Gas Dynamics, UNIT 2

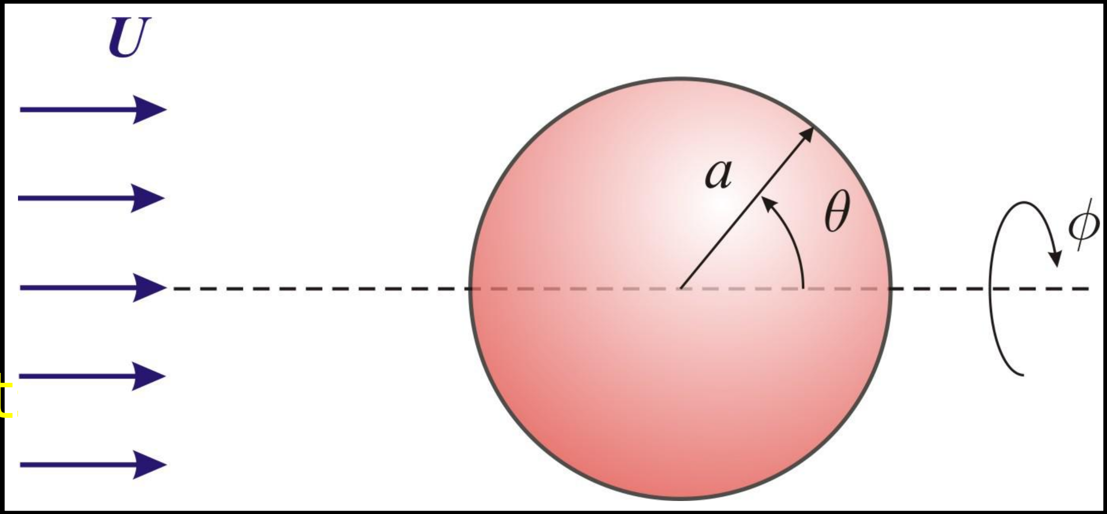
Overview

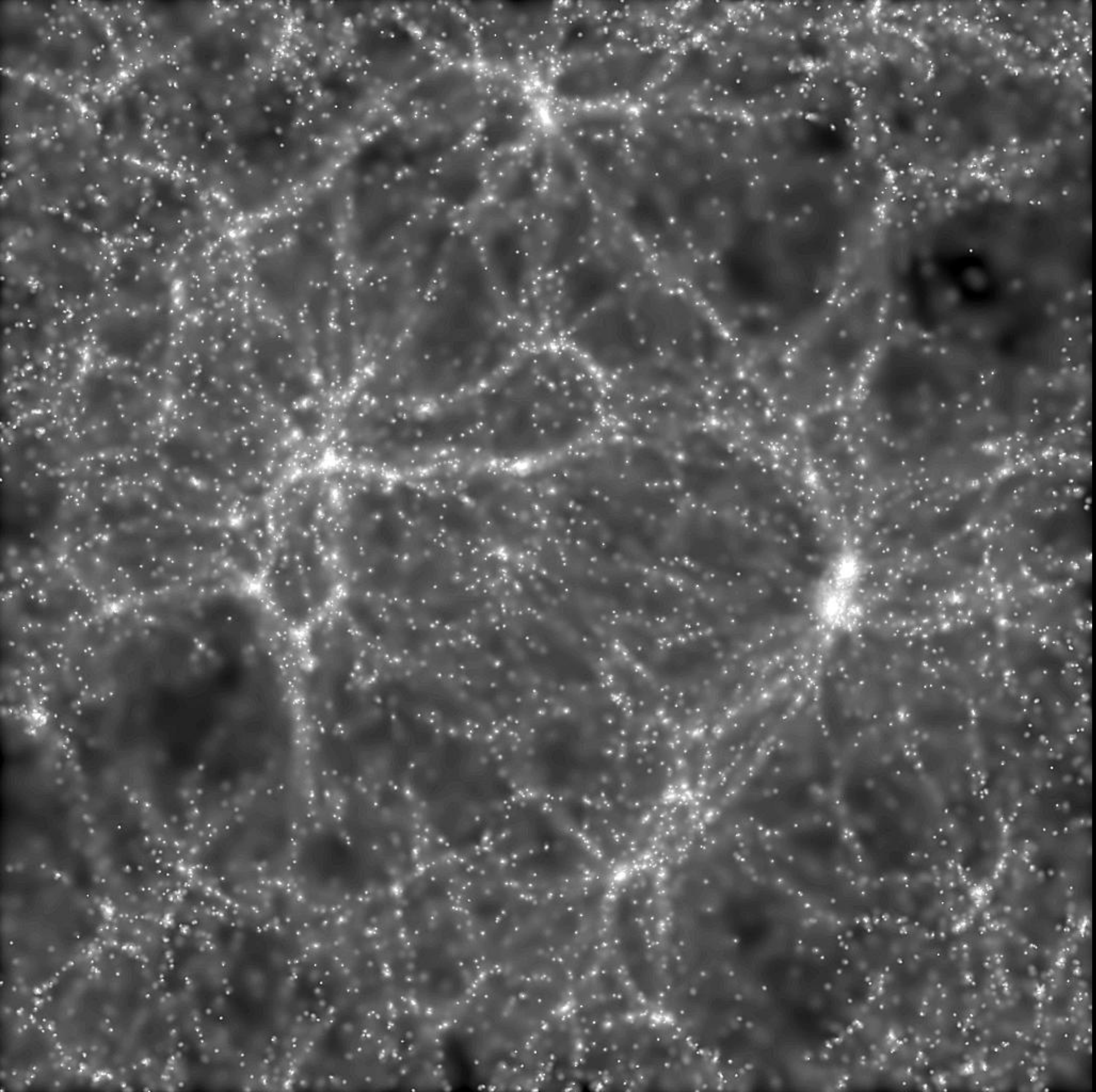
What will we treat during this course？

- Basic equations of gas dynamics
  - Equation of motion
  - Mass conservation
  - Equation of state
- Fundamental processes in a gas
  - Steady Flows
  - Self-gravitating gas
  - Wave phenomena
  - Shocks and Explosions
  - Instabilities: Jeans’ Instability

