# FLOW OF COMPRESSIBLE FLUID

**Introduction**:

- At normal conditions, we assumed that the flow of liquid is incompressible. But the effect of compressibility of fluid must be considered if there is significant change in volume of the fluid i.e. $\Delta\rho/\rho$ ≃ ≃≙
- The compressible fluid flow problems are much more difficult than the incompressible because of thermodynamic consideration.

**Classification of Compressible Flow**:

- At low velocity (M < ≃ ▷≃) flow is always considered incompressible.
- As fluid velocity approaches to the sonic velocity, the compressibility effect of the fluid acquire importance.
- So Mach number is most important parameter in compressible flow and compressible flow of fluid is categorized by the Mach number.

$$\text{Mach number, } M = \frac{\text{Velocity of fluid or aircraft}}{\text{Velocity of sound}}$$

# FLOW OF COMPRESSIBLE FLUID

**Classification of Compressible Flow (continue):**

Based^on^the^value^of^Mach^number ^there^are^six^types^ flow^defined^as^follows

| Types of Flow | Mach No. | Example |
|---|---|---|
| Subsonic incompressible flow | M <= 0.30 | Fan, Blowers,Hydraulics |
| Subsonic compressible flow | 0.3 < M < 1.0 | Aircraft Turbomachines |
| Transsonic flow | 0.9 < M < 1.0 | Compressor blades |
| Sonic flow | M =1 | Velocity of Sound |
| Supersonic flow | 1 < M < 3 | Mig 21 flight |
| Hypersonic flow | M > 3 | Rockets, Missiles |

## FLOW OF COMPRESSIBLE FLUID

**Fundamental Equations of Compressible Flow:**

- Continuity equation
- Energy equation
- Momentum equation
- P-v-t relation for different process
- Equation of State
- $C_p - C_v = R$
- $C_p / C_v = k$

# FLOW OF COMPRESSIBLE FLUID

TAYLOR^^SERIES

$$f(x) = \rho_X = \rho$$

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{f''(x)}{2!}\cdot h^2 + \frac{f'''(x)}{3!}\cdot h^3 + -------$$

$$\rho_{x+\frac{dx}{2}} = \rho + \left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} + \left(\frac{\delta^2\rho}{\delta x^2}\right)\cdot\frac{1}{2!}\cdot\left(\frac{dx}{2}\right)^2 + \left(\frac{\delta^3\rho}{\delta x^3}\right)\cdot\frac{1}{3!}\cdot\left(\frac{dx}{2}\right)^3 + ---------$$

$$\rho_{x+\frac{dx}{2}} = \rho + \left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} \qquad \text{(Neglecting higher order terms)}$$

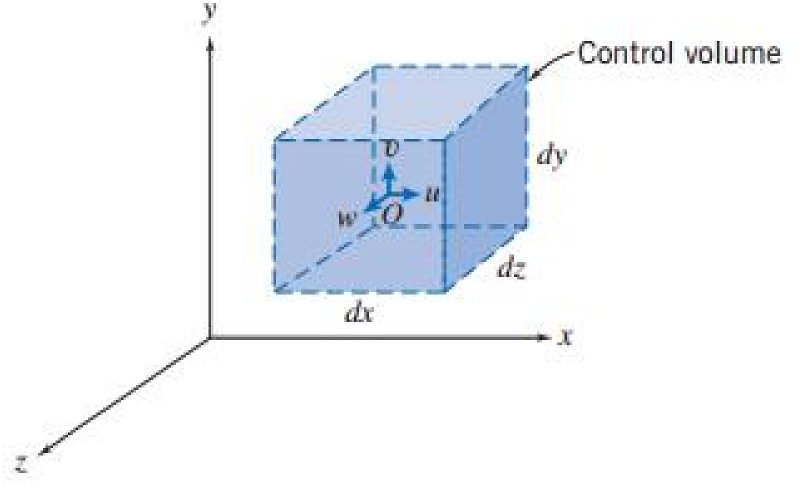
Here $h = \frac{dx}{2}$ and $f(x) = \rho_x = \rho$

