be obtained by a linear superposition of the two normal modes:

$$\underbrace{\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)}$$ where c_1 and c_2 are constants.

unknowns: $X_1^{(1)}, X_1^{(2)}, \phi_1, \phi_2, c_1, c_2$

To reduce the number of unknowns, we can choose $c_1 = c_2 = 1$ with no loss of generality. So,

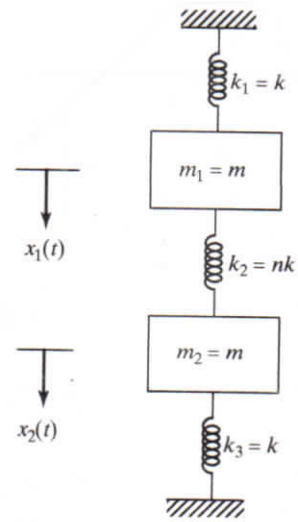
$$\begin{cases} x_1(t) = x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ x_2(t) = x_2^{(1)}(t) + x_2^{(2)}(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{cases}$$

In the above equations, $X_1^{(1)}, X_1^{(2)}, \phi_1$, and ϕ_2 can be determined from the initial conditions:

$$\begin{cases} x_1(t=0) = x_1(0), & \dot{x}_1(t=0) = \dot{x}_1(0) \\ x_2(t=0) = x_2(0), & \dot{x}_2(t=0) = \dot{x}_2(0) \end{cases}$$ $\rightarrow$ By applying these I.C.s, we can obtain the following:

$$X_1^{(1)} = [\{X_1^{(1)} \cos \phi_1\}^2 + \{X_1^{(1)} \sin \phi_1\}^2]^{1/2}$$
$$= \frac{1}{(r_2 - r_1)} \left[\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2 \dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2} \right]^{1/2}$$
$$X_1^{(2)} = [\{X_1^{(2)} \cos \phi_2\}^2 + \{X_1^{(2)} \sin \phi_2\}^2]^{1/2}$$
$$= \frac{1}{(r_2 - r_1)} \left[\{-r_1 x_1(0) + x_2(0)\}^2 + \frac{\{r_1 \dot{x}_1(0) - \dot{x}_2(0)\}^2}{\omega_2^2} \right]^{1/2}$$
$$\phi_1 = \tan^{-1} \left\{ \frac{X_1^{(1)} \sin \phi_1}{X_1^{(1)} \cos \phi_1} \right\} = \tan^{-1} \left\{ \frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1 [r_2 x_1(0) - x_2(0)]} \right\}$$
$$\phi_2 = \tan^{-1} \left\{ \frac{X_1^{(2)} \sin \phi_2}{X_1^{(2)} \cos \phi_2} \right\} = \tan^{-1} \left\{ \frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2 [-r_1 x_1(0) + x_2(0)]} \right\}$$

- Example: Find the natural frequencies and mode shapes of the following spring-mass system, which is constrained to move in the vertical direction only. Take $n=1$.



Solution: We measure x_1 and x_2 from the static equilibrium positions of the masses m_1 and m_2, respectively.

As was shown before:

$$\text{EOMs}\begin{cases} m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=f_1 \\ m_2\ddot{x}_2-c_2\dot{x}_1+(c_2+c_3)\dot{x}_2-k_2x_1+(k_2+k_3)x_2=f_2 \end{cases}$$

$$\text{In this case: }\begin{cases} m_1=m_2=m \\ c_1=c_2=c_3=0 \\ k_1=k_2=k_3=k \\ f_1=f_2=0 \end{cases} \xrightarrow{\text{So}}$$

$$\text{EOMs: }\begin{cases} m\ddot{x}_1+2kx_1-kx_2=0 \\ m\ddot{x}_2-kx_1+2kx_2=0 \end{cases}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We assume harmonic solution: $x_i(t) = X_i \cos(\omega t + \phi)$ $i=1,2$

The frequency equation is obtained by substituting $x_1(t)$ and $x_2(t)$ into the EOMs:

$$\begin{vmatrix} (-m\omega^2+2k) & (-k) \\ (-k) & (-m\omega^2+2k) \end{vmatrix} = 0 \rightarrow m^2\omega^4 - 4km\omega^2 + 3k^2 = 0 \xrightarrow{\text{Solve}}$$

$$* \begin{cases} \omega_1 = \sqrt{\dfrac{4km - [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2}} = \sqrt{\dfrac{k}{m}} \\ \omega_2 = \sqrt{\dfrac{4km + [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2}} = \sqrt{\dfrac{3k}{m}} \end{cases} \rightarrow \text{natural frequencies}$$

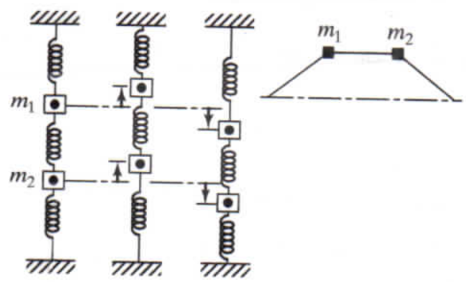
The amplitude ratios are:

$$\begin{cases} r_1 = \dfrac{X_2^{(1)}}{X_1^{(1)}} = \dfrac{-m\omega_1^2+2k}{k} = \dfrac{k}{-m\omega_1^2+2k} = 1 \\ r_2 = \dfrac{X_2^{(2)}}{X_1^{(2)}} = \dfrac{-m\omega_2^2+2k}{k} = \dfrac{k}{-m\omega_2^2+2k} = -1 \end{cases} \quad \text{(how ?)}$$

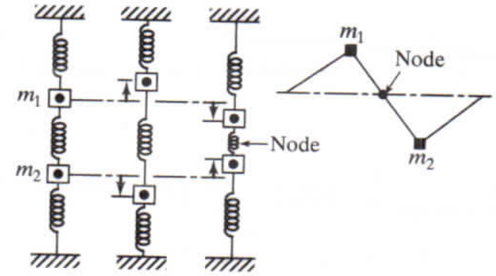
So, the natural modes are given by:

$$* \begin{cases} \text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) \end{Bmatrix} \\ \\ \text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix} \end{cases}$$

First mode: When system vibrates in its first mode, the amplitudes of the two masses remain the same. So, motions of m_1 and m_2 are in phase!



Second mode: When the system vibrates in its second mode, the two masses have the same magnitude with opposite signs. So, m_1 and m_2 are 180° out of phase!



The general solution of the system can be expressed as:

$$\begin{cases} x_1(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ \\ x_2(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{cases}$$

– Example: Find the initial conditions that need to be applied to the spring-mass system in previous example so as to make it vibrate in:

(a) the first mode, and

(b) the second mode.

Solution: The general solution of the system was given at the end of the previous example. We assume the following I.C.s:

$$\begin{cases} x_1(t=0) = x_1(0) \\ \dot{x}_1(t=0) = \dot{x}_1(0) \\ x_2(t=0) = x_2(0) \\ \dot{x}_2(t=0) = \dot{x}_2(0) \end{cases}$$

→ By applying these initial conditions and using $r_1 = 1$ and $r_2 = -1$, we can get:

$$X_1^{(1)} = -\frac{1}{2}\left\{[x_1(0) + x_2(0)]^2 + \frac{m}{k}[\dot{x}_1(0) + \dot{x}_2(0)]^2\right\}^{1/2}$$

$$X_1^{(2)} = -\frac{1}{2}\left\{[-x_1(0) + x_2(0)]^2 + \frac{m}{3k}[\dot{x}_1(0) - \dot{x}_2(0)]^2\right\}^{1/2}$$

$$\phi_1 = \tan^{-1}\left\{\frac{-\sqrt{m}\,[\dot{x}_1(0) + \dot{x}_2(0)]}{\sqrt{k}\,[x_1(0) + x_2(0)]}\right\}$$

$$\phi_2 = \tan^{-1}\left\{\frac{\sqrt{m}\,[\dot{x}_1(0) - \dot{x}_2(0)]}{\sqrt{3k}\,[-x_1(0) + x_2(0)]}\right\}$$

(verify!)

refer to the eqs. of $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 found earlier!

