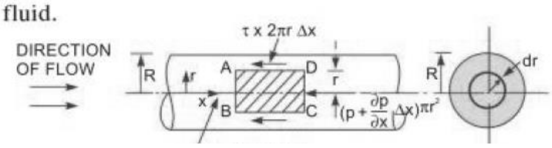
FLOW OF VISCOUS FLUID THROUGH CIRCULAR PIPE

For the flow of viscous fluid through circular pipe, the velocity distribution across a section, the ratio of maximum velocity to average velocity, the shear stress distribution and drop of pressure for a given length is to be determined. The flow through the circular pipe will be viscous or laminar, if the Reynolds number (R_e*) is less than 2000. The expression for Reynold number is given by

$$R_e = \frac{\rho VD}{\mu}$$

where ρ = Density of fluid flowing through pipe
V = Average velocity of fluid
D = Diameter of pipe and
μ = Viscosity of fluid.



Consider a horizontal pipe of radius R. The viscous fluid is flowing from left to right in the pipe as shown in Fig. (a). Consider a fluid element of radius r, sliding in a cylindrical fluid element of radius $(r + dr)$. Let the length of fluid element be Δx. If 'p' is the intensity of pressure on the face AB, then the intensity of pressure on face CD will be $\left(p + \frac{\partial p}{\partial x}\Delta x\right)$. Then the forces acting on the fluid element are :

1. The pressure force, $p \times \pi r^2$ on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2$ on face CD.
3. The shear force, $\tau \times 2\pi r\Delta x$ on the surface of fluid element. As there is no acceleration, hence the summation of all forces in the direction of flow must be zero *i.e.*,

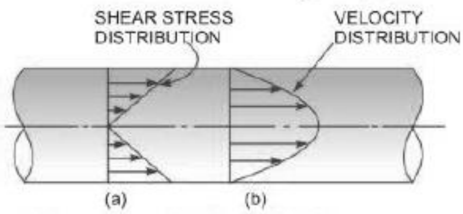
$$p\pi r^2 - \left(p + \frac{\partial p}{\partial x}\Delta x\right)\pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or $$-\frac{\partial p}{\partial x}\Delta x\pi r^2 - \tau \times 2\pi r \times \Delta x = 0$$

or $$-\frac{\partial p}{\partial x} . r - 2\tau = 0$$

$$\therefore \quad \tau = -\frac{\partial p}{\partial x}\frac{r}{2} \quad \text{.............................1}$$

The shear stress τ across a section varies with 'r' as $\frac{\partial p}{\partial x}$ across a section is constant. Hence shear stress distribution across a section is linear as shown in Fig. (a).



Shear stress and velocity distribution across a section.

(*i*) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \frac{du}{dy}$ is substituted in equation **1**

But in the relation $\tau = \mu \frac{du}{dy}$, y is measured from the pipe wall. Hence

$$y = R - r \quad \text{and} \quad dy = -dr$$

$$\therefore \quad \tau = \mu \frac{du}{-dr} = -\mu \frac{du}{dr}$$

$$-\mu \frac{du}{dr} = -\frac{\partial p}{\partial x}\frac{r}{2} \quad \text{or} \quad \frac{du}{dr} = \frac{1}{2\mu}\frac{\partial p}{\partial x} r$$

Integrating this above equation w.r.t. 'r', we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 + C$$

where C is the constant of integration and its value is obtained from the boundary condition that at $r = R$, $u = 0$.

$$\therefore \quad 0 = \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 + C$$

$$\therefore \quad C = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

Substituting this value of C in equation (9.2), we get

$$u = \frac{1}{4\mu}\frac{\partial p}{\partial x} r^2 - \frac{1}{4\mu}\frac{\partial p}{\partial x} R^2$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2]$$