# FLOW OF COMPRESSIBLE FLUID

## Derivation of Continuity Equation

Consider an infinitesimal control volume with sides of length dx, dy and dz . The density of fluid at the center of the cube 'o' is 'ρ' and velocity is

assumed to be $\vec{v} = u \cdot \hat{i} + v \cdot \hat{j} + w \cdot \vec{k}$



# FLOW OF COMPRESSIBLE FLUID

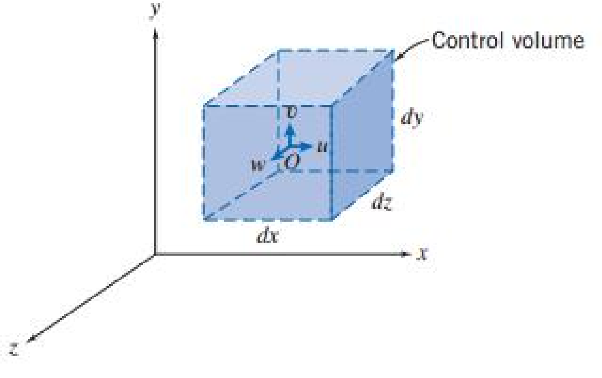
## Derivation of Continuity Equation (Continue)

Now velocity and density of fluid at right surface (along x-axis) of the control volume

$$u_{x+\frac{dx}{2}} = u + \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2} \quad \text{and} \quad \rho_{x+\frac{dx}{2}} = \rho + \left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2}$$

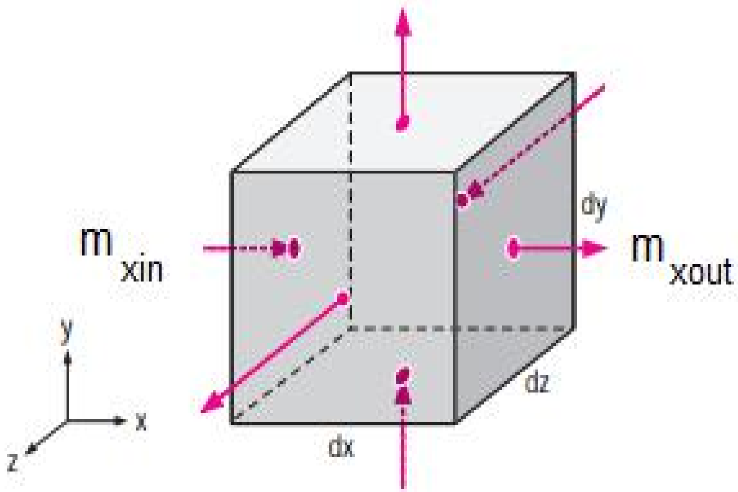
Again velocity and density of fluid at left surface (along x-axis) of the control volume

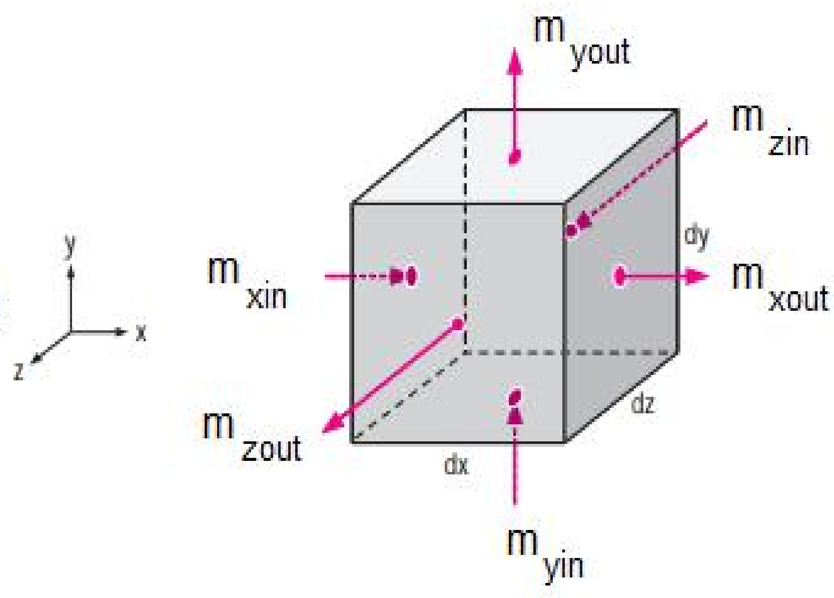
$$u_{x+\frac{dx}{2}} = u - \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2} \quad \text{and} \quad \rho_{x+\frac{dx}{2}} = \rho - \left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2}$$