(a) From the previous example:

First normal mode: $$\vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)}\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \\ X_1^{(1)}\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_2\right) \end{Bmatrix}$$

Compare the above equation with the general solution of the system (provided at the end of the previous example). The motion of the system will be identical with the first normal mode only if $X_1^{(2)} = 0$. So, $\underline{x_1(0) = x_2(0)}$ and $\underline{\dot{x}_1(0) = \dot{x}_2(0)}$.

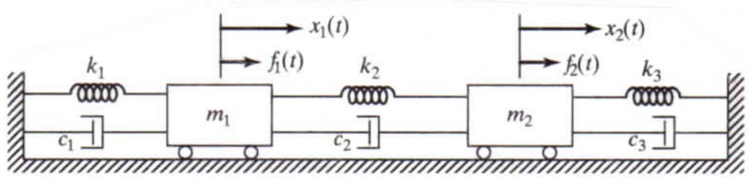
(b) From the previous example:

$$\text{Second normal mode: } \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix}$$

Compare the above equation with the general solution of the system (provided at the end of the previous example). The motion of the system will coincide with the second normal mode only if $X_1^{(1)} = 0$. So, $\underline{x_1(0) = -x_2(0)}$ and $\underline{\dot{x}_1(0) = -\dot{x}_2(0)}$.

– Example: (Free-Vibration Response of a Two-Degree-of-Freedom System)

Find the free-vibration response of the following system with:



$k_1 = 30$, $k_2 = 5$, $k_3 = 0$

$m_1 = 10$, $m_2 = 1$

$c_1 = c_2 = c_3 = 0$

I.C.s: $x_1(0) = 1$, $\dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$

Solution: As discussed before, the EOMs of the system are:

$$\begin{cases} m_1\ddot{x}_1(t)+(k_1+k_2)x_1(t)-k_2x_2(t)=0 \\ m_2\ddot{x}_2(t)-k_2x_1(t)+(k_2+k_3)x_2(t)=0 \end{cases} \xrightarrow[\text{harmonic motion}]{\text{Assume}} \begin{cases} x_1(t)=X_1\cos(\omega t+\phi) \\ x_2(t)=X_2\cos(\omega t+\phi) \end{cases}$$

$$\rightarrow \begin{cases} \{-m_1\omega^2+(k_1+k_2)\}X_1-k_2X_2=0 \\ -k_2X_1+\{-m_2\omega^2+(k_2+k_3)\}X_2=0 \end{cases} \rightarrow \begin{bmatrix} -m_1\omega^2+k_1+k_2 & -k_2 \\ -k_2 & -m_2\omega^2+k_2+k_3 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \underbrace{\begin{bmatrix} -10\omega^2+35 & -5 \\ -5 & -\omega^2+5 \end{bmatrix}}_{det=0}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow \text{Eq. } *$$

$\hookrightarrow$ frequency equation: $10\omega^4-85\omega^2+150=0 \rightarrow \begin{cases} \omega_1^2=2.5 \rightarrow \omega_1=1.5811 \\ \omega_2^2=6 \rightarrow \omega_2=2.4495 \end{cases}$

eigenvalues

natural frequencies

The substitution of $\omega^2=\omega_1^2=2.5$ in Eq. $*$ Leads to $X_2^{(1)}=2X_1^{(1)}$.

The substitution of $\omega^2=\omega_2^2=6.0$ in Eq. $*$ Leads to $X_2^{(2)}=-5X_1^{(2)}$.

So, the normal modes (eigenvectors) are given by:

eigenvectors

$$\vec{X}^{(1)}=\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix}=\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}X_1^{(1)} \quad \text{and} \quad \vec{X}^{(2)}=\begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix}=\begin{Bmatrix} 1 \\ -5 \end{Bmatrix}X_1^{(2)}$$

The free-vibration responses of the masses m_1 and m_2 are:

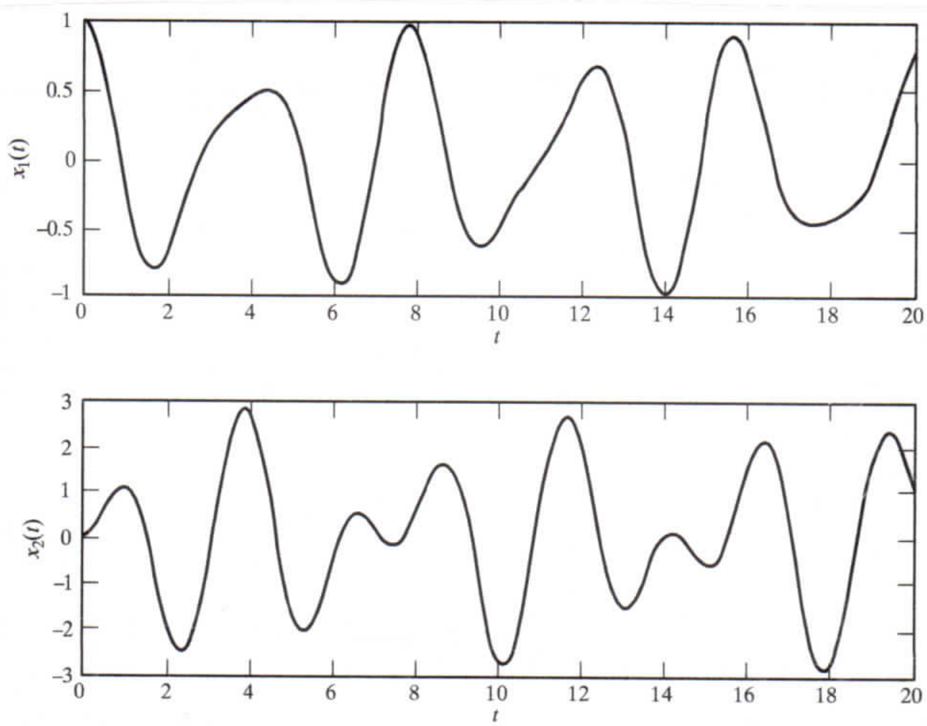
$$\begin{cases} x_1(t)=X_1^{(1)}\cos(1.5811t+\phi_1)+X_1^{(2)}\cos(2.4495t+\phi_2) \\ x_2(t)=2X_1^{(1)}\cos(1.5811t+\phi_1)-5X_1^{(2)}\cos(2.4495t+\phi_2) \end{cases} \rightarrow$$

$X_1^{(1)}, X_1^{(2)}, \phi_1$, and ϕ_2 must be determined from initial conditions.

$$\text{I.C.s}\begin{cases} x_1(t=0)=1 \\ \dot{x}_1(t=0)=0 \\ x_2(t=0)=0 \\ \dot{x}_2(t=0)=0 \end{cases} \xrightarrow[\text{these, we get}]{\text{By applying}} \begin{cases} X_1^{(1)}=5/7 \\ X_1^{(2)}=2/7 \\ \phi_1=0 \\ \phi_2=0 \end{cases} \quad \text{(verify!)}$$

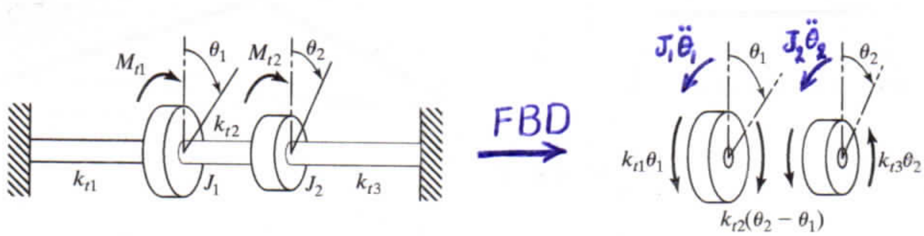
$$\text{Finally, }\begin{cases} x_1(t)=\frac{5}{7}\cos(1.5811t)+\frac{2}{7}\cos(2.4495t) \\ x_2(t)=\frac{10}{7}\cos(1.5811t)-\frac{10}{7}\cos(2.4495t) \end{cases}$$

The graphical representation of the above responses is provided below.



