Unit: 5 VECTOR CALCULUS

1 Vector Function

A vector function is a vector whose magnitude and/or direction depends on values of certain variables. Based on the input variable, we have different types of functions:

1. A vector function $\vec{F}$ of a scalar t written as $\vec{F} = \vec{F}(t)$, where the input variable is t, a scalar.
   Example: Position of a particle in space with a position vector say $\vec{r}$ dependent on time t and written as $\vec{r} = \vec{r}(t)$.

2. A scalar function $\phi(\vec{r})$ of a vector $\vec{r}$ written as $\phi = \phi(\vec{r})$, where the input variable is $\vec{r}$, a vector.
   Example: Temperature T of a heated body in steady state at any point $\vec{r}$ given by $T = T(\vec{r})$.

3. A vector function $\vec{F}$ of another vector $\vec{r}$ written as $\vec{F} = \vec{F}(\vec{r})$. where the input variable is $\vec{r}$, a vector.
   Example: Linear velocity $\vec{V}$ of a rotating body with position vector $\vec{r}$ and constant angular velocity $\vec{w}$ written as $\vec{V} = \vec{w} \times \vec{r}$.

1.1 Vector Differentiation

For a vector function $\vec{r} = \vec{r}(t)$, the derivative of $\vec{r}(t)$ (if it exists) is defined by:

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{r}(t + \delta t) - \vec{r}(t)}{\delta t}$$

In terms of Cartesian co-ordinates, $\vec{r}(t)$ is differentiable at a point t if and only if all the components of $\vec{r}(t)$ are differentiable at t. Thus, if $\vec{r}(t) = (x(t), y(t), z(t))$ or $x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in directions of x, y, z axes respectively.

1.2 Higher Order Differentiation

If $\vec{r}(t)$ is a vector function, then $\dfrac{d\vec{r}}{dt}$ is the first order derivative of $\vec{r}$.
The second order derivatiave of $\vec{r}(t)$ is given by $\dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right) = \dfrac{d^2\vec{r}}{dt^2}$.
Similarly, higher order derivatives of $\vec{r}$ can be obtained.

1.3 Geometrical Interpretation

Geometrically, for a vector function $\vec{r}(t)$, $\dfrac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at any point P on the curve. If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then the magnitude of the vector along the tangent at point P is given by $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$.
If $\left|\dfrac{d\vec{r}}{dt}\right| = 1$, then $\dfrac{d\vec{r}}{dt}$ is the unit vector along the tangent at point P.

1.4 Velocity and Acceleration

Let the scalar variable t denote time and the vector $\vec{r}$ represent the position vector of a moving particle P relative to origin O.

The velocity vector of the particle at P is given by: $\vec{V} = \dfrac{d\vec{r}}{dt}$ and its direction is along the tangent at P.

The acceleration vector of the particle at P is given by: $\vec{a} = \dfrac{d\vec{V}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

1.5 Some Useful Results

1. $\dfrac{d\,\vec{c}}{dt} = \vec{O}$, when $\vec{c}$ is a constant vector and $\vec{O}$ is a zero vector.

2. $\dfrac{d}{dt}\left(\vec{a} \pm \vec{b}\right) = \dfrac{d\vec{a}}{dt} \pm \dfrac{d\vec{b}}{dt}$, when $\vec{a}$ and $\vec{b}$ are vector functions of t.

3. $\dfrac{d\vec{f}}{dt} = \dfrac{d\vec{f}}{d\phi}\dfrac{d\phi}{dt}$, when $\vec{f}$ is a vector function of ϕ and ϕ is a scalar function of t.

4. $\dfrac{d(\phi\vec{V})}{dt} = \phi\dfrac{d\vec{V}}{dt} + \dfrac{d\phi}{dt}\vec{V}$, when ϕ is a scalar function of t and $\vec{V}$ is a vector function of t.

5. $\dfrac{d(\vec{a}\cdot\vec{b})}{dt} = \vec{a}\cdot\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}\cdot\vec{b}$, when $\vec{a}$ and $\vec{b}$ are vector functions of t.