Applications

- Isothermal sphere & Globular Clusters
- Special flows and drag forces
- Solar & Stellar Winds
- Sound waves and surface waves on water
- Shocks
- Point Explosions, Blast waves & Supernova Remnants



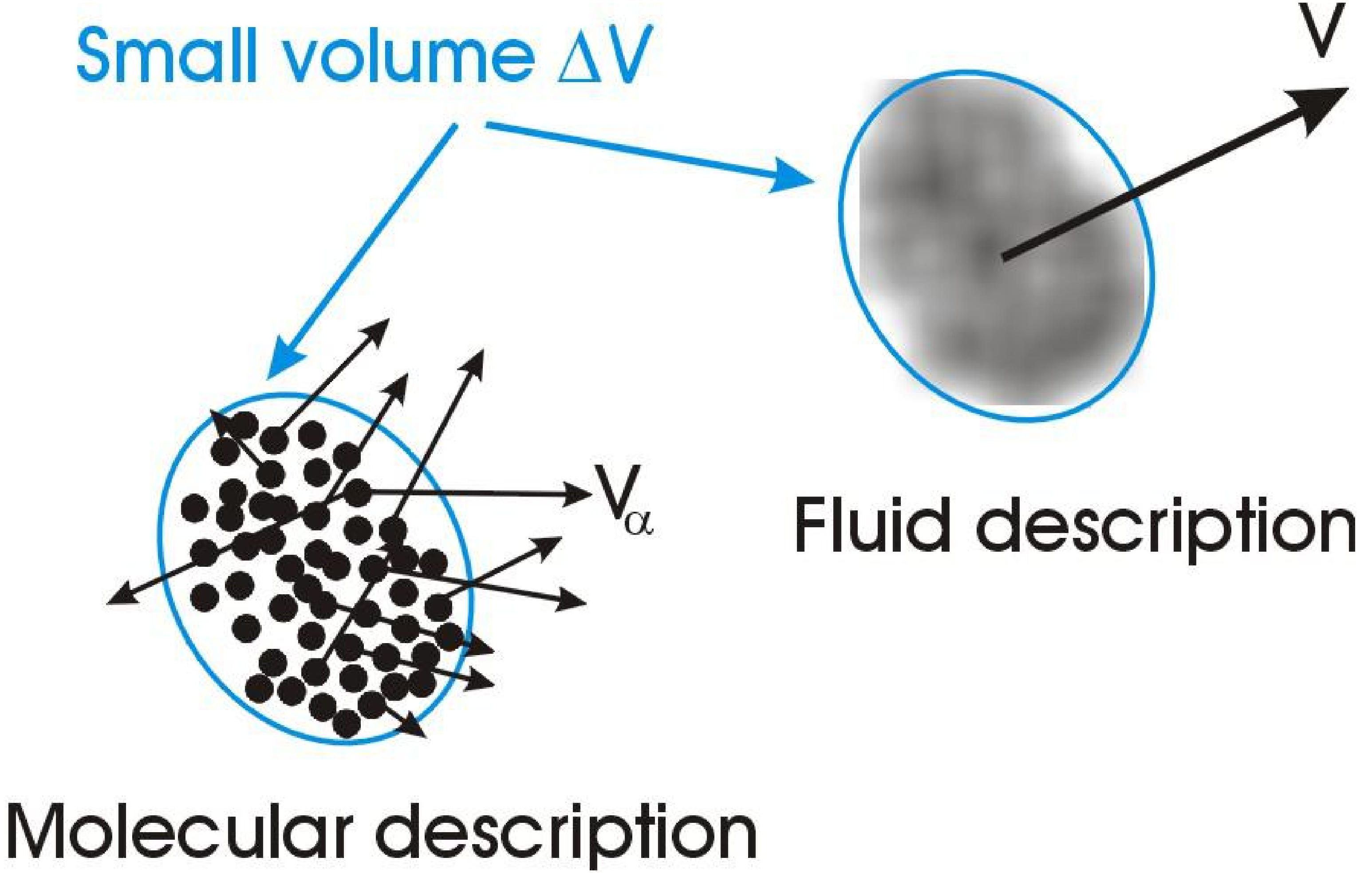


LARGE
SCALE
STRUCTURE

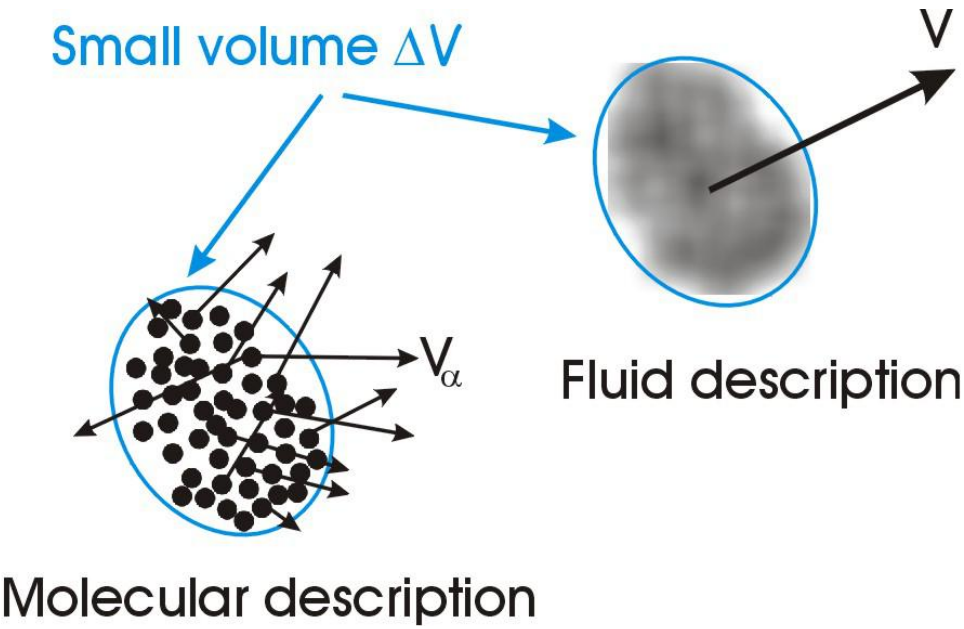
Classical Mechanics vs. Fluid Mechanics

Single-particle (classical) Mechanics	Fluid Mechanics
Deals with single particles with a fixed mass	Deals with a continuum with a variable mass-density
Calculates a single particle trajectory	Calculates a collection of flow lines (flow field) in space
Uses a position *vector* and velocity *vector*	Uses a *fields* : Mass density, velocity field....