* Torsional System

Consider the following torsional system consisting of two discs mounted on a shaft, as shown. The three segments of the shaft have rotational spring constants k_{t1}, k_{t2}, and k_{t3}, as indicated in the figure. Also shown are the discs mass moments of inertia J_1 and J_2, the applied torques M_{t1} and M_{t2}, and the rotational degrees of freedom θ_1 and θ_2.



The differential equations of rotational motion for the discs J_1 and J_2 can be derived as:

$$\begin{cases} J_1\ddot{\theta}_1 + (k_{t1}+k_{t2})\theta_1 - k_{t2}\theta_2 = M_{t1} \\ J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2}+k_{t3})\theta_2 = M_{t2} \end{cases} \qquad \text{(how?)}$$

For the free-vibration of the system, we have:

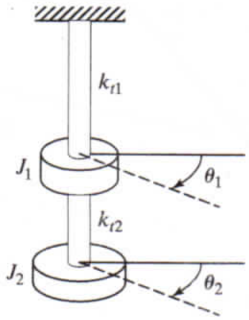
$$\begin{cases} J_1\ddot{\theta}_1 + (k_{t1}+k_{t2})\theta_1 - k_{t2}\theta_2 = 0 \\ J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2}+k_{t3})\theta_2 = 0 \end{cases} \rightarrow \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2}+k_{t3} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The analysis presented in case of translational systems is also applicable to torsional systems.

– Example: Find the natural frequencies and mode shapes for the torsional system shown in the following for $J_1 = J_0$, $J_2 = 2J_0$, and $k_{t1} = k_{t2} = k_t$.

Solution: The differential equations of motion for $\begin{cases} J_1 = J_0 \\ J_2 = 2J_0 \\ k_{t1} = k_{t2} = k_t \\ k_{t3} = 0 \end{cases}$ are:

previous page!



$$\begin{cases} J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0 \\ 2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0 \end{cases} \quad \text{(verify!)}$$

Consider the following harmonic solution:

$$\theta_i(t) = \Theta_i \cos(\omega t + \phi) \qquad i = 1, 2$$

By applying the harmonic solution on the EOMs we get the following frequency equation:

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0 \quad \text{(verify!)}$$

The solution of the frequency equation yields:

$$* \begin{cases} \omega_1 = \sqrt{\dfrac{k_t}{4J_0}(5 - \sqrt{17})} \\ \omega_2 = \sqrt{\dfrac{k_t}{4J_0}(5 + \sqrt{17})} \end{cases}$$

The amplitude ratios are:

$$\begin{cases} r_1 = \dfrac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = 2 - \dfrac{5 - \sqrt{17}}{4} \\ r_2 = \dfrac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = 2 - \dfrac{5 + \sqrt{17}}{4} \end{cases}$$

The normal modes of the system are given by:

$$\vec{\Theta}^{(1)} = \begin{Bmatrix} \Theta_1^{(1)} \\ \Theta_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 - \frac{5-\sqrt{17}}{4} \end{Bmatrix} \Theta_1^{(1)} \rightarrow \vec{\theta}^{(1)}(t) = \begin{Bmatrix} \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) \\ \Theta_2^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} \quad \text{First Mode}$$

$$\vec{\Theta}^{(2)} = \begin{Bmatrix} \Theta_1^{(2)} \\ \Theta_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 - \frac{5+\sqrt{17}}{4} \end{Bmatrix} \Theta_1^{(2)} \rightarrow \vec{\theta}^{(2)}(t) = \begin{Bmatrix} \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \\ \Theta_2^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} \quad \text{Second Mode}$$

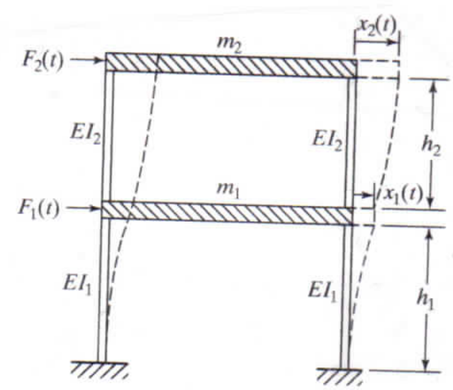
So, the free-vibration responses of the two discs are as following:

$$* \begin{cases} \theta_1(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \\ \theta_2(t) = \Theta_2^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_2^{(2)} \cos(\omega_2 t + \phi_2) \end{cases}$$

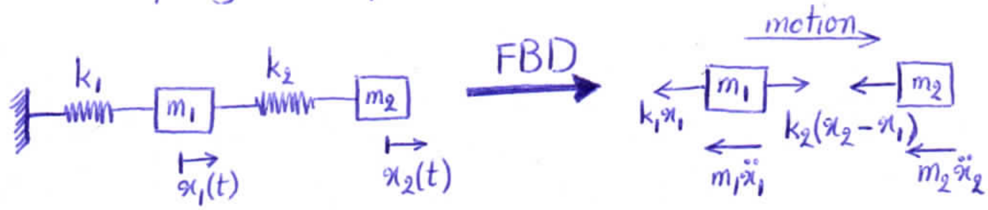
$\rightarrow$ In these equations, $\Theta_1^{(1)}$, $\Theta_1^{(2)}$, ϕ_1, and ϕ_2 can be found from I.C.s

* Additional Examples

– Example: Consider the two-story building shown. The girders are assumed to be rigid, and the columns have flexural rigidities EI_1 and EI_2, with negligible masses. The stiffness of each column can be computed as $\frac{24EI_i}{h_i^3}$ $i=1,2$. For $m_1=2m$, $m_2=m$, $h_1=h_2=h$, and $EI_1=EI_2=EI$, determine the natural frequencies and mode shapes of the frame.



Solution: The equivalent spring-mass system is shown below:



$$k_i = 2\left(\frac{24EI_i}{h_i^3}\right) \quad i=1,2 \rightarrow k_1 = k_2 = k = \frac{48EI}{h^3}$$

$$\text{EOMs}\begin{cases} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \end{cases} \quad \text{(how?)}$$

Consider harmonic motion: $x_i(t) = X_i\cos(\omega t + \phi) \quad i=1,2$

So, we can get:

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{(verify!)}$$

The frequency equation is:

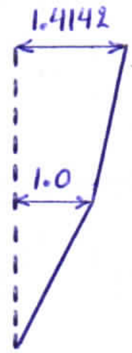
$$m_1m_2\omega^4 - (m_2k_1 + m_2k_2 + m_1k_2)\omega^2 + k_1k_2 = 0$$

$$\underrightarrow{\text{solve}}\begin{cases} \omega_1 = 0.5412\sqrt{\frac{k}{m}} = 3.7495\sqrt{\frac{EI}{mh^3}} \\ \omega_2 = 1.3066\sqrt{\frac{k}{m}} = 9.0524\sqrt{\frac{EI}{mh^3}} \end{cases} \quad \text{(verify!)} \ *$$

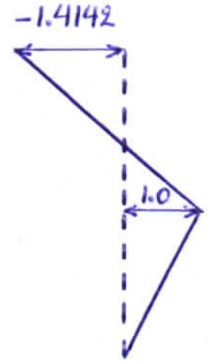
$$\rightarrow\begin{cases} r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + k_1 + k_2}{k_2} = \frac{-2m\omega_1^2 + 2k}{k} = \frac{-2(0.2929k) + 2k}{k} = 1.4142 \\ r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + k_1 + k_2}{k_2} = \frac{-2m\omega_2^2 + 2k}{k} = \frac{-2(1.7071) + 2k}{k} = -1.4142 \end{cases}$$

The mode shapes are: $\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.4142 \end{Bmatrix} X_1^{(1)}$ and $\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} X_1^{(2)}$