6. $\dfrac{d(\vec{a}\times\vec{b})}{dt} = \vec{a}\times\dfrac{d\vec{b}}{dt} + \dfrac{d\vec{a}}{dt}\times\vec{b}$, when $\vec{a}$ and $\vec{b}$ are vector functions of t.

7. If $\vec{a}, \vec{b}, \vec{c}$ are vector functions of t, then $\dfrac{d}{dt}\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\dfrac{d\,\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \dfrac{d\,\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \dfrac{d\,\vec{c}}{dt}\right]$

8. $\dfrac{d}{dt}\left[\vec{a}\times(\vec{b}\times\vec{c})\right] = \dfrac{d\,\vec{a}}{dt}\times\left(\vec{b}\times\vec{c}\right) + \vec{a}\times\left(\dfrac{d\,\vec{b}}{dt}\times\vec{c}\right) + \vec{a}\times\left(\vec{b}\times\dfrac{d\,\vec{c}}{dt}\right)$, if $\vec{a}, \vec{b}, \vec{c}$ are vector functions of t.

1.6 Component of any vector along a given direction

Let $\vec{V}$ be any vector and $\vec{a}$ be a given constant vector. Then the component of $\vec{V}$ in the direction of $\vec{a}$ is given by $\vec{V}\cdot\hat{a}$, where $\hat{a}$ is a unit vector corresponding to $\vec{a}$.

Recall:

1. If $\vec{v} = (v_1, v_2, v_3)$ is any vector, then $\hat{v} = \dfrac{1}{|v|}(v_1, v_2, v_3)$ is a unit vector in the direction of $\vec{v}$ where $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the magnitude of $\vec{v}$.

2. If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are two vectors, then

   (a) $\vec{a}\cdot\vec{b} = a_1\cdot b_1 + a_2\cdot b_2 + a_3\cdot b_3$ is the dot product or scalar product between $\vec{a}$ and $\vec{b}$.

   (b) $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$

   is the cross product or vector product of $\vec{a}$ and $\vec{b}$.

(c) If $\vec{c} = (c_1, c_2, c_3)$, then $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left[\vec{a}\,\vec{b}\,\vec{c}\right]$ is called the box product and is given by:

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3. If $\vec{a}$ and $\vec{b}$ are two vectors, then the angle between them is given by $\cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$.

4. If vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then $\vec{a} \cdot \vec{b} = 0$.

5. If vectors $\vec{a}$ and $\vec{b}$ are parallel to each other, then $\vec{a} = k \cdot \vec{b}$ for some scalar k.

2 Scalar and Vector Fields

A variable quantity whose value at any point in a region of space depends on the position of the point is called a **point function**. There are two types of point functions.

1. If to each point (x, y, z) of a region R in space there corresponds a number or a scalar $\phi = \phi(x, y, z)$, then ϕ is called the **scalar function** or **scalar point function**. The region R is called the **scalar field**.
   Examples on scalar point function are: (i) temperature distribution in a medium, (ii) density of the body, (iii) distribution of atmospheric pressure in space, etc.

2. If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{V} = \vec{V}(x, y, z)$, then $\vec{V}$ is called the **vector function** or **vector point function**. The region R is called the **vector field**.
   Examples on vector point function are: (i) velocity of moving fluid at any time, (ii) electric field density, (iii) magnetic field density, etc.

A vector field which is independent of time is called **stationay** or **steady-state** vector field.

2.1 Level Surfaces and vector differential operator

Let a scalar point function $\phi(x, y, z)$ be defined on a certain region R of space. The set of points satisfying $\phi(x, y, z) = k$ for some fixed value of k defines a surface and is called **level surface**.
For different values of k, different level surfaces will be obtained and no two level surfaces will intersect.
The **vector differential operator** read as del or nabla is given by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

∇ operates on scalar functions as well as vector functions.

2.2 Gradient of a Scalar Field

For a given scalar function $\phi(x, y, z)$, the gradient of ϕ written as $grad(\phi)$ or $\nabla\phi$ is a vector function defined as $\nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$.

We note that $\nabla\phi$ is always normal(perpendicular) to the surface $\phi(x, y, z) = c$.