Deals only with externally applied forces (e.g. gravity, friction etc)	Deals with internal AND external forces
Is formally linear (so: there is a *superposition principle* for solutions)	Is intrinsically non-linear No superposition principle in general!

Basic Definitions



Mass, mass-density and velocity



Mass density →:

$$\boxed{\rho(\boldsymbol{x}\,,\,t) = \lim_{\Delta\mathcal{V}\to 0} \frac{\Delta m}{\Delta \mathcal{V}}}$$

Mass ƛm in volume ƛV

$$\Delta m = \sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha$$

Mean velocity ***V***(***x*** **,** *t*) is defined as:

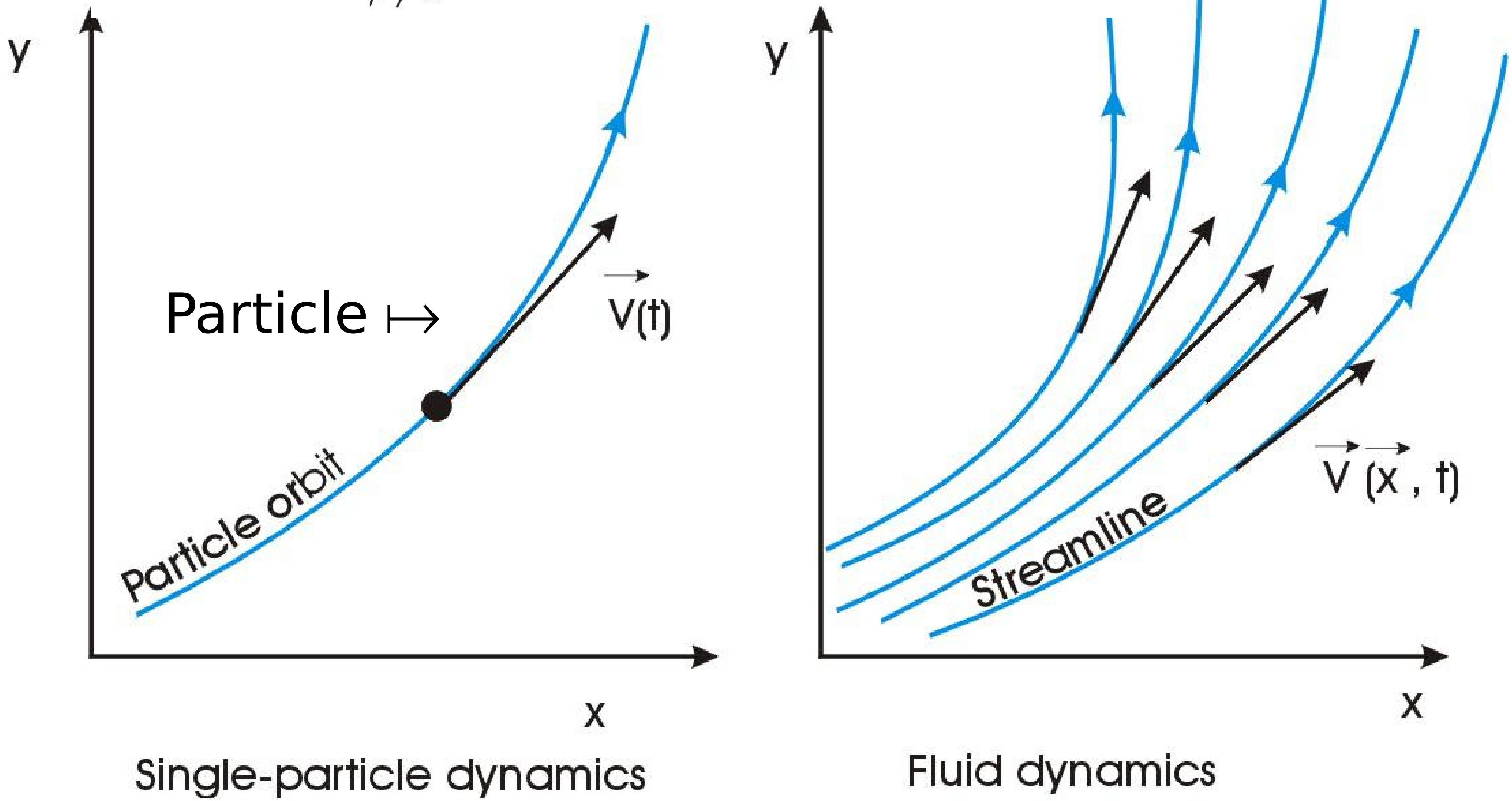
$$\boxed{\boldsymbol{V} = \frac{\sum_{\boldsymbol{x}_\alpha \text{ in } \Delta\mathcal{V}} m_\alpha \boldsymbol{V}_\alpha}{\Delta m}}$$