*

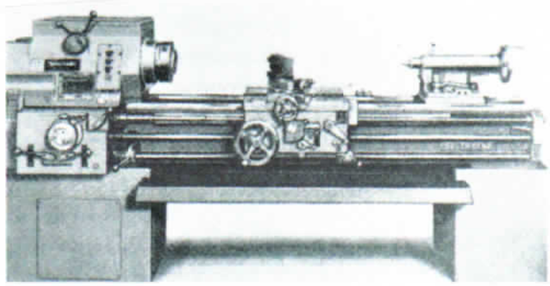


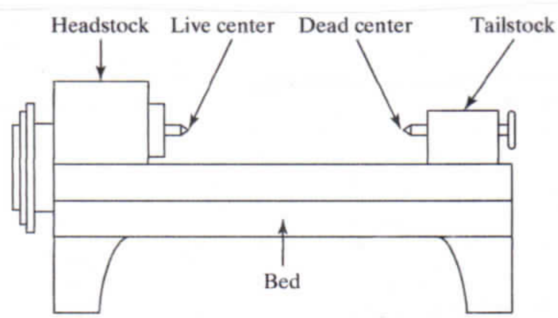
First mode

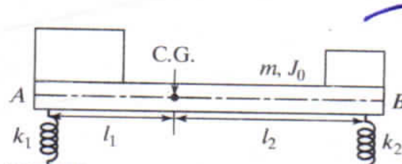


Second mode

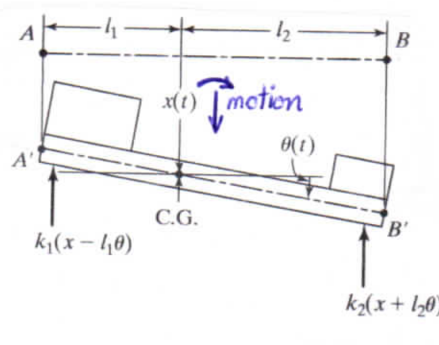
– Example: Consider the lathe shown below. The lathe bed is replaced by an elastic beam supported on short elastic columns and the headstock and tailstock are replaced by two lumped masses as shown. The lathe can be modeled as a two-degree-of-freedom system. The deflection $x(t)$ of the C.G. and rotation $\Theta(t)$ can be used to describe the motion of the system. Determine the equations of motion of the system using $x(t)$ and $\Theta(t)$.







FBD



Solution: From the free-body diagram shown, we get:

$$\begin{cases} \overset{+}{\uparrow}\Sigma F_y = 0 \rightarrow m\ddot{x} + k_1(x - \ell_1\theta) + k_2(x + \ell_2\theta) = 0 \\ \overset{\curvearrowleft}{+}\Sigma M_{C.G.} = 0 \rightarrow J_0\ddot{\theta} - k_1(x - \ell_1\theta)\ell_1 + k_2(x + \ell_2\theta)\ell_2 = 0 \end{cases}$$

$\downarrow$ rearrange

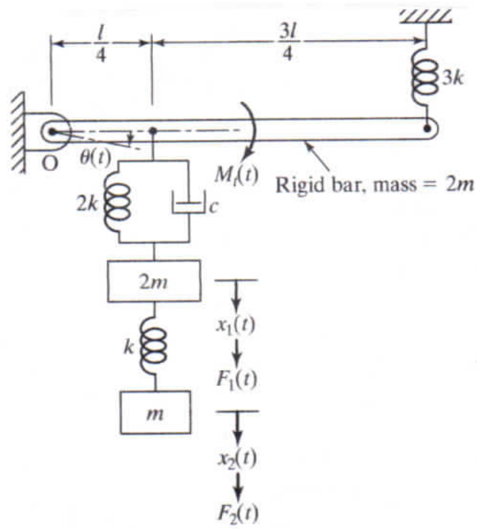
$$\begin{cases} m\ddot{x} + (k_1 + k_2)x + (-k_1\ell_1 + k_2\ell_2)\theta = 0 \\ J_0\ddot{\theta} + (-k_1\ell_1 + k_2\ell_2)x + (k_1\ell_1^2 + k_2\ell_2^2)\theta = 0 \end{cases}$$

$\downarrow$ write in matrix form

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1\ell_1 + k_2\ell_2 \\ -k_1\ell_1 + k_2\ell_2 & k_1\ell_1^2 + k_2\ell_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

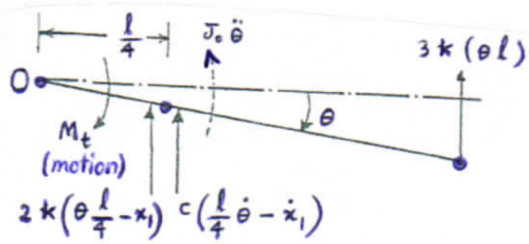
It is seen that each of the above equations contain 'x' and 'θ'. They become independent of each other if the coupling term $(k_1\ell_1 - k_2\ell_2)$ is equal to zero, i.e. $k_1\ell_1 = k_2\ell_2$. If $k_1\ell_1 \neq k_2\ell_2$, the resultant motion of the lathe AB is both translational and rotational when either a displacement or torque is applied through the C.G. of the body as an initial condition. In other words, the lathe rotates in the vertical plane and has vertical motion as well, unless $k_1\ell_1 = k_2\ell_2$. This is known as 'elastic or static coupling'.

– Example: Derive the equations of motion for the system shown.

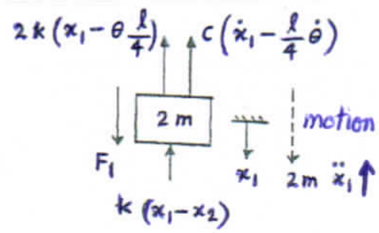


Note: $J_O = \frac{2}{3} m \ell^2$

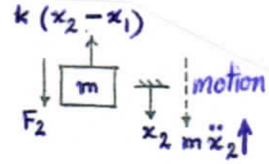
Solution: The equations of equilibrium are as following:



$$\overset{+}{\curvearrowright} \Sigma M_O = 0 \rightarrow J_o\ddot{\theta} + 2k\left(\frac{\ell\theta}{4} - x_1\right)\frac{\ell}{4} + c\left(\frac{\ell\dot{\theta}}{4} - \dot{x}_1\right)\frac{\ell}{4} + 3k(\ell\theta)\ell - M_t = 0$$



$$\overset{+}{\uparrow} \Sigma F_y = 0 \rightarrow 2m\ddot{x}_1 + 2k\left(x_1 - \frac{\ell\theta}{4}\right) + c\left(\dot{x}_1 - \frac{\ell\dot{\theta}}{4}\right) + k(x_1 - x_2) - F_1 = 0$$



$$\overset{+}{\uparrow}\Sigma F_y = 0 \rightarrow m\ddot{x}_2 + k(x_2 - x_1) - F_2 = 0$$

These equations can be stated in matrix form as following:

$$\begin{bmatrix} \frac{2}{3}m\ell^2 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{c\ell^2}{16} & -\frac{c\ell}{4} & 0 \\ -\frac{c\ell}{4} & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{25k\ell^2}{8} & -\frac{k\ell}{2} & 0 \\ -\frac{k\ell}{2} & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \theta \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} M_t \\ F_1 \\ F_2 \end{Bmatrix} \quad \text{(verify!)}$$

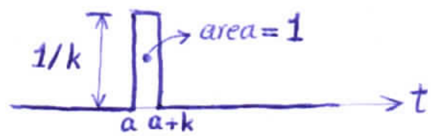
In "Appendix 2: Multi-Degree-of-Freedom Systems", additional examples using a slightly different approach as well as vibration of continuous systems are discussed.

* Appendix 1: Short Impulses and Dirac's Delta Function

Phenomena of an impulsive nature, such as the action of forces over short intervals of time, arise in various applications, for example, if a mechanical system is hit by a hammerblow. We use 'Dirac's delta function' to model such problems.