$\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$ is a unit vector normal to the surface $\phi(x, y, z) = c$.

The gradient of a scalar field ϕ is a vector normal to the surface $\phi(x, y, z) = c$. It is in the direction of maximum rate of change of ϕ.

2.2.1 Directional Derivative

$\nabla\phi \cdot \hat{a}$ is the **directional derivative** of a scalar function $\phi(x, y, z)$ in the direction of $\vec{a}$.

This gives the rate of change of ϕ at any point (x, y, z) in the direction of $\vec{a}$.

$\nabla\phi$ gives the maximum rate of change(directional derivative) of ϕ with magnitude $|\nabla\phi|$.

2.3 Divergence of a Vector Field

For a differentiable vector function $\vec{V}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, the divergence of $\vec{V}$ written as $div\vec{V}$ or $\nabla \cdot \vec{V}$ is a scalar function defined as

$$\nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

The divergence of a $\vec{V}$ gives the rate of change of outward flow of a fluid per unit volume at a given point.

2.3.1 Solenoidal Function

If there is no gain of the fluid anywhere, then $div\vec{V} = \nabla \cdot \vec{V} = 0$.

This is called **continuity equation for incompressible fluid** or **condition of incompressibility**.This is also known as **law of conservation of mass**.

A vector function $\vec{V}$ is said to be a **solenoidal function** or **solenoidal** if $div\vec{V} = 0$.

2.4 Curl of a Vector Field

For a differentiable vector function $\vec{V}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, the curl or rotation of $\vec{V}$ written as $curl\vec{V}$ or $\nabla \times \vec{V}$ is a vector function defined as

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

The angular velocity of rotation of a body at any point is equal to half the curl of the linear velocity at that point.

2.4.1 Irrotational vector and conservative field

If the vector function (field) $\vec{F}$ is that due to a moving fluid, then the field tend to rotate in the region if $curl\vec{F} \neq \vec{0}$. The region where $curl\vec{F} = \vec{0}$ will have no rotation and the field is called an irrotational field.
Thus, $\vec{F}$ is said to be **irrotational** if $\nabla \times \vec{F} = \vec{0}$. In this case, there exists a scalar function ϕ called **scalar potential** such that $\nabla\phi = \vec{F}$. The field $\vec{F}$ is called **conservative vector field**.

3 Properties of Gradient, Divergence and Curl

3.1 Properties of gradient, divergence and curl

Let $\vec{u}$, $\vec{v}$ be two vector functions and ϕ, ψ be two scalar functions. Then,

1. grad $(\phi \pm \psi) = $ grad $(\phi) \pm$ grad (ψ) or $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$,
2. div $(\vec{u} \pm \vec{v}) =$ div $(\vec{u}) \pm$ div $(\vec{v})$ or $\nabla \cdot (\vec{u} \pm \vec{v}) = \nabla \cdot \vec{u} \pm \nabla \cdot \vec{v}$,
3. curl $(\vec{u} \pm \vec{v}) =$ curl $\vec{u} \pm$ curl $\vec{v}$ or $\nabla \times (\vec{u} \pm \vec{v}) = \nabla \times \vec{u} \pm \nabla \times \vec{v}$,
4. grad $(\phi\psi) = \phi$ grad $(\psi) + \psi$ grad (ϕ) or $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$,
5. grad $\left(\dfrac{\phi}{\psi}\right) = \dfrac{\psi \text{ grad } (\phi) - \phi \text{ grad } (\psi)}{\psi^2}$ or $\nabla\left(\dfrac{\phi}{\psi}\right) = \dfrac{\psi\,\nabla\phi - \phi\,\nabla\psi}{\psi^2}$, where $\psi \neq 0$.
6. div $(\phi\vec{u}) = \phi$ div $(\vec{u}) +$ grad $(\phi) \cdot \vec{u}$ or $\nabla \cdot (\phi\vec{u}) = \phi\nabla \cdot \vec{u} + \nabla\phi \cdot \vec{u}$.
7. div $(\vec{u} \times \vec{v}) = ($curl $\vec{u}) \cdot \vec{v} - \vec{u} \cdot ($curl $\vec{v})$ or $\nabla \cdot (\vec{u} \times \vec{v}) = (\nabla \times \vec{u}) \cdot \vec{v} - \vec{u} \cdot (\nabla \times \vec{v})$
8. curl $(\phi\vec{u}) = ($grad $\phi) \times \vec{u} + \phi$ curl $\vec{u}$ or $\nabla \times (\phi\vec{u}) = (\nabla\phi) \times \vec{u} + \phi(\nabla \times \vec{u})$,
9. curl $(\vec{u} \times \vec{v}) = ($div $\vec{v})\vec{u} - ($div $\vec{u})\vec{v} + (\vec{v} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{v}$
10. grad $(\vec{u} \cdot \vec{v}) = \vec{u} \times$ curl $\vec{v} + \vec{v} \times$ curl $\vec{u} + (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u}$