Equation of Motion: from Newton to Navier–Stokes/Euler

$$m_\alpha \frac{\mathrm{d}\boldsymbol{V}_\alpha}{\mathrm{d}t} = \sum_{\beta \neq \alpha} \boldsymbol{F}_{\alpha\beta}$$

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$

Particle $\mapsto$

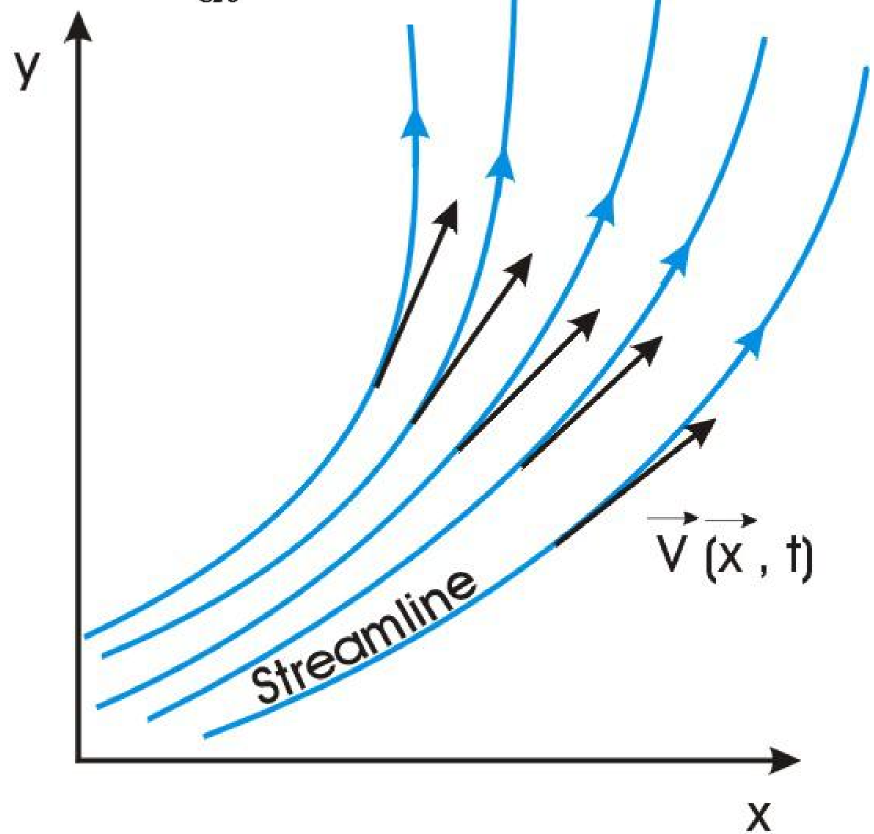


Equation of Motion: from Newton to Navier–Stokes/Euler

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{f}$$

You have to work with a velocity <u>field</u> that depends on position *and* time!

$$\boldsymbol{V} = (V_\mathrm{x}\,,\ V_\mathrm{y}\,,\ V_\mathrm{z}) = \boldsymbol{V}(\boldsymbol{x}\,,\ t)$$



Fluid dynamics

Derivatives, derivatives…

Eulerian change: $\delta Q = Q(\boldsymbol{x}\,,\, t+\Delta t) - Q(\boldsymbol{x}\,,\, t) \approx \dfrac{\partial Q}{\partial t}\,\Delta t$

Derivatives, derivatives…

Eulerian change:
evaluated at a
<u>fixed position</u>

$$\delta Q = Q(\boldsymbol{x}\ ,\ t + \Delta t) - Q(\boldsymbol{x}\ ,\ t) \approx \frac{\partial Q}{\partial t}\,\Delta t$$

Lagrangian change:
evaluated at a
<u>shifting position</u>

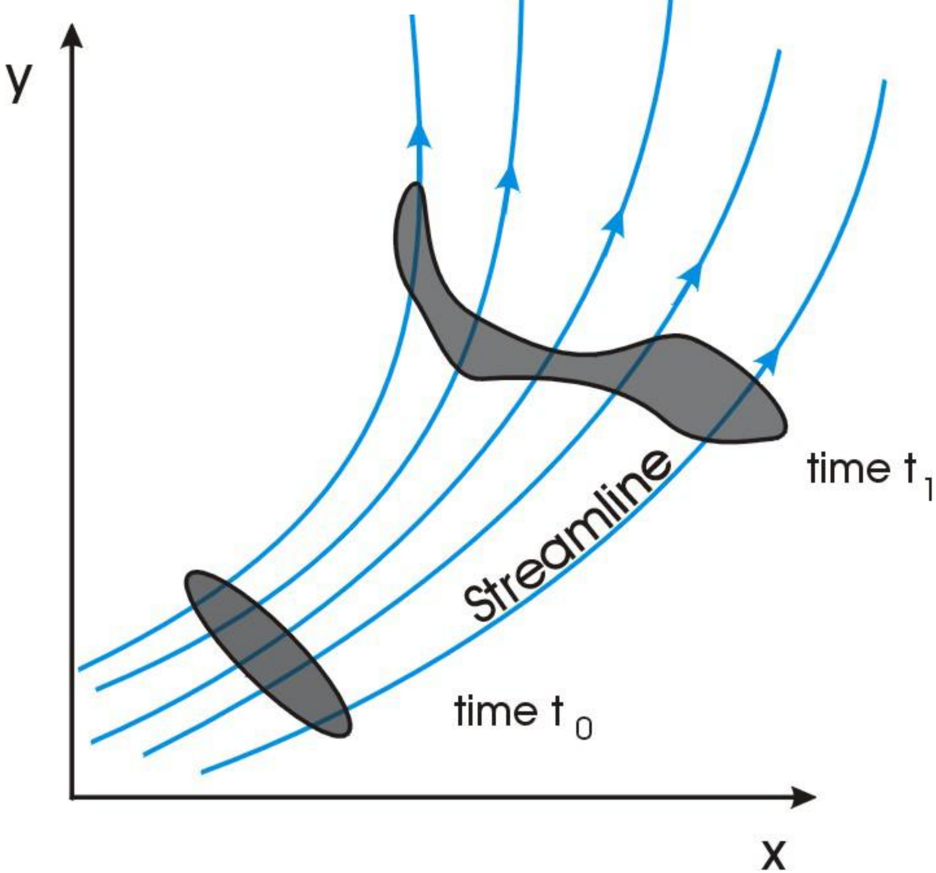
$$\Delta Q = Q(\boldsymbol{x} + \Delta\boldsymbol{x}\ ,\ t + \Delta t) - Q(\boldsymbol{x}\ ,\ t) \approx \frac{\mathrm{d}Q}{\mathrm{d}t}\,\Delta t$$