To model situations of that type, we consider the function

$$f_k(t-a)=\begin{cases} 1/k & \text{if } a\le t\le a+k \\ 0 & \text{otherwise} \end{cases}$$



(and Later its limit as $k\to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t=a$ to $t=a+k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a\le t\le a+k$ is called the 'Impulse' of the force. The impulse of f_k is:

$$I_k=\int_0^{\infty} f_k(t-a)\,dt=\int_a^{a+k}\frac{1}{k}\,dt=1.0$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k\to 0$ $(k>0)$. This limit is denoted by $\delta(t-a)$, that is:

$$\delta(t-a)=\lim_{k\to 0} f_k(t-a).$$

$\delta(t-a)$ is called the 'Dirac delta function' or the 'unit impulse function'.

We can write:

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^\infty \delta(t-a)\,dt = 1.0$$

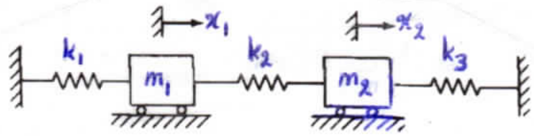
Note that $\delta(t-a)$ is not a function in the ordinary sense as used in calculus. From calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function. In particular, for a continuous function $g(t)$ one uses the property:

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a).$$

Reference: Erwin Kreyszig
Advanced Engineering Mathematics
John Wiley & Sons, Inc.

* Appendix 2: Multi-Degree-of-Freedom Systems

* 2 DOF System:

Consider:  Assume: $x_2 > x_1$

EOMs: $\begin{cases} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = 0 \\ m_2\ddot{x}_2 - k_2 x_1 + (k_2+k_3)x_2 = 0 \end{cases}$ (why?) matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Special case: $\begin{cases} k_1 = k_2 = k_3 = k \\ m_1 = m_2 = m \end{cases} \rightarrow \underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}_{[m]\{\ddot{x}\} + [k]\{x\} = \{0\}}$

If: $\{x\} = \{X \sin(\omega_n t + \phi)\} \rightarrow \{\ddot{x}\} = -\omega_n^2 \{x\}$

So, $-\omega_n^2 m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\underbrace{\left(-m\omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)}_{\det = 0}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \longrightarrow \text{divide both sides by } k$$

$$\left| \frac{-m\omega_n^2}{k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \right| = 0 \xrightarrow[\lambda = \frac{m\omega_n^2}{k}]{\text{Assume}} \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \rightarrow$$

$$(2-\lambda)(2-\lambda) - (-1)(-1) = 0 \rightarrow \begin{cases} \lambda_1 = 1 \rightarrow \frac{m}{k}\omega_{n,1}^2 = 1 \rightarrow \omega_{n,1}^2 = \frac{k}{m} \\ \lambda_2 = 3 \rightarrow \frac{m}{k}\omega_{n,2}^2 = 3 \rightarrow \omega_{n,2}^2 = \frac{3k}{m} \end{cases}$$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substitute $\lambda_1 = 1 \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow \begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \rightarrow x_1 = x_2 \rightarrow \{\Phi_1\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Substitute $\lambda_2 = 3 \rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow \begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \rightarrow x_1 = -x_2 \rightarrow \{\Phi_2\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

$$[\Phi] = [\Phi_1 \quad \Phi_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Orthogonality: $\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} = [\Phi]^T [m] [\Phi]$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}$$

$$\begin{bmatrix} \ddots & & \\ & K_r & \\ & & \ddots \end{bmatrix} = [\Phi]^T [k] [\Phi]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}$$

Check: $\begin{cases} \omega_{n,1}^2 = \dfrac{K_{r,11}}{M_{r,11}} = \dfrac{2k}{2m} = \dfrac{k}{m} \rightarrow \text{OK!} \\ \omega_{n,2}^2 = \dfrac{K_{r,22}}{M_{r,22}} = \dfrac{6k}{2m} = \dfrac{3k}{m} \rightarrow \text{OK!} \end{cases}$

Example: if $\begin{cases} k_1 = 10^5 \text{ N/m}, \ k_2 = 2\times10^5 \text{ N/m}, \ k_3 = 4\times10^5 \text{ N/m} \\ m_1 = 10 \text{ kg}, \ m_2 = 20 \text{ kg} \end{cases} \rightarrow \begin{cases} \lambda_1 = 1.5858 \\ \lambda_2 = 4.4142 \end{cases}$

(verify!) $\rightarrow [\Phi] = \begin{bmatrix} 1 & 1 \\ 0.7071 & -0.7071 \end{bmatrix}$

* Free Vibration:

Consider: $[m]\{\ddot{x}\}+[k]\{x\}=\{0\}$ Undamped!

I.C.s: $\{x_0\}$ and $\{\dot{x}_0\}$

$$[\ddots M_r \ddots]=[\Phi]^T[m][\Phi]=\begin{bmatrix} M_{r,11} & & 0 \\ & M_{r,22} & \\ 0 & \ddots & M_{r,nn} \end{bmatrix}$$

$$[\ddots k_r \ddots]=[\Phi]^T[k][\Phi]=\begin{bmatrix} K_{r,11} & & 0 \\ & K_{r,22} & \\ 0 & \ddots & K_{r,nn} \end{bmatrix}$$

Introduce $\{q\}$ such that $\{x\}=[\Phi]\{q\}$

$$\{\dot{x}\}=[\Phi]\{\dot{q}\}$$

$$\{\ddot{x}\}=[\Phi]\{\ddot{q}\}$$

So, $[m]\{\ddot{x}\}+[k]\{x\}=\{0\}$

$$\underbrace{[m][\Phi]\{\ddot{q}\}+[k][\Phi]\{q\}=\{0\}}_{\text{premultiply both sides by } [\Phi]^T}$$

$$\underline{[\Phi]^T[m][\Phi]}\{\ddot{q}\}+\underline{[\Phi]^T[k][\Phi]}\{q\}=[\Phi]^T\{0\}$$

$$[\ddots M_r \ddots]\{\ddot{q}\}+[\ddots K_r \ddots]\{q\}=\{0\}$$

If we consider the 2DOF system of the previous example:

$$\begin{bmatrix} M_{r,11} & 0 \\ 0 & M_{r,22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} K_{r,11} & 0 \\ 0 & K_{r,22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \begin{cases} 2m\ddot{q}_1 + 2kq_1 = 0 \rightarrow q_1(t) = q_{10}\cos(\omega_{n,1}t) + \dfrac{\dot{q}_{10}}{\omega_{n,1}}\sin(\omega_{n,1}t) \\ 2m\ddot{q}_2 + 6kq_2 = 0 \rightarrow q_2(t) = q_{20}\cos(\omega_{n,2}t) + \dfrac{\dot{q}_{20}}{\omega_{n,2}}\sin(\omega_{n,2}t) \end{cases}$$

Note: $\{x\} = [\Phi]\{q\} \rightarrow \{q\} = [\Phi]^{-1}\{x\}$

So, $\{q_0\} = \begin{Bmatrix} q_{10} \\ q_{20} \end{Bmatrix} = [\Phi]^{-1}\{x_0\} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix}$

$\{\dot{q}_0\} = \begin{Bmatrix} \dot{q}_{10} \\ \dot{q}_{20} \end{Bmatrix} = [\Phi]^{-1}\{\dot{x}_0\} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{Bmatrix}$

I.C.s

$$\{x\} = [\Phi]\{q\} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \rightarrow \begin{cases} x_1(t) = q_1(t) + q_2(t) \\ x_2(t) = q_1(t) - q_2(t) \end{cases}$$

Example: if $\begin{cases} k_1 = 10^5\ N/m,\ k_2 = 2\times10^5\ N/m,\ k_3 = 4\times10^5\ N/m & x_{10} = 0.05\ m,\ x_{20} = 0 \\ m_1 = 10\ kg,\ m_2 = 20\ kg & ,\ \dot{x}_{10} = 0,\ \dot{x}_{20} = 0 \end{cases}$

$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},\quad \begin{bmatrix} \ddots & & \\ & K_r & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} 3.2\times10^5 & 0 \\ 0 & 8.8\times10^5 \end{bmatrix}$$

