Limits of several variables

Functions of two variables: If three variables *x*, *y* and *z* are so related that the value of *z* depends upon the values of *x* and *y*, then *z* is called a function of two variables *x* and *y*. It is denoted by $z = f(x, y)$.

Limit: If a function $f(x, y)$ has a limit say '*L*' at a point (a, b) then it is denoted as $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Note:

1. Limit may or may not exist.
2. If a limit exists, it must be unique.

Algebra of Limits: If $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$, then

1. $\lim_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = L \pm M$
2. $\lim_{(x,y)\to(a,b)} f(x,y) \cdot g(x,y) = L \cdot M$
3. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, provided $M \neq 0$

Partial Derivatives of first order: Let $z = f(x, y)$

1. **The derivative of *z* with respect to *x*, if it exists when *x* alone varies and *y* remains constant is called the partial derivative of *z* with respect to *x*.**
   It is denoted as $\frac{\partial z}{\partial x}$ **or** $\frac{\partial f}{\partial x}$ **or** z_x **or** f_x.
2. **The derivative of *z* with respect to *y*, if it exists when *y* alone varies and *x* remains constant is called the partial derivative of *z* with respect to *y*.**
   It is denoted as $\frac{\partial z}{\partial y}$ **or** $\frac{\partial f}{\partial y}$ **or** z_y **or** f_y.

Partial Derivatives of second order: If $z = f(x, y)$, then the partial derivatives of f_x and f_y gives the second order partial derivatives as follows:

1. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
2. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$
3. $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$
4. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Note: f_{xy} and f_{yx} are called mixed partial derivatives and they are equal if the first partials f_x and f_y are continuous.

Differentiation of a function of a function: If *u* is a function of *t* and *t* is a function of *x* and *y*, then $\frac{\partial u}{\partial x} = \frac{du}{dt}\frac{\partial t}{\partial x}$ **and** $\frac{\partial u}{\partial y} = \frac{du}{dt}\frac{\partial t}{\partial y}$.

Homogeneous Function: A function $u = f(x, y)$ is said to be homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Similarly a function $u = f(x, y, z)$ is said to be homogeneous of degree n if $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$.

Euler's Theorem: If $u = f(x, y)$ is a homogeneous function of in variables x and y of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Similarly, if $u = f(x, y, z)$ is a homogeneous function in three variables of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$.

Jacobian: If $u = f(x, y)$ and $v = g(x, y)$, then the Jacobian of u and v with respect to x and y is denoted by $J = \frac{\partial(u,v)}{\partial(x,y)}$ **and is defined as**

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

If $u = f(x, y, z)$, $v = g(x, y, z)$ and $w = h(x, y, z)$, then the Jacobian of u, v and w with respect to x, y and z is denoted by $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ **and is defined as**

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties of Jacobian:

(i) **If u, v, w are functions of x, y and z with** $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ **&** $J' = \frac{\partial(x,y,z)}{\partial(u,v,w)}$**,**

then $JJ' = 1$.

(ii) **If u, v, w are functions of $r(x, y, z)$, $s(x, y, z)$ and $t(x, y, z)$ then**

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$$