Shift along
streamline:

$$\boxed{\Delta\boldsymbol{x} = \boldsymbol{V}\,\Delta t}$$

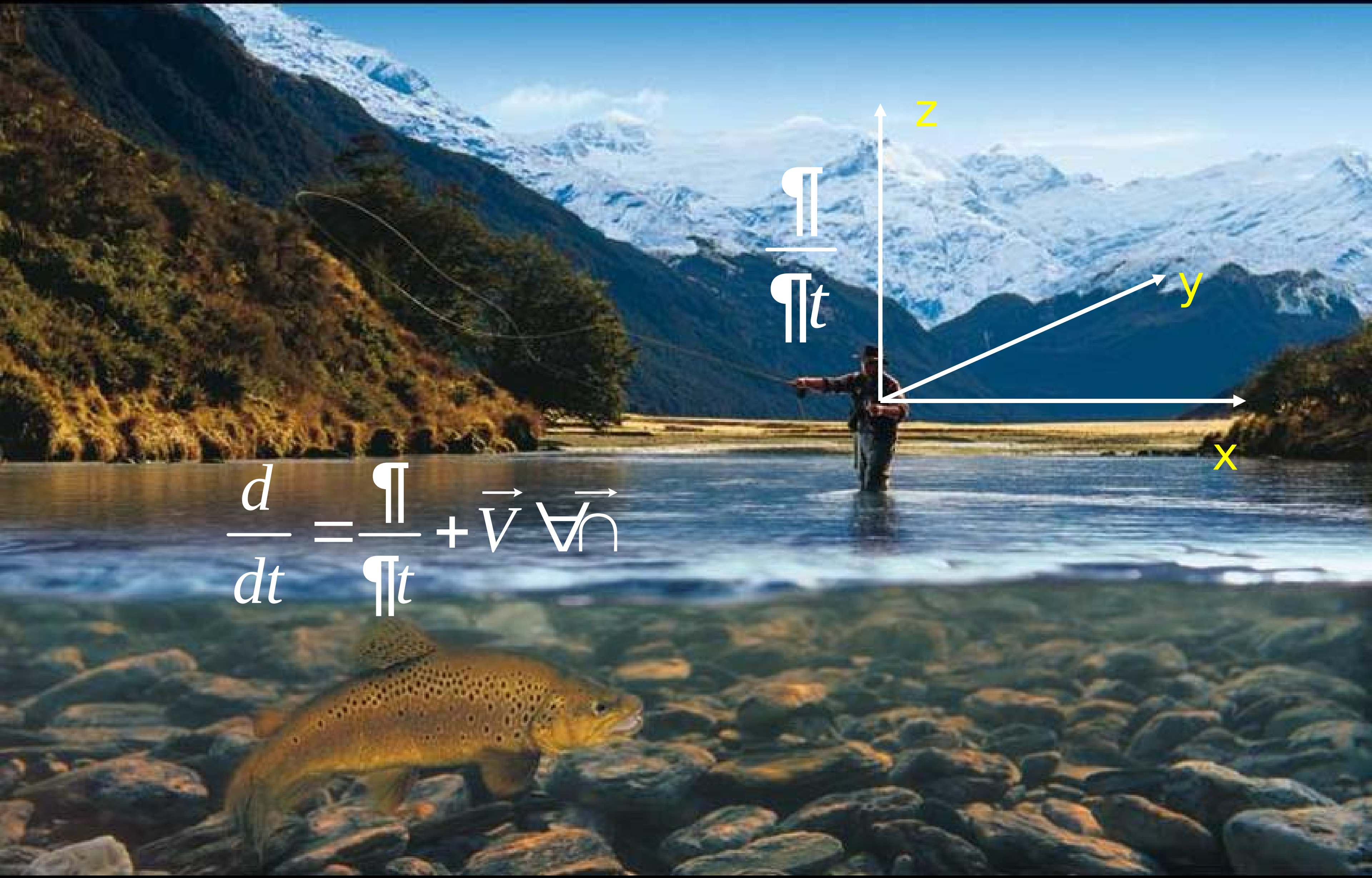
Comoving derivative d/dt

$$\Delta \boldsymbol{x} = \boldsymbol{V}\,\Delta t$$



$$\begin{aligned}
\Delta Q &= Q(t+\Delta t\,,\ \boldsymbol{x}+\Delta\boldsymbol{x}) - Q(t\,,\ \boldsymbol{x}) \\
&\approx \frac{\partial Q}{\partial t}\Delta t + (\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla})Q \\
&= \left[\frac{\partial Q}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})Q\right]\Delta t \\
&\equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)\,\Delta t\ .
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})}$$



z
y
x
¶/¶t
d/dt = ¶/¶t + V⃗ ∀∩⃗

Notation: working with the gradient operator

Gradient operator is a 'machine' that converts a scalar into a vector:

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right)$$

For scalar $Q(\boldsymbol{x}\ ,\ t)$:

$$\boldsymbol{\nabla} Q = \frac{\partial Q}{\partial x}\,\hat{\boldsymbol{e}}_x + \frac{\partial Q}{\partial y}\,\hat{\boldsymbol{e}}_y + \frac{\partial Q}{\partial z}\,\hat{\boldsymbol{e}}_z$$

Related operators: turn scalar into scalar, vector into vector….

$$\Delta\boldsymbol{x}\cdot\boldsymbol{\nabla} \equiv \Delta x\,\frac{\partial}{\partial x} + \Delta y\,\frac{\partial}{\partial y} + \Delta z\,\frac{\partial}{\partial z}$$

$$\boldsymbol{V}\cdot\boldsymbol{\nabla} \equiv V_{\mathrm{x}}\frac{\partial}{\partial x} + V_{\mathrm{y}}\frac{\partial}{\partial y} + V_{\mathrm{z}}\frac{\partial}{\partial z}$$

GRADIENT OPERATOR AND VECTOR ANALYSIS (See Appendix A)

scalar into vector: $\boldsymbol{g} = -\tilde{\boldsymbol{N}}\ \pm$

vector into scalar: $\tilde{\boldsymbol{N}}\ \forall \boldsymbol{g} = -4pGr$

vector into vector: $\tilde{\boldsymbol{N}}\ ' \ \boldsymbol{B} = \frac{4p}{c}\boldsymbol{J}$

tensor into vector: $\tilde{\boldsymbol{N}}\ \forall \mathrm{T} = -\ \boldsymbol{f}$

Useful relations: $\forall\left(\quad ' \ \boldsymbol{B}\right) = 0\ ,\qquad ' \qquad \pm = 0\ ,\qquad \forall\left(\quad \pm\right) = \cap^{2}\pm$

Program for uncovering the basic equations:

1. Define the fluid acceleration and formulate the equation of motion by analogy with single particle dynamics;

2. Identify the forces, such as pressure force;

3. Find equations that describe the response of the other fluid properties

Equation of motion for a fluid:

$$\boxed{\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}\right] = \boldsymbol{f}}$$

Equation of motion for a fluid:

$$\boxed{\rho\,\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right] = \boldsymbol{f}}$$

The acceleration of a fluid element is defined as:

$$\boxed{\boldsymbol{a} \equiv \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}}$$

Equation of motion for a fluid:

$$\rho\,\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right] = \boldsymbol{f}$$

This equation states:

mass density × acceleration = force density

note: GENERALLY THERE IS NO FIXED MASS IN FLUID MECHANICS!

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$$\boxed{\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} \equiv \rho \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} \right] = \boldsymbol{f}}$$

<u>Non-linear term!</u>

Makes it much more difficult
To find 'simple' solutions.

Prize you pay for working with
a velocity-<u>field</u>

$$\boldsymbol{V} = (V_{\mathrm{x}}\,,\, V_{\mathrm{y}}\,,\, V_{\mathrm{z}}) = \boldsymbol{V}(\boldsymbol{x}\,,\, t)$$

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<u>Force-density</u>

This force densitycan be:

- internal:
  - pressure force
  - viscosity (friction)
  - self-gravity
- external
  - For instance: external gravitational force

Pressure force and thermal motions

Split velocities into the
average velocity

$V(\boldsymbol{x}, t)$,

and an isotropically distributed
deviation from average, the
random velocity:

$(\boldsymbol{x}, t)$

Individual particle:

$$\boldsymbol{v}_\alpha = \boldsymbol{V}(\boldsymbol{x}\,,\ t) + \boldsymbol{\sigma}_\alpha(\boldsymbol{x}\,,\ t)\ .$$

Average properties of random velocity $\boldsymbol{\sigma}$:

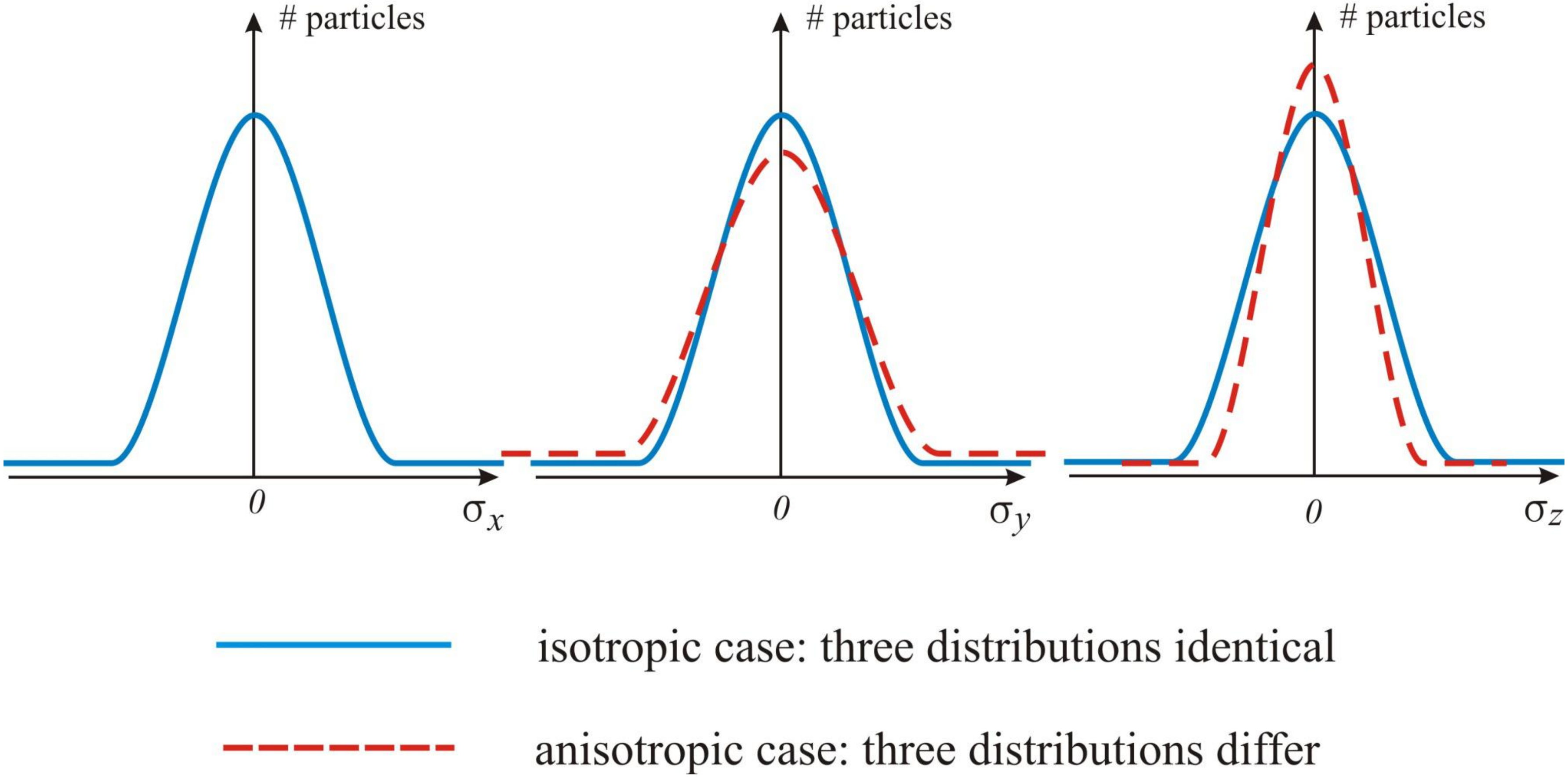
$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{v}} - \boldsymbol{V} = \boldsymbol{0}\ ;$$