(verify!)

$$\begin{cases} x_1(t) = 0.025\cos(125.93t) + 0.025\cos(210.1t) \\ x_2(t) = 0.0177\cos(125.93t) - 0.0177\cos(210.1t) \end{cases}$$

As another example, consider

$$\begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 75 & -25 \\ -25 & 50 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\omega_{n,1}^2 = 5.82764$$

$$\omega_{n,2}^2 = 7.55264$$

$$[\Phi] = \begin{bmatrix} 1 & 1 \\ 2.2247 & -0.2247 \end{bmatrix} \rightarrow [\Phi]^{-1} = \begin{bmatrix} 0.0918 & 0.4082 \\ 0.9082 & -0.4082 \end{bmatrix}$$

I.C.s: $x_{10} = x_{20} = 0$, $\dot{x}_{10} = 1$, $\dot{x}_{20} = 2$

$$\{x\} = [\Phi]\{q\} \rightarrow \{q_0\} = [\Phi]^{-1}\{x_0\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\{\dot{q}_0\} = [\Phi]^{-1}\{\dot{x}_0\} = \begin{Bmatrix} 0.9083 \\ 0.0917 \end{Bmatrix}$$

$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix}\{\ddot{q}\} + \begin{bmatrix} \ddots & & \\ & K_r & \\ & & \ddots \end{bmatrix}\{q\} = \{0\} \rightarrow \begin{cases} M_{r,11}\ddot{q}_1 + K_{r,11}q_1 = 0 \\ M_{r,22}\ddot{q}_2 + K_{r,22}q_2 = 0 \end{cases} \rightarrow$$

$$\begin{cases} \ddot{q}_1 + \omega_{n,1}^2 q_1 = 0 \longrightarrow q_1(t) = q_{10}\cos(\omega_{n,1}t) + \dfrac{\dot{q}_{10}}{\omega_{n,1}}\sin(\omega_{n,1}t) \\ \ddot{q}_2 + \omega_{n,2}^2 q_2 = 0 \longrightarrow q_2(t) = q_{20}\cos(\omega_{n,2}t) + \dfrac{\dot{q}_{20}}{\omega_{n,2}}\sin(\omega_{n,2}t) \end{cases}$$

$$\rightarrow \begin{cases} q_1(t) = 0.376\sin(2.414t) \\ q_2(t) = 0.033\sin(2.748t) \end{cases}$$

$$\{x(t)\} = [\Phi]\{q(t)\}$$

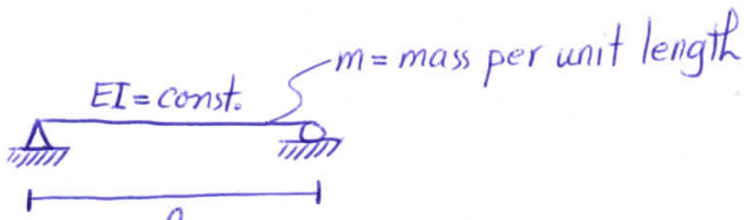
$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 2.2247 & -0.2247 \end{bmatrix} \begin{Bmatrix} 0.376 \sin(2.414t) \\ 0.033 \sin(2.748t) \end{Bmatrix}$$

As a result,

$$\begin{cases} x_1(t) = 0.376 \sin(2.414t) + 0.033 \sin(2.748t) \\ x_2(t) = 0.836 \sin(2.414t) - 7.415 \times 10^{-3} \sin(2.748t) \end{cases}$$

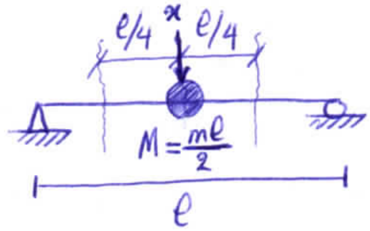
* Continuous Systems:

Consider a simply-supported beam as a continuous system. In order to study the vibration of the system, we may model/idealize it as a SDOF system, or a 2DOF system, or in general a NDOF system. Through consideration of the idealized model(s), the vibration properties of the system can be approximated. Consideration of higher number of degrees of freedom will result in more accurate results. Here we consider SDOF and 2DOF idealizations and evaluate the accuracy of the predicted frequency of the system.

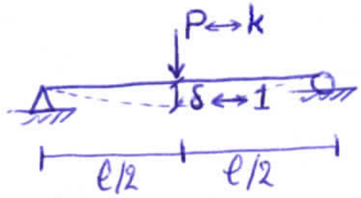


$$\omega_{n,exact} = \pi^2\sqrt{\frac{EI}{m\ell^4}} = 9.869\sqrt{\frac{EI}{m\ell^4}}$$

Model as a SDOF System:



$\rightarrow$

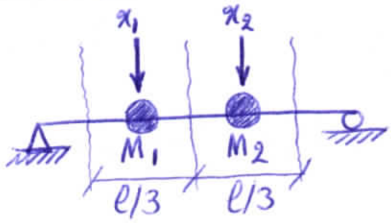


$$\delta = \frac{P\ell^3}{48EI} \rightarrow P = \underbrace{\frac{48EI}{\ell^3}}_{k}\delta$$

$$\text{So, } k = \frac{48EI}{\ell^3},\ M = \frac{m\ell}{2} \rightarrow M\ddot{x} + kx = 0 \rightarrow \omega_n^2 = \frac{k}{M} = \frac{\frac{48EI}{\ell^3}}{\frac{m\ell}{2}} = \frac{96EI}{m\ell^4}$$

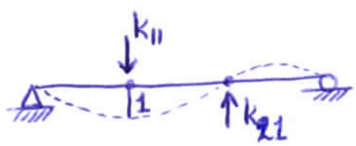
$$\rightarrow \omega_{n,SDOF} = 9.798\sqrt{\frac{EI}{m\ell^4}} \xleftrightarrow[\text{with}]{\text{Compare}} \omega_{n,exact} = 9.869\sqrt{\frac{EI}{m\ell^4}}$$

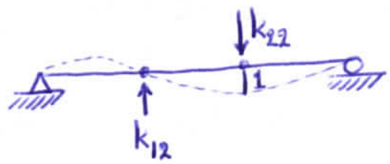
Model as a 2DOF System:



$$\begin{cases} M_1 = \dfrac{m\ell}{3} \\ M_2 = \dfrac{m\ell}{3} \end{cases} \rightarrow [m] = \frac{m\ell}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: In $[k]_{n\times n}$, k_{ij} = the force at 'i' due to a unit displacement at 'j'





$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$\downarrow$$

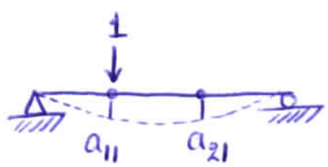
$$[a] = [k]^{-1}$$

$$\downarrow$$

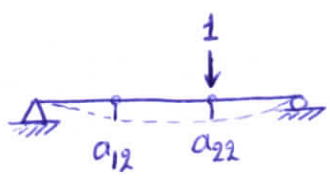
Note: In $[a]_{n\times n}$, a_{ij} =
the displacement at 'i'
due to a unit force at 'j'

$$[a] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

So,



$$\rightarrow a_{11} = \frac{8\ell^3}{486EI}, \quad a_{21} = \frac{7\ell^3}{486EI}$$



$$\rightarrow a_{12} = \frac{7\ell^3}{486EI}, \quad a_{22} = \frac{8\ell^3}{486EI}$$

$$[a] = [k]^{-1} = \frac{\ell^3}{486EI}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \rightarrow [k]^{-1}[m]\{\ddot{x}\} + [k]^{-1}[k]\{x\} = [k]^{-1}\{0\} \rightarrow$$

$$[a][m]\{\ddot{x}\} + [I]\{x\} = \{0\}$$

$$\frac{\ell^3}{486EI}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \frac{m\ell}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