3.2 The Laplacian Operator ∇^2

The operator $\nabla \cdot \nabla = \nabla^2$ read as **del square** is defined as $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ and is called the **Laplacian Operator**.

If ϕ is a scalar function, then $\nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

3.3 Laplace's Equation and Harmonic function

For a function f, $\nabla^2 f = 0$ is called the **Laplace's Equation**. A function satisfying Laplace's equation is called **Harmonic Function**.

3.4 Properties involving Laplacian operator

Let ϕ be a scalar function and $\vec{u}$ be a vector function. Then,

1. div (grad ϕ) $= \nabla \cdot (\nabla\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$
2. $(\nabla \cdot \nabla)\vec{u} = \nabla^2\vec{u}$
3. grad (div $\vec{u}$) $= \nabla(\nabla \cdot \vec{u})$
4. curl (grad ϕ) $= \nabla \times \nabla\phi = \vec{0}$
5. div (curl $\vec{u}$) $= \nabla \cdot (\nabla \times \vec{u}) = 0$
6. curl curl $\vec{u} = \nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2\vec{u}$

4 Line Integral

Any integral which is evaluated along a curve is called **Line Integral**.

1. If $f(x, y, z)$ is a scalar field defined on a smooth curve C from A to B, then the line integral of f along curve C can be defined as:

   (a) $\int_A^B f(x, y, z)\,ds = \int_A^B f\left(x(t), y(t), z(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$
   when $x = x(t); y = y(t); z = z(t)$

   (b) $\int_A^B f(x, y, z)\,d\vec{r} = \int_A^B f(x, y, z)\hat{\imath}dx + f(x, y, z)\hat{\jmath}dy + f(x, y, z)\hat{k}dz$,
   as $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \implies d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$

2. If $\vec{F}(x, y, z)$ is a vector field defined on a smooth curve C from A to B, then the line integral of $\vec{F}$ along curve C can be defined as:

   (a) $\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B (f_1dx + f_2dy + f_3dz)$, where $\vec{F} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$

   (b) $\int_A^B \vec{F} \times d\vec{r} = \int_A^B (f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}) \times (\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$

Note: When path of integration is a closed path C, the integral sign is denoted by $\int_C$ or $\oint_C$.

4.1 Application of Line Integral

1. The work done by a force $\vec{F}$ along a curve C is given by $\int_C \vec{F} \cdot d\vec{r}$.

2. The circulation of the velocity $\vec{v}$ of a fluid particle along a closed curve C is given by: $\oint_C \vec{v} \cdot d\vec{r}$.

4.2 Line Integrals Independent Of Path

Theorem 4.1. *The necessary and sufficient condition that the line integral* $\int_A^B \vec{F} \cdot d\vec{r}$ *be independent of the path is that* $\vec{F}$ *is the gradient of some scalar function* ϕ.

In this case, $\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B d\phi = [\phi]_A^B = \phi_B - \phi_A$

Corollary 4.2. *If* $\vec{F} = \nabla\phi$, *then curl* $\vec{F}$ = *curl grad* $\phi = \vec{0}$.

Corollary 4.3. *If* $\int_C \vec{F} \cdot d\vec{r}$ *is independent of path , then* $\oint_C \vec{F} \cdot d\vec{r} = 0$ *along any closed curve* C.

