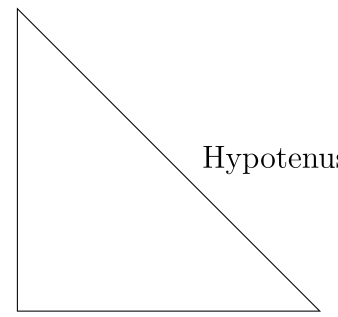
1 Introduction to Trigonometric Ratios



- For a right angled triangle ABC, right angled at B, with respect to the angle C, we have the following trigonometric ratios.

$$\sin C = \frac{\text{Opposite Side}}{\text{Hypotenuse}} = \frac{AB}{AC} \qquad \cos C = \frac{\text{Adjacent Side}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\tan C = \frac{\text{Opposite Side}}{\text{Adjacent Side}} = \frac{AB}{BC} \qquad \cot C = \frac{\text{Adjacent Side}}{\text{Opposite Side}} = \frac{BC}{AB}$$

$$\sec C = \frac{\text{Hypotenuse}}{\text{Opposite Side}} = \frac{AC}{AB} \qquad \text{cosec } C = \frac{\text{Hypotenuse}}{\text{Adjacent Side}} = \frac{AC}{BC}$$

- **Relation between trigonometric ratios:** For an acute angle θ,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\tan\theta\cdot\cot\theta = 1 \quad \sin\theta\cdot\text{cosec }\theta = 1 \quad \cos\theta\cdot\sec\theta = 1$$

- **Values trigonometric ratios for some basic angles:**

Angle	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\text{cosec }\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

- **Basic Trigonometric Identities:**

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \text{cosec }^2\theta$$

- **Trigonometric Ratios of complementary angles:**

$\sin(90^\circ - \theta) = \cos\theta$

$\cos(90^\circ - \theta) = \sin\theta$

$\tan(90^\circ - \theta) = \cot\theta$

$\cot(90^\circ - \theta) = \tan\theta$

$\sec(90^\circ - \theta) = \text{cosec }\theta$

$\text{cosec }(90^\circ - \theta) = \sec\theta$

- **Some more relations between trigonometric ratios:** For an acute angle θ,

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$	$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$
$\sin(\pi - \theta) = \sin\theta$	$\cos(\pi - \theta) = -\cos\theta$	$\tan(\pi - \theta) = -\tan\theta$	$\cot(\pi - \theta) = -\cot\theta$
$\sin(\pi + \theta) = -\sin\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\tan(\pi + \theta) = \tan\theta$	$\cot(\pi + \theta) = \cot\theta$
$\sin(2\pi - \theta) = -\sin\theta$	$\cos(2\pi - \theta) = \cos\theta$	$\tan(2\pi - \theta) = -\tan\theta$	$\cot(2\pi - \theta) = -\cot\theta$
$\sin(2\pi + \theta) = \sin\theta$	$\cos(2\pi + \theta) = \cos\theta$	$\tan(2\pi + \theta) = \tan\theta$	$\cot(2\pi + \theta) = \cot\theta$

- **Addition Formulae:**

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$$

$$\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$$

- **Sum to Product:**

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

- **Product to Sum:**

$$2\sin A\cos B = \sin(A+B) + \sin(A-B) \qquad 2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B) \qquad 2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

- **Multiple and Submultiple Identities:**

  (1) $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

  (2) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

  (3) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

  (4) $\sin 3x = 3\sin x - 4\sin^3 x$

  (5) $\cos 3x = 4\cos^3 x - 3\cos x$

  (6) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

  (7) $\sin^2\left(\dfrac{x}{2}\right) = \dfrac{1-\cos x}{2}$

  (8) $\cos^2\left(\dfrac{x}{2}\right) = \dfrac{1+\cos x}{2}$

  (9) $\tan^2\left(\dfrac{x}{2}\right) = \dfrac{1-\cos x}{1+\cos x}$

- **Height and Distance:**

  (1) **Line os sight:** It is a line drawn from the eye of an observer to the point in the object viewed by the observer.

  (2) **The angle of elevation:** This is the angle fromed by the line of sight with the horizontal when it is above the horizontal level.

  (3) **The angle of depression:** This is the angle formed by the line of sight with the horizontal when viewed below the horizontal level.

- **Inverse Trigonometry**

(1) The domain and ranges of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (principal value branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$
$y = \sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \text{cosec}^{-1} x$	$R - (-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - 0$

(2) For suitable values of domain, we have:

$\sin(\sin^{-1} x) = x$ $\qquad$ $\sin^{-1}(\sin x) = x$

$\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$ $\qquad$ $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$ $\qquad$ $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$

$\sin^{-1}(-x) = -\sin^{-1} x$ $\qquad$ $\tan^{-1}(-x) = -\tan^{-1} x$ $\qquad$ $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$ $\qquad$ $\cot^{-1}(-x) = \pi - \cot^{-1} x$ $\qquad$ $\sec^{-1}(-x) = \pi - \sec^{-1} x$

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ $\qquad$ $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ $\qquad$ $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$

(3) **Some more useful formulae:**

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(ii) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(iii) $\sin^{-1} x = \cos^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(iv) $\cos^{-1} x = \sin^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

(v) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$