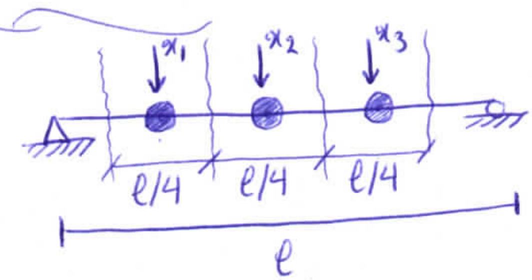
$$\left| \frac{-\omega_n^2 m\ell^4}{1458EI}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \rightarrow \left| \begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} - \underbrace{\frac{1458EI}{\omega_n^2 m\ell^4}}_{\lambda}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 8-\lambda & 7 \\ 7 & 8-\lambda \end{vmatrix} = 0 \rightarrow (\lambda - 15)(\lambda - 1) = 0$$

$$\begin{cases} \lambda_1 = 15 = \dfrac{1458EI}{\omega_{n,1}^2 m\ell^4} \rightarrow \omega_{n,1}^2 = \dfrac{1458EI}{15m\ell^4} \rightarrow \omega_{n,1} = 9.859\sqrt{\dfrac{EI}{m\ell^4}} \\[2ex] \lambda_2 = 1 = \dfrac{1458EI}{\omega_{n,2}^2 m\ell^4} \rightarrow \omega_{n,2}^2 = \dfrac{1458EI}{m\ell^4} \rightarrow \omega_{n,2} = 30.184\sqrt{\dfrac{EI}{m\ell^4}} \end{cases}$$

$$\rightarrow \omega_{n,2DOF} = 9.859\sqrt{\frac{EI}{m\ell^4}} \xleftrightarrow[\text{with}]{\text{Compare}} \omega_{n,exact} = 9.869\sqrt{\frac{EI}{m\ell^4}}$$

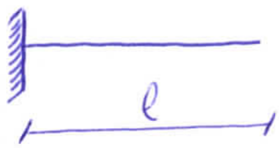
Exercise: Model as a 3DOF system:



Note: $EI = \text{const.}$

$$[m] = \frac{m\ell}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

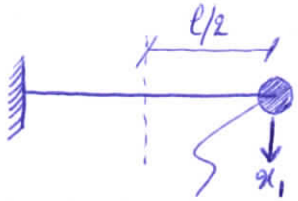
As another example, we consider a cantilever beam, and model it as a SDOF and a 2DOF system.



$EI = \text{Const.}$

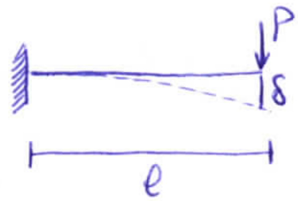
m = mass per unit length

Model as a SDOF System:



$M = \frac{m\ell}{2}$

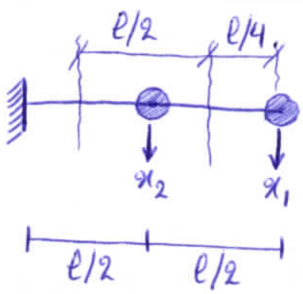
$\rightarrow$



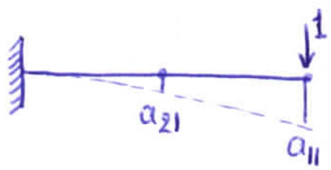
$$\delta = \frac{P\ell^3}{3EI} \rightarrow P = \underbrace{\frac{3EI}{\ell^3}}_{k} \cdot \delta$$

$$\omega_n^2 = \frac{k}{M} = \frac{3EI/\ell^3}{m\ell/2} = \frac{6EI}{m\ell^4} \rightarrow \omega_{n,SDOF} = 2.449\sqrt{\frac{EI}{m\ell^4}}$$

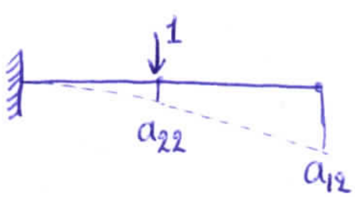
Model as a 2DOF System:



$$M_1 = \frac{m\ell}{4}, \quad M_2 = \frac{m\ell}{2} \rightarrow [m] = \frac{m\ell}{4}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$



$$\rightarrow a_{11} = \frac{\ell^3}{3EI} = \frac{16\ell^3}{48EI}, \quad a_{21} = \frac{5\ell^3}{48EI}$$



$$\rightarrow a_{12} = \frac{5\ell^3}{48EI}, \quad a_{22} = \frac{2\ell^3}{48EI}$$

$$[a] = \frac{\ell^3}{48EI}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$$

So, $\left| -\omega_n^2 [a][m] + [I] \right| = 0$

$$\left| \frac{-m\ell^4 \omega_n^2}{192EI} \begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 16 & 10 \\ 5 & 4 \end{bmatrix} - \underbrace{\frac{192EI}{\omega_n^2 m\ell^4}}_{\lambda} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \rightarrow \begin{vmatrix} 16-\lambda & 10 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$$\rightarrow \begin{cases} \lambda_1 = 19.274 = \dfrac{192EI}{\omega_{n,1}^2 m\ell^4} \longrightarrow \omega_{n,1} = 3.156\sqrt{\dfrac{EI}{m\ell^4}} \\ \lambda_2 = 0.726 = \dfrac{192EI}{\omega_{n,2}^2 m\ell^4} \longrightarrow \omega_{n,2} = 16.262\sqrt{\dfrac{EI}{m\ell^4}} \end{cases}$$

$\lambda_1 = 19.274$

$$\begin{bmatrix} 16-19.274 & 10 \\ 5 & 4-19.274 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \longrightarrow x_1 = 1, \; x_2 = 0.3274$$

$$\downarrow$$

$$\{\Phi_1\} = \begin{Bmatrix} 1 \\ 0.3274 \end{Bmatrix}$$

$\lambda_2 = 0.726$

$$\begin{bmatrix} 16-0.726 & 10 \\ 5 & 4-0.726 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \longrightarrow x_1 = 1, \; x_2 = -1.5274$$

$$\downarrow$$

$$\{\Phi_2\} = \begin{Bmatrix} 1 \\ -1.5274 \end{Bmatrix}$$

$$[\Phi] = \begin{bmatrix} 1 & 1 \\ 0.3274 & -1.5274 \end{bmatrix}$$

Note that:

$$\begin{bmatrix} \ddots & & \\ & M_r & \\ & & \ddots \end{bmatrix} = [\Phi]^T [m] [\Phi]$$

$$\begin{bmatrix} \ddots & & \\ & K_r & \\ & & \ddots \end{bmatrix} = [\Phi]^T [k] [\Phi]$$

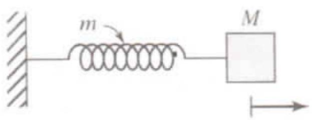
(verify orthogonality!)

where $[k] = [a]^{-1}$

* Tables

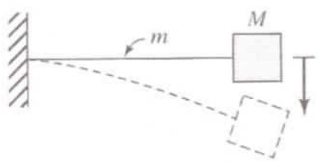
Equivalent Masses, Springs and Dampers

Equivalent masses



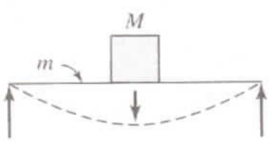
Mass (M) attached at end of spring of mass m

$$m_{eq} = M + \frac{m}{3}$$



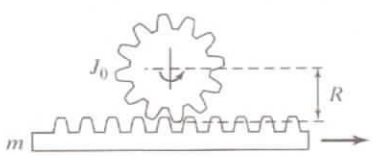
Cantilever beam of mass m carrying an end mass M