Corollary 4.4. *Let* $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$, *then* $\int_C (F_1dx + F_2dy)$ *is independent of its path if* $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

5 Green's Theorem in Plane

Theorem 5.1. *If $M(x,y)$ and $N(x,y)$, $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ are continuous everywhere in a region R of $XY-$ plane bounded by a closed curve C, then $\oint\limits_C (Mdx + Ndy) = \iint\limits_R \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) dxdy$, where C is traversed in anti-clockwise direction.*

5.1 Green's Theorem in Vector Form

Let $\vec{F} = M\hat{i} + N\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j}$, then $\vec{F}\cdot d\vec{r} = Mdx + Ndy$ and $\nabla \times \vec{F} = \hat{k}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$
Thus, $(\nabla \times \vec{F})\cdot \hat{k} = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$
Thus in vector form, Green's theorem can be written as:

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_R (\nabla \times \vec{F})\cdot d\vec{A}$$

where $d\vec{A} = dA\hat{k} = \hat{k}dxdy$

5.2 Area of a plane region R bounded by a simple closed curve

The area of a region R bounded by a simple closed curve C is given by:

$$\iint\limits_R dxdy = \frac{1}{2}\oint\limits_C [-ydx + xdy]$$

In polar form, the area is given by: $\dfrac{1}{2}\displaystyle\int\limits_C r^2 d\theta$, where $x = r\cos\theta$ and $y = r\sin\theta$

6 Parametric Equations of some known curves

Name of curve	Equation of curve	counter-clockwise	clockwise
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos t,\ y = b\sin t,$ $0 \le t \le 2\pi$	$x = a\cos t,\ y = -b\sin t,$ $0 \le t \le 2\pi$
Circle	$x^2 + y^2 = a^2$	$x = a\cos t,\ y = a\sin t,$ $0 \le t \le 2\pi$	$x = a\cos t,\ y = -a\sin t,$ $0 \le t \le 2\pi$

- Parametric equation of a line segment joining (x_1, y_1, z_1) to (x_2, y_2, z_2) is given by: $x = (1-t)x_1 + tx_2$; $y = (1-t)y_1 + ty_2$; $z = (1-t)z_1 + tz_2$; $0 \le t \le 1$

7 Surface Integral

The integral which can be evaluated over a surface is called **surface integral**.
Let $\vec{F}$ be a continuous vector function defined over a surface S. Let $\hat{n}$ be a unit normal vector to the surface at any point P on a small area δS drawn outward if the surface is closed or always towards the same side of the surface if open. The surface integral of $\vec{F}$ over S is defined by

$$\iint\limits_S \vec{F}\cdot\hat{n}\, ds = \iint\limits_S \vec{F}\cdot d\vec{s}$$

7.1 Evaluation of Surface Integral

A surface integral is evaluated by reducing it into a double integral by projecting the given surface S onto one the co-ordinate planes.
If the projection of δs, D_1 is taken over $XY-$ plane, then $ds = \dfrac{dxdy}{|\hat{n}\cdot\hat{k}|}$.
If the projection of δs, D_2 is taken over $YZ-$ plane, then $ds = \dfrac{dydz}{|\hat{n}\cdot\hat{i}|}$.
If the projection of δs, D_3 is taken over $ZX-$ plane, then $ds = \dfrac{dxdz}{|\hat{n}\cdot\hat{j}|}$.
Thus, we have:

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{D_1} \vec{F}\cdot\hat{n}\frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \iint_{D_2} \vec{F}\cdot\hat{n}\frac{dydz}{|\hat{n}\cdot\hat{i}|} = \iint_{D_3} \vec{F}\cdot\hat{n}\frac{dxdz}{|\hat{n}\cdot\hat{j}|}$$

7.2 Other types of Surface Integrals

Other types of Surface integrals are given by:

1. $\displaystyle\iint_S \phi\, d\vec{s}$
2. $\displaystyle\iint_S \vec{F}\times\hat{n}\, d\vec{s}$
3. $\displaystyle\iint_S \phi\cdot\hat{n}\, d\vec{s}$