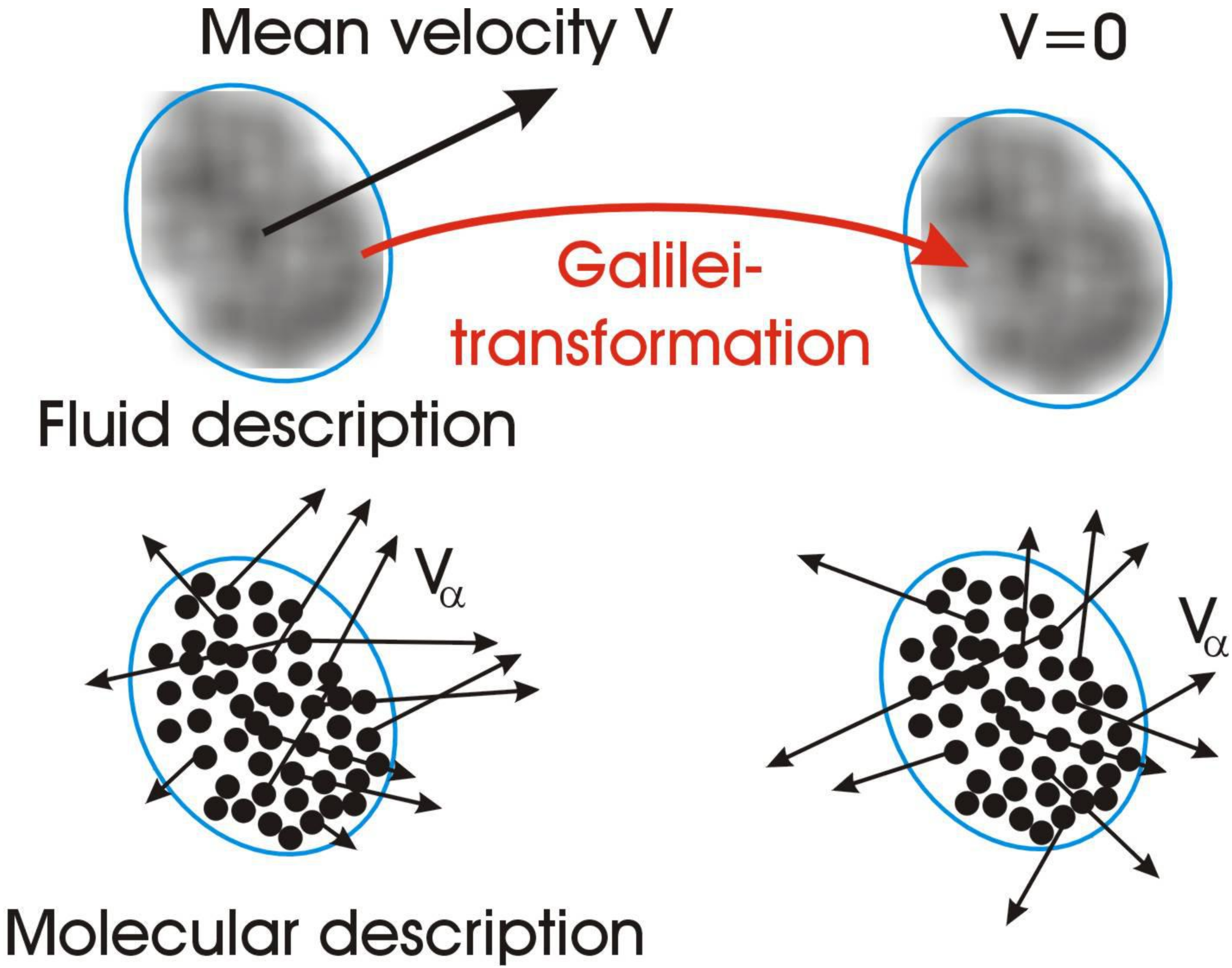
$$\overline{\sigma_x^2} = \overline{\sigma_y^2} = \overline{\sigma_z^2} = \tfrac{1}{3}\overline{\sigma^2}\ ,$$

and

$$\overline{\sigma_x\sigma_y} = \overline{\sigma_x\sigma_z} = \overline{\sigma_y\sigma_z} = \cdots = 0\ .$$