$$m_{eq} = M + 0.23\,m$$



Simply supported beam of mass m carrying a mass M at the middle

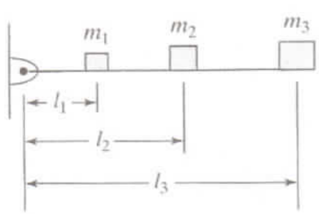
$$m_{eq} = M + 0.5\,m$$



Coupled translational and rotational masses

$$m_{eq} = m + \frac{J_0}{R^2}$$

$$J_{eq} = J_0 + mR^2$$



Masses on a hinged bar

$$m_{eq_1} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3$$

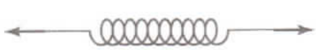
Equivalent springs

Rod under axial load (l = length, A = cross sectional area)

$$k_{eq} = \frac{EA}{l}$$

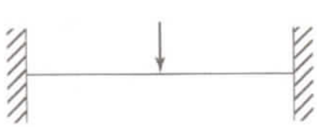
Tapered rod under axial load (D, d = end diameters)

$$k_{eq} = \frac{\pi EDd}{4l}$$



Helical spring under axial load (d = wire diameter, D = mean coil diameter, n = number of active turns)

$$k_{eq} = \frac{Gd^4}{8nD^3}$$

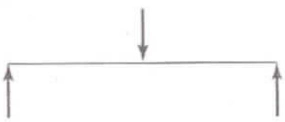


Fixed-fixed beam with load at the middle

$$k_{eq} = \frac{192EI}{l^3}$$

Cantilever beam with end load

$$k_{eq} = \frac{3EI}{l^3}$$

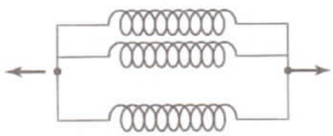


Simply supported beam with load at the middle

$$k_{eq} = \frac{48EI}{l^3}$$

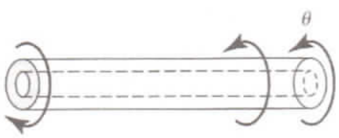
Springs in series

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$



Springs in parallel

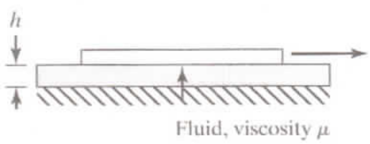
$$k_{eq} = k_1 + k_2 + \cdots + k_n$$



Hollow shaft under torsion
(l = length, D = outer diameter,
d = inner diameter,

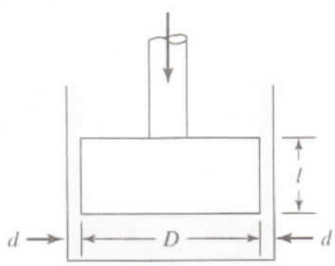
$$k_{eq} = \frac{\pi G}{32l}(D^4 - d^4)$$

Equivalent viscous dampers



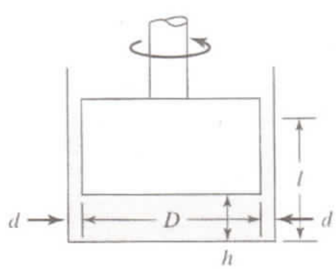
Relative motion between parallel surfaces
(A = area of smaller plate)

$$c_{eq} = \frac{\mu A}{h}$$



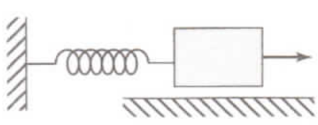
Dashpot (axial motion of a piston in a cylinder)

$$c_{eq} = \mu \frac{3\pi D^3 l}{4d^3}\left(1 + \frac{2d}{D}\right)$$



Torsional damper

$$c_{eq} = \frac{\pi \mu D^2 (l - h)}{2d} + \frac{\pi \mu D^3}{32h}$$



Dry friction (Coulomb damping)
(fN = friction force,
ω = frequency,
X = amplitude of vibration)

$$c_{eq} = \frac{4fN}{\pi \omega X}$$

Quantity	SI Equivalence	English Equivalence
Work or energy	1 in. $-$ $lb_f = 0.11298484$ J 1 ft $-$ $lb_f = 1.355818$ J 1 Btu $= 1055.056$ J 1 kWh $= 3.6 \times 10^6$ J	1 J $= 8.850744$ in. $-$ lb_f 1 J $= 0.737562$ ft $-$ lb_f $= 0.947817 \times 10^{-3}$ Btu $= 0.277778$ kWh
Power	1 in $-$ lb_f/sec $= 0.1129848$ W 1 ft $-$ lb_f/sec $= 1.355818$ W $= 0.0018182$ hp 1 hp $= 745.7$ W	1 W $= 8.850744$ in. $-$ lb_f/sec 1 W $= 0.737562$ ft $-$ lb_f/sec $= 1.34102 \times 10^{-3}$ hp
Area moment of inertia or second moment of area	1 $in^4 = 41.6231 \times 10^{-8}$ m^4 1 $ft^4 = 86.3097 \times 10^{-4}$ m^4	1 $m^4 = 240.251 \times 10^4$ in^4 $= 115.862$ ft^4
Mass moment of inertia	1 in $-$ lb_f $-$ $sec^2 = 0.1129848$ $m^2 \cdot kg$	1 $m^2 \cdot kg = 8.850744$ in. $-$ lb_f $-$ sec^2
Spring constant:		
translatory	1 lb_f/in. $= 175.1268$ N/m 1 lb_f/ft $= 14.5939$ N/m	1 N/m $= 5.71017 \times 10^{-3}$ lb_f/in. $= 68.5221 \times 10^{-3}$ lb_f/ft
torsional	1 in. $-$ lb_f/rad $= 0.1129848$ m $\cdot$ N/rad	1 m $\cdot$ N/rad $= 8.850744$ in-lb_f/rad $= 0.737562$ lb_f $-$ ft/rad
Damping constant:		
translatory	1 lb_f $-$ sec/in $= 175.1268$ N $\cdot$ s/m	1 N $\cdot$ s/m $= 5.71017 \times 10^{-3}$ lb_f $-$ sec/in.
torsional	1 in $-$ lb_f $-$ sec/rad $= 0.1129848$ m $\cdot$ N $\cdot$ s/rad	1 m $\cdot$ N $\cdot$ s/rad $= 8.850744$ lb_f $-$ in $-$ sec/rad
Angles	1 rad $= 57.295754$ degrees; 1 rpm $= 0.0166667$ rev/sec $= 0.104720$ rad/sec;	1 degree $= 0.0174533$ rad; 1 rad/sec $= 9.54909$ rpm

Trigonometric Basic Equations

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
$\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$	$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
$\sin(2\alpha)=2\sin\alpha\cos\alpha=\dfrac{2\tan\alpha}{1+\tan^2\alpha}$ $\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1$ $=1-2\sin^2\alpha=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$ $\tan(2\alpha)=\dfrac{2\tan\alpha}{1-\tan^2\alpha}$ $\cot(2\alpha)=\dfrac{\cot^2\alpha-1}{2\cot\alpha}$	$\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ $\cos\left(\frac{\alpha}{2}\right)=\pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ $\tan\left(\frac{\alpha}{2}\right)=\dfrac{1-\cos\alpha}{\sin\alpha}=\dfrac{\sin\alpha}{1+\cos\alpha}=\pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$ $\cot\left(\frac{\alpha}{2}\right)=\dfrac{1+\cos\alpha}{\sin\alpha}=\dfrac{\sin\alpha}{1-\cos\alpha}=\pm\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}$
$\sin^2\alpha+\cos^2\alpha=1$ $\tan^2\alpha+1=\sec^2\alpha$ $1+\cot^2\alpha=\csc^2\alpha$	$\sin(3\alpha)=3\sin\alpha-4\sin^3\alpha$ $\cos(3\alpha)=4\cos^3\alpha-3\cos\alpha$ $\tan(3\alpha)=\dfrac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}$
$\sin(4\alpha)=4\sin\alpha\cos\alpha\,(2\cos^2\alpha-1)$ $\cos(4\alpha)=8\cos^4\alpha-8\cos^2\alpha+1$	$\sin(5\alpha)=5\sin\alpha-20\sin^3\alpha+16\sin^5\alpha$ $\cos(5\alpha)=16\cos^5\alpha-20\cos^3\alpha+5\cos\alpha$