(*ii*) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when $r = 0$ in equation Thus maximum velocity, U_{max} is obtained as

$$U_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2 \qquad \text{..................(A)}$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge of the fluid across the section by the area of the pipe (πR^2). The discharge (Q) across the section is obtained by considering the flow through a circular ring element of radius r and thickness dr as shown in Fig. . The fluid flowing per second through this elementary ring

$$dQ = \text{velocity at a radius } r \times \text{area of ring element}$$

$$= u \times 2\pi r\, dr$$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial x}[R^2 - r^2] \times 2\pi r\, dr$$

$$\therefore \quad Q = \int_0^R dQ = \int_0^R -\frac{1}{4\mu}\frac{\partial p}{\partial x}(R^2 - r^2) \times 2\pi r\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \int_0^R (R^2 - r^2)\, r dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \int_0^R (R^2 r - r^3)\, dr$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{1}{4\mu}\left(\frac{-\partial p}{\partial x}\right) \times 2\pi \times \frac{R^4}{4} = \frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^4$$

$$\therefore \quad \text{Average velocity,} \quad \bar{u} = \frac{Q}{\text{Area}} = \frac{\frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^4}{\pi R^2}$$

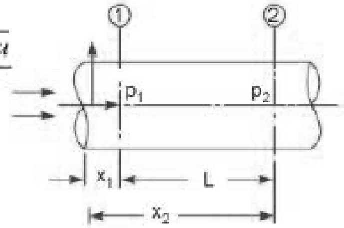
$$\text{or} \quad \bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^2 \qquad \text{......................(B)}$$

Dividing equation **A** by equation **B**

$$\frac{U_{max}}{\bar{u}} = \frac{-\frac{1}{4\mu}\frac{\partial p}{\partial x} R^2}{\frac{1}{8\mu}\left(-\frac{\partial p}{\partial x}\right) R^2} = 2.0$$

(*iii*) **Drop of Pressure for a given Length (*L*) of a pipe**

From equation **B** we have



$$\bar{u} = \frac{1}{8\mu}\left(\frac{-\partial p}{\partial x}\right) R^2 \quad \text{or} \quad \left(\frac{-\partial p}{\partial x}\right) = \frac{8\mu\bar{u}}{R^2}$$

Integrating the above equation w.r.t. *x*, we get

$$-\int_2^1 dp = \int_2^1 \frac{8\mu\bar{u}}{R^2}\, dx$$

$\therefore$
$$-[p_1 - p_2] = \frac{8\mu\bar{u}}{R^2}[x_1 - x_2] \text{ or } (p_1 - p_2) = \frac{8\mu\bar{u}}{R^2}[x_2 - x_1]$$

$$= \frac{8\mu\bar{u}}{R^2} L \qquad \{\because \; x_2 - x_1 = L \text{ from Fig.}\}$$

$$= \frac{8\mu\bar{u}L}{(D/2)^2} \qquad \left\{\because \; R = \frac{D}{2}\right\}$$

$$(p_1 - p_2) = \frac{32\mu\bar{u}L}{D^2},$$ where $p_1 - p_2$ is the drop of pressure.

$\therefore$ Loss of pressure head $= \dfrac{p_1 - p_2}{\rho g}$

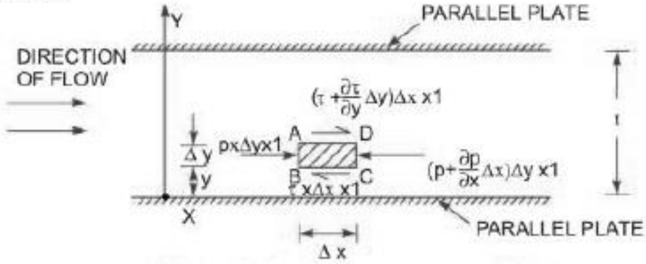
$\therefore$
$$\frac{p_1 - p_2}{\rho g} = h_f = \frac{32\mu\bar{u}L}{\rho g D^2}$$

is called **Hagen Poiseuille Formula.**

Ref: R.K. Bansal

FLOW OF VISCOUS FLUID BETWEEN TWO PARALLEL PLATES

In this case also, the shear stress distribution, the velocity distribution across a section ; the ratio of maximum velocity to average velocity and difference of pressure head for a given length of parallel plates, are to be calculated.



Viscous flow between two parallel plates.