## FLOW OF COMPRESSIBLE FLUID

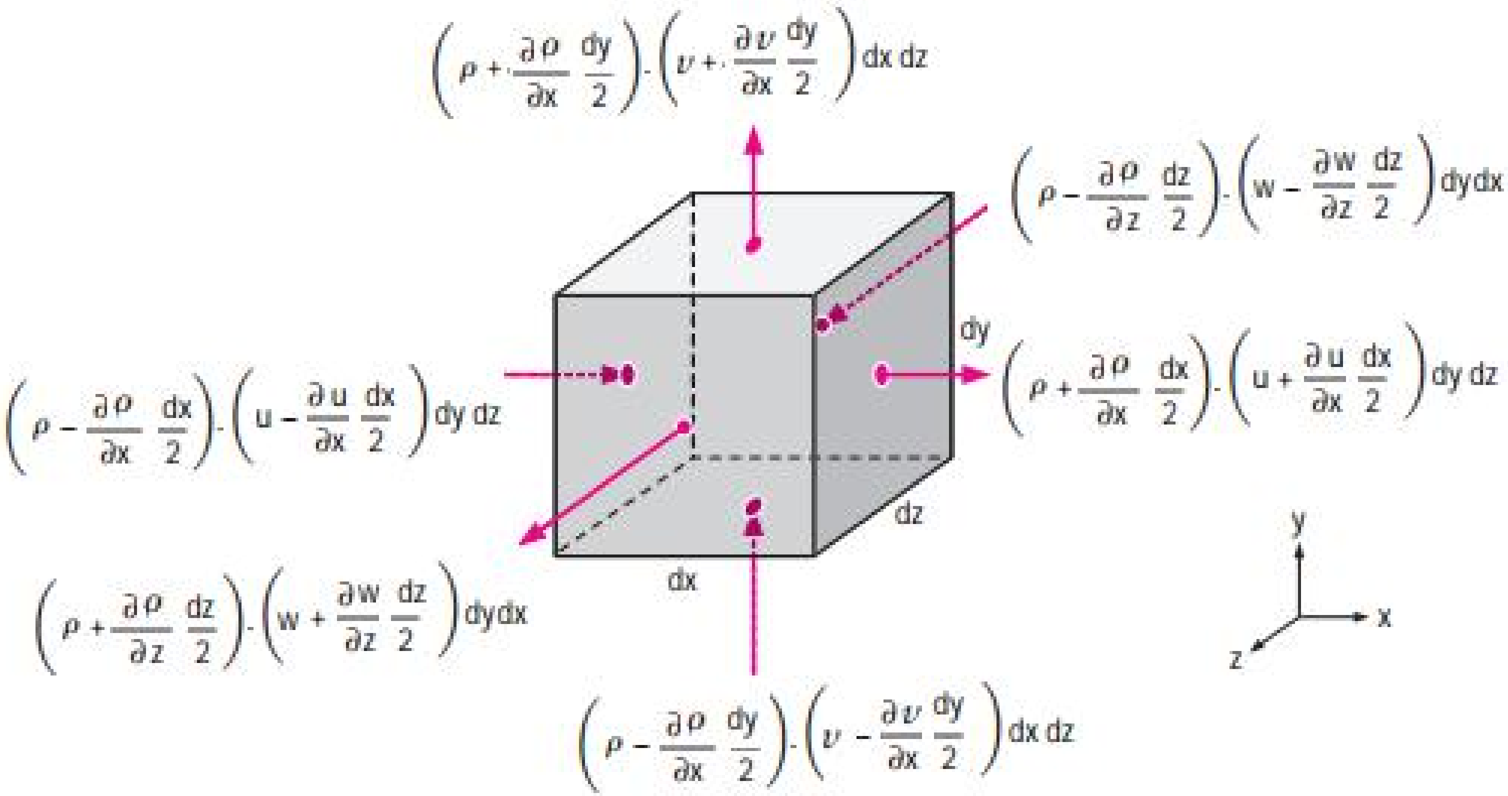
## Derivation of Continuity Equation (Continue)