7.3 Surface area of a curved surface

Let S be a surface represented by $f(x,y,z)=c$. Then the unit normal to the surface S is given by:

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{f_x\hat{i}+f_y\hat{j}+f_z\hat{k}}{\sqrt{f_x^2+f_y^2+f_z^2}}$$

If D is the projection of S onto the $XY-$ plane, then the surface area of S is given by:

$$\iint_S dS = \iint_D \frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \iint_D \frac{\sqrt{f_x^2+f_y^2+f_z^2}}{|f_z|}\,dx\,dy$$

7.4 Flux

The flux of $\vec{F}$ along the surface S is given by: $\displaystyle\int_S \vec{F}\cdot\hat{n}\,dS$.

Here $\vec{F}=\rho\vec{V}$, where ρ and $\vec{V}$ are respectively the density and the velocity of the fluid flowing across a surface S. The flux of $\vec{F}$ gives the total quantity of the fluid flowing in unit time through the surface S in positive direction. $\hat{n}$ is the unit outward normal to the surface S.

8 Volume Integral

The integral which can be evaluated over a volume is called a **volume integral**.
Let a volume V be bounded by a closed surface S in space. The volume integral can be defined as:

1. $\iiint\limits_V \phi(x,y,z)\, d\vec{V}$ for a scalar field ϕ defined on V;

2. $\iiint\limits_V \vec{F}(x,y,z)\, d\vec{V}$ for a vector field $\vec{F}$ defined on V.

9 Stoke's Theorem

If $\vec{F}$ is a continuous differentiable function defined on an open surface S bounded by a closed curve C, then

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \iint\limits_S \nabla\times\vec{F}\cdot\hat{n}\, ds,$$

where C is traversed in anticlockwise direction and $\hat{n}$ is the outward drawn unit normal vector to the surface S.

10 Gauss Divergence Theorem

If $\vec{F}$ is a continuous differentiable function defined over a volume V bounded by a closed surface S, then

$$\iint\limits_S \vec{F}\cdot\hat{n}\, ds = \iiint\limits_V \nabla\cdot\vec{F}\, dV,$$

where $\hat{n}$ is the outward drawn unit normal vector to the surface S.

11 Parametric Equations of some known curves

Name of curve	Equation of curve	counter-clockwise	clockwise
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x=a\cos t,\ y=b\sin t,$ $0\le t\le 2\pi$	$x=a\cos t,\ y=-b\sin t,$ $0\le t\le 2\pi$
Circle	$x^2+y^2=a^2$	$x=a\cos t,\ y=a\sin t,$ $0\le t\le 2\pi$	$x=a\cos t,\ y=-a\sin t,$ $0\le t\le 2\pi$

- Parametric equation of a line segment joining (x_1,y_1,z_1) to (x_2,y_2,z_2) is given by: $x=(1-t)x_1+tx_2$; $y=(1-t)y_1+ty_2$; $z=(1-t)z_1+tz_2$; $0\le t\le 1$

12 Some useful Results

1. $\iint\limits_R f(x,y)\, dy\, dx = \iint\limits_D g(u,v)|J|\, du\, dv$, when x,y are functions of u,v and $J=\dfrac{\partial(x,y)}{\partial(u,v)}$.

2. $\iiint\limits_{V_1} f(x,y,z)\, dz\, dy\, dx = \iiint\limits_{V_2} g(u,v,w)|J|\, du\, dv\, dw$, when x,y,z are functions of u,v,w and $J=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

3. $\iint f(x,y)\, dy\, dx = \iint f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$, when x,y are converted into polar co-ordinates by $x=r\cos\theta$; $y=r\sin\theta$ as $J=r$.

4. $\iiint\limits_R f(x,y,z)\,dx\,dy\,dz = \iiint\limits_D f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz$, when the Cartesian co-ordinates are changed into cylindrical co-ordinates under the relation $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.

5. $\iiint\limits_R f(x,y,z)\,dx\,dy\,dz = \iiint\limits_D f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) r^2 \sin\theta\,dr\,d\theta\,d\phi$, when the Cartesian co-ordinates are changed into spherical co-ordinates under the relation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi,\ z = r\cos\theta$.