Double Integral

1. When Limits are constants, i.e. rectangular region

- Double integral in rectangular region: $a \le x \le b$ and $c \le y \le d$ is denoted as:

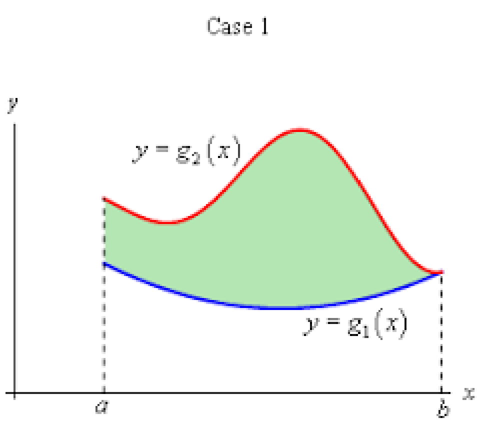
$$\iint_R f(x,y)\,dxdy = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dxdy$$

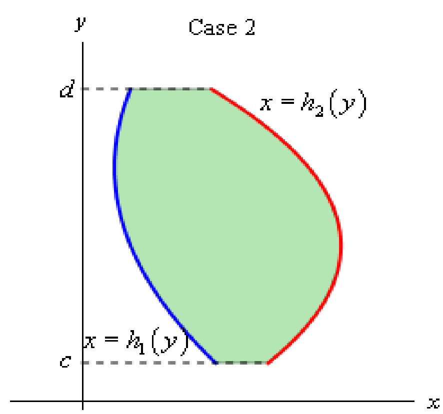
- We can evaluate the double integral by integrating with respect to one variable at a time and treating other as constant.
- The order of integration can be interchanged as per the convenience. Thus,

$$\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dxdy = \int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy \text{ OR } \int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx$$

2. When Limits are not constant

Case: 1 As shown in the figure, below, the region is bounded by x = a, x = b, $y = g_1(x)$ and $y = g_2(x)$. Here we first integrate with respect to y as y is a function of x and then integrate with respect to x. It is denoted as $\int_{a}^{b}\int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$.





Case: 2 As shown in figure above, the region is bounded by y = c, y = d, $x = h_1(y)$ and $x = h_2(y)$. Here we first integrate with respect to x and then with respect to y. It is denoted as $\int_{c}^{d}\int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$.

Remember:

(i) Always integrate from inside outwards.
(ii) If limits of inner integrals are functions of x, then integrate w.r.t. y and if limits of inner integrals are functions of y, then integrate w.r.t. x.
(iii) Limits of outer integrals are always constant.

3. When Limits are not given

1. Trace the curve. Locate the region of integration (ROI) and the co-ordinates of the points of intersection.
2. To integrate inner integral w.r.t. y, choose a strip parallel to y axis of width dx in the region.
   a. Values of lower and upper ends of the strips are expressed as functions of x in the inner integral.
   b. For outer integral, moving the strip dx, from left to right, end values are obtained.
3. To integrate the inner integral w.r.t. x, choose a strip parallel to x axis of width dy in the region.
   a. Values of left and right ends of the strip are expressed as functions of y and become the limit for the inner integral.
   b. For outer integral, the lower and upper end of the region gives the limit.

4. Application of integration to area and volume

- The area A of a region R in XY-plane bounded by the curves $y = g_1(x)$ and $y = g_2(x)$ and the lines x = a and y = b is given by: $\int_a^b \int_{g_1(x)}^{g_2(x)} dydx$.
- The volume of solid is given by: $\iint_R z\, dxdy$.

Triple integral

It is expressed as $\iiint_R f(x,y,z)\,dxdydz$

By suitably arranging the terms we can express it as $\int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dxdydz$