## FLOW OF COMPRESSIBLE FLUID

## Derivation of Continuity Equation (Continue)



# FLOW OF COMPRESSIBLE FLUID

## Derivation of Continuity Equation (Continue)

$$m_{x\_in} = \left[u - \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot\left[\rho - \left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho - u\cdot\left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2} - \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2} + \left(\frac{\delta u}{\delta x}\right)\cdot\left(\frac{\delta \rho}{\delta x}\right)\cdot\left(\frac{dx}{2}\right)^2\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho - u\cdot\left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2} - \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

$$m_{x\_out} = \left[u + \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot\left[\rho + \left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho + u\cdot\left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2} + \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2} + \left(\frac{\delta u}{\delta x}\right)\cdot\left(\frac{\delta \rho}{\delta x}\right)\cdot\left(\frac{dx}{2}\right)^2\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho + u\cdot\left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2} + \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

FLOW OF COMPRESSIBLE FLUID

## Derivation of Continuity Equation (Continue)

$$m_{x\_net} = m_{x\_in} - m_{x\_out} = -\left[u\cdot\left(\frac{\delta\rho}{\delta x}\right) + \rho\cdot\left(\frac{\delta u}{\delta x}\right)\right](dx\cdot dy\cdot dz)$$

$$= -\frac{\delta}{\delta x}(\rho\, u)\cdot(dx\cdot dy\cdot dz)$$

$$m_{y\_net} = -\frac{\delta}{\delta y}(\rho\, v)\cdot(dx\cdot dy\cdot dz)$$

$$m_{z\_net} = -\frac{\delta}{\delta z}(\rho\, w)\cdot(dx\cdot dy\cdot dz)$$

Rate^of^change^of^mass^within^control^volume^
dx dy dz^

$$m_{change} = \frac{\delta}{\delta t}(\rho\cdot dx\cdot dy\cdot dz) = \frac{\delta\rho}{\delta t}(dx\cdot dy\cdot dz)$$

## FLOW OF COMPRESSIBLE FLUID

## Derivation of Continuity Equation (Continue)

As per principle of conservation of mass,

$$m_{x\_net} + m_{y\_net} + m_{z\_net} = m_{change}$$

$$-\frac{\delta}{\delta x}(\rho u)\cdot(dx\cdot dy\cdot dz) - \frac{\delta}{\delta y}(\rho v)\cdot(dx\cdot dy\cdot dz) - \frac{\delta}{\delta z}(\rho w)\cdot(dx\cdot dy\cdot dz) = \frac{\delta\rho}{\delta t}(dx\cdot dy\cdot dz)$$

$$\frac{\delta}{\delta x}(\rho u) + \frac{\delta}{\delta y}(\rho v) + \frac{\delta}{\delta z}(\rho w) + \frac{\delta\rho}{\delta t} = 0$$

Again,

$$\vec{V} = u\,\hat{i} + v\,\hat{j} + w\,\vec{k} \qquad \nabla = \hat{i}\cdot\frac{\delta}{\delta x} + \hat{j}\cdot\frac{\delta}{\delta y} + \vec{k}\cdot\frac{\delta}{\delta z}$$

$$\vec{V}\cdot\nabla = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} = \text{Div}\,\vec{V} \qquad \rho\vec{V} = \hat{i}(u\rho) + \hat{j}(v\rho) + \vec{k}(w\rho)$$

$$\nabla\cdot\rho\vec{V} = \frac{\delta}{\delta x}(\rho u) + \frac{\delta}{\delta y}(\rho v) + \frac{\delta}{\delta z}(\rho w)$$

Continuity equation become, $\nabla\cdot\rho\vec{V} + \frac{\delta\rho}{\delta t} = 0$

# FLOW OF COMPRESSIBLE FLUID

## Derivation of Continuity Equation (Continue)

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Alternative form of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla} \rho}_{\text{Material derivative of } \rho} + \rho \vec{\nabla} \cdot \vec{V} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{V} = 0$$

## FLOW OF COMPRESSIBLE FLUID

**Q. Derive Energy Equation for Compressible Fluid Considering Both Isothermal and Adiabatic Process:**

**Solution:**

For compressible fluid density depends on pressure and temperature. The internal energy is also affected by the change of heat. Energy equation can be applied to any gas or system. Considering a fluid flow in a streamline from point 1 to point 2.

**Assumption:**

- One dimensional flow
- Steady flow
- Frictionless flow
- Equation of state is applicable.