DISTRIBUTION OF RANDOM VELOCITIES ALONG THE THREE COORDINATE AXES




Mean velocity V
V=0
Galilei-
transformation
Fluid description
V_α
V_α
Molecular description

Acceleration of particle $\mapsto$

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{v}_\alpha}{\mathrm{d}t} &= \frac{\partial \boldsymbol{v}_\alpha}{\partial t} + (\boldsymbol{v}_\alpha \cdot \boldsymbol{\nabla})\boldsymbol{v}_\alpha \\
&= \frac{\partial(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)}{\partial t} + ((\boldsymbol{V} + \boldsymbol{\sigma}_\alpha)\cdot \boldsymbol{\nabla})\,(\boldsymbol{V} + \boldsymbol{\sigma}_\alpha) \\
&= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\frac{\partial \boldsymbol{\sigma}_\alpha}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}_\alpha}_{\text{linear in } \boldsymbol{\sigma}}} + \boxed{\underbrace{(\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\nabla})\,\boldsymbol{\sigma}_\alpha}_{\text{quadratic in } \boldsymbol{\sigma}}}
\end{aligned}
$$

Acceleration of particle $\mapsto$ (II)

Effect of average over many particles in small volume:

$$
\begin{aligned}
\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} &= \overline{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}} \\
&= \underbrace{\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}}_{\text{total derivative mean flow}} + \boxed{\underbrace{\left(\frac{\partial}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\right)\overline{\boldsymbol{\sigma}}}_{\text{vanishes: }\overline{\boldsymbol{\sigma}}=0!}} + \boxed{\underbrace{\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\boldsymbol{\sigma}}}_{\text{remains: quadratic in }\boldsymbol{\sigma}}}
\end{aligned}
$$

Average equation of motion:

$$\rho\,\overline{\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}} = \overline{\boldsymbol{f}}$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = \underbrace{\overline{\boldsymbol{f}}}_{\text{mean ext. force}} - \rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\boldsymbol{\sigma}}$$

For isotropic fluid: $$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P$$

Some tensor algebra:

Vector

$$\boldsymbol{A} \equiv A_i\, \boldsymbol{e}_i = A_x\, \boldsymbol{e}_1 + A_y \boldsymbol{e}_2 + A_z\, \boldsymbol{e}_3 = \begin{pmatrix} A_x \\ \\ A_y \\ \\ A_z \end{pmatrix}$$

Three notations for the same animal!

Some tensor algebra: the divergence of a vector in cartesian (x, y, z) coordinates

Vector

$$\boldsymbol{A} \equiv A_i\, \boldsymbol{e}_i = A_x\, \boldsymbol{e}_1 + A_y \boldsymbol{e}_2 + A_z\, \boldsymbol{e}_3 = \begin{pmatrix} A_x \\ \\ A_y \\ \\ A_z \end{pmatrix}$$

$\downarrow$

Scalar

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Rank 2 Tensor

Rank 2 tensor

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j == \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Rank 2 Tensor and Tensor Divergence

Rank 2 tensor ***T***

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j == \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Vector

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = \left(\frac{\partial T_{ij}}{\partial x_i}\right) \boldsymbol{e}_j = \begin{pmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{pmatrix}$$

Special case:
Dyadic Tensor = Direct Product of two Vectors

$$\boldsymbol{A} \otimes \boldsymbol{B} \equiv A_i B_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\nabla} \cdot \boldsymbol{A}) \, \boldsymbol{B} + (\boldsymbol{A} \cdot \boldsymbol{\nabla}) \boldsymbol{B}$$

This is the product rule for differentiation!

Application: Pressure Force (I)

Tensor divergence: $(\rho\,\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma} = \boldsymbol{\nabla}\cdot\,(\rho\,\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}) - (\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}))\;\boldsymbol{\sigma}$

Isotropy of the random velocities:

$$\boxed{\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\,(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}})}$$

Second term = scalar x vector!

This <u>must</u> vanish upon averaging!!

Application: Pressure Force (II)

$$\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\left(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}\right)$$

Isotropy of the random velocities

$$\overline{\sigma_i\sigma_j} = \frac{1}{3}\overline{\sigma^2}\,\delta_{ij} = \begin{cases} \frac{1}{3}\overline{\sigma^2} & \text{when } i=j \\ 0 & \text{when } i\neq j \end{cases}$$

$$\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}} = \rho\begin{pmatrix} \frac{1}{3}\overline{\sigma^2} & 0 & 0 \\ 0 & \frac{1}{3}\overline{\sigma^2} & 0 \\ 0 & 0 & \frac{1}{3}\overline{\sigma^2} \end{pmatrix} = \frac{\rho\overline{\sigma^2}}{3}\,\boldsymbol{I}$$

Diagonal Pressure Tensor

Pressure force, conclusion:

$$\boxed{\rho\,\overline{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma})} = \boldsymbol{\nabla}\cdot\left(\rho\,\overline{\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}}\right) = \boldsymbol{\nabla}\left(\frac{\rho\overline{\sigma^2}}{3}\right) \equiv \boldsymbol{\nabla}P}$$

Equation of motion for frictionless ('ideal') fluid:

$$\rho\left(\frac{\partial\boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\right) = -\boldsymbol{\nabla}P + \text{other (external) forces}$$

$$P(\boldsymbol{x}\,,\,t) \equiv \tfrac{1}{3}\rho\,\overline{\sigma^2}$$

Summary:

- We know how to interpret the time-derivative d/dt
- We know what the equation of motion looks like;
- We know where the pressure force comes from (thermal motions), and how it looks: $\boldsymbol{f} = -\nabla P$.
- **We still need**
  - A way to link the pressure to density and temperature: $P = P(\rho, T)$;
  - A way to calculate how the density ρ of the fluid changes.