Assume $P_1$, $T_1$ and $\rho_1$ are pressure, temperature and density of fluid at point 1 and $P_2$, $T_2$ and $\rho_2$ are pressure, temperature and density of fluid at point 2.

**(SEE HAND ANALYSIS FOR DERIVATION FOR BOTH PROCESS)**

# FLOW OF COMPRESSIBLE FLUID

**Q: Derive expression for mass flow rate of flow of compressible fluid through a converging nozzle from a large reservoir to a receiver. What is the condition for maximum flow rate for air ? Deduce also an expression for maximum flow rate.**

Derivation:

Assume $P_1$, $V_1$ and $\rho_1$ are pressure, velocity and density of fluid at section 1 and $P_2$, $V_2$ and $\rho_2$ are pressure, velocity and density of fluid at section 2.

**Assumptions**:

- Adiabatic process
- Frictionless flow
- Steady flow
- One dimensional flow
- Equation of state is applicable.

(SEE HAND NOTE ANALYSIS FOR DERIVATION)

## FLOW OF COMPRESSIBLE FLUID

**Q: Derive expression for the area velocity relationship for one dimensional compressible flow. Show the effects of variation of area on subsonic, sonic, and supersonic flows.**

**Deriv**ation:

Assume P, V and ρ are pressure, velocity and density of the fluid, M is the Mach number and C is the velocity of sound.

**Assumptions**:

- One dimensional flow
- Steady flow
- Frictionless flow
- Adiabatic conditions

**(SEE HAND NOTE ANALYSIS FOR DERIVATION)**

## MACH WAVE WHEN BODY MOVING AT A SONIC , SUBSONIC AND SUPERSONIC VELOCITY

### Subsonic Velocit

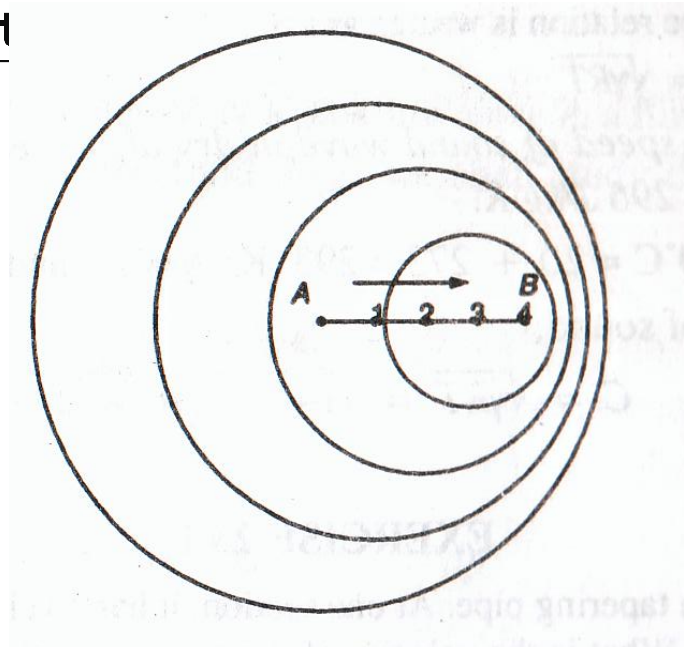



Fig. 23·3. Subsonic velocity.

The velocity 'v' of projectile is always less than velocity of wave 'c'. The nose of the projectile is always behind the wave front of the pressure wave. The projectile is always penetrating in an area of disturbed fluid. The disturbed fluid has some effect on the fluid resistance to the motion of the projectile.

**MACH WAVE WHEN BODY MOVING AT A SONIC , SUBSONIC AND SUPERSONIC VELOCITY**

## Sonic Velocity:

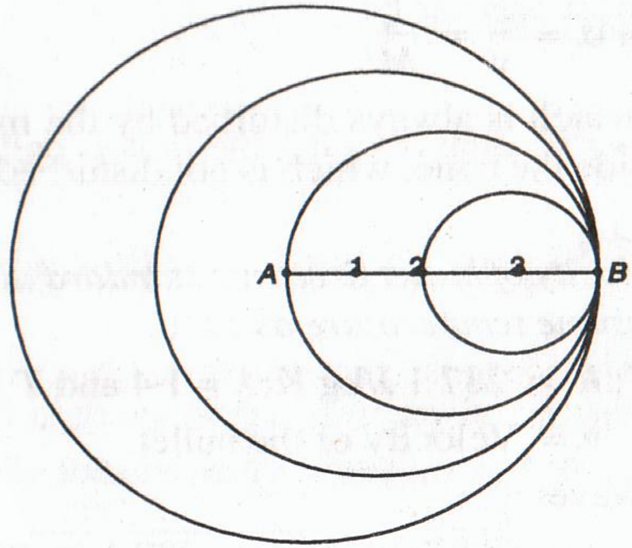



Fig. 23·4. Sonic velocity.

The^^velocity^^^‘v’^of^the^projectile^^is^equal^to^the^velocity^of^wave^‘c’ ^^
The^^nose^of^the^projectile^and^the^wave^front^always^move^through^a^
common^point ^The^nose^of^the^projectile^pushes^wave^with^intense^
pressure ^which^moves^along^with^it^and^is^known^as^shock^wave ^It^
causes^a^great^resistance^to^the^motion^of^the^projectile

**MACH WAVE WHEN BODY MOVING AT A SONIC , SUBSONIC AND SUPERSONIC VELOCITY**

## Supersonic Velocity:

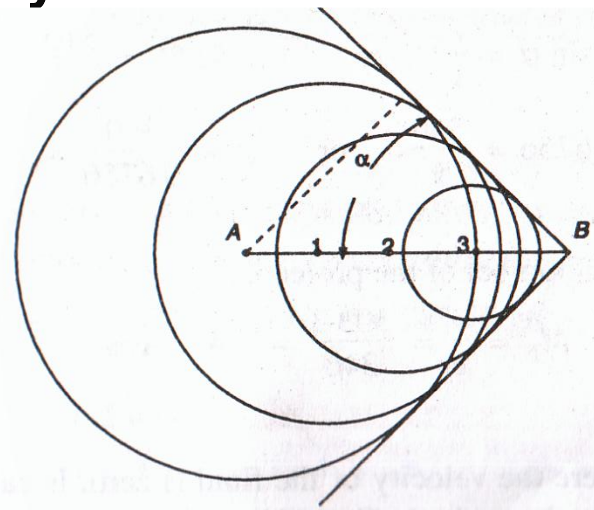



Fig. 23·5. Supersonic velocity.

The velocity 'v' of the projectile greater than the velocity of wave 'c'. The pressure wave always behind the nose of projectile. Projectile moves ahead of pressure wave. The entire system of spherical pressure waves forms a conical figure with vertex at point B. The half of angle α of the cone vertex is called Mach angle. From geometry of the figure,

$$\text{SIN}\alpha = c/v = 1/M$$

## NON–DIMENSIONAL CONSTANT

**Problem–1:**

A^certain^gas^is^flowing^through^a^duct^in^which^change^in^cross‿
section^occurs^and^at^a^particular^cross‿section^the^velocity^of^gas^is^^
⋟⋉≜^m⁓s^‿^^absolute^pressure^^∐≪^KN⁓m≪^^and^^temperature^^⊳∐≂^^C^^
respectively‿^Assuming^isentropic^conditions‿^Calculate^the^velocity^
and^Mach^number^where^the^pressure^is^◁⊳≪^^KN⁓m≪^^absolute‿^R^⌐^
≃‿≪⋀≐^KJ⁓Kg^^≂K^^^^and^^K^⌐^◁‿⋟‿

**SEE HAND ANALYSIS**

## NON-DIMENSIONAL CONSTANT

**Problem-2:**

Air is flowing through a duct with a velocity of ▹≙≃ m/s at a certain section of the duct, the temperature and pressure are ≫≃≃ C and ⨿≃ KPa respectively. Considering isentropic flow, Find the velocity and temperature at another section where the absolute pressure is ◃⋘≃ KPa. Also find the Mach number at the two sections. Assume R = ◃⨿⁝≃ J/kg K and k = ◃.≫.

## NON-DIMENSIONAL CONSTANT

**Problem-3:**

Air at ≙ƛ≃ KPa absolute and ▷∐≈C in a large tank enters a converging-diverging nozzle. The exit flow from the nozzle discharges into atmosphere. The nozzle exit diameter is ≙≃ mm. Calculate the max. mass flow rate and the throat diameter.