Consider two parallel fixed plates kept at a distance 't' apart as shown in Fig. . A viscous fluid is flowing between these two plates from left to right. Consider a fluid element of length Δx and thickness Δy at a distance y from the lower fixed plate. If p is the intensity of pressure on the face AB of the fluid element then intensity of pressure on the face CD will be $\left(p + \frac{\partial p}{\partial x} \Delta x\right)$. Let τ is the shear stress acting on the face BC then the shear stress on the face AD will be $\left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right)$. If the width of the element in the direction perpendicular to the paper is unity then the forces acting on the fluid element are :

1. The pressure force, $p \times \Delta y \times 1$ on face AB.
2. The pressure force, $\left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y \times 1$ on face CD.
3. The shear force, $\tau \times \Delta x \times 1$ on face BC.
4. The shear force, $\left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right) \Delta x \times 1$ on face AD.

For steady and uniform flow, there is no acceleration and hence the resultant force in the direction of flow is zero.

$$\therefore \quad p\Delta y \times 1 - \left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y \times 1 - \tau \Delta x \times 1 + \left(\tau + \frac{\partial \tau}{\partial y} \Delta y\right) \Delta x \times 1 = 0$$

or

$$-\frac{\partial p}{\partial x} \Delta x \Delta y + \frac{\partial \tau}{\partial x} \Delta y \Delta x = 0$$

Dividing by $\Delta x \Delta y$, we get $-\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = 0$ or $\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y}$ ....(A)

(*i*) **Velocity Distribution.** To obtain the velocity distribution across a section, the value of shear stress $\tau = \mu \frac{du}{dy}$ from Newton's law of viscosity for laminar flow is substituted in equation **A**

$$\therefore \quad \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{du}{dy}\right) = \mu \frac{\partial^2 u}{dy^2}$$

$$\therefore \quad \frac{\partial^2 u}{dy^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

Integrating the above equation w.r.t. y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1 \qquad \left\{\because \frac{\partial p}{\partial x} \text{ is constant}\right\}$$

Integrating again $\qquad u = \frac{1}{\mu} \frac{\partial p}{\partial x} \frac{y^2}{2} + C_1 y + C_2$ ....(B) where C_1 and C_2 are constants of integration. Their values are obtained from the two boundary conditions that is (*i*) at $y = 0$, $u = 0$ (*ii*) at $y = t$, $u = 0$.

The substitution of $y = 0, u = 0$ in equation (9.8) gives

$$0 = 0 + C_1 \times 0 + C_2 \text{ or } C_2 = 0$$

The substitution of $y = t, u = 0$ in equation (9.8) gives

$$0 = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{t^2}{2} + C_1 \times t + 0$$

$$\therefore \quad C_1 = -\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{t^2}{2 \times t} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}t$$

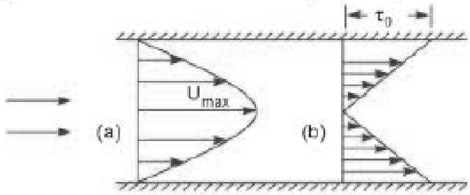
Substituting the values of C_1 and C_2 in equation (9.8)

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + y\left(-\frac{1}{2\mu}\frac{\partial p}{\partial x}t\right)$$

or

$$u = -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2] \qquad \text{.......................(C)}$$

In the above equation, μ, $\frac{\partial p}{\partial x}$ and t are constant. It means u varies with the square of y. Hence equation **C** is a equation of a parabola. Hence velocity distribution across a section of the parallel plate is parabolic. This velocity distribution is shown in Fig



(*ii*) **Ratio of Maximum Velocity to Average Velocity.** The velocity is maximum, when $y = t/2$. Substituting this value in equation **C** , we get

$$U_{max} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[t \times \frac{t}{2} - \left(\frac{t}{2}\right)^2\right]$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{t^2}{2} - \frac{t^2}{4}\right] = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{t^2}{4} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2 \qquad \text{... (D)}$$

The average velocity, $\bar{u}$, is obtained by dividing the discharge (Q) across the section by the area of the section ($t \times 1$). And the discharge Q is obtained by considering the rate of flow of fluid through the strip of thickness dy and integrating it. The rate of flow through strip is

$$dQ = \text{Velocity at a distance } y \times \text{Area of strip}$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2] \times dy \times 1$$

$$\therefore \quad Q = \int_0^t dQ = \int_0^t -\frac{1}{2\mu}\frac{\partial p}{\partial x}[ty - y^2]\,dy$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{ty^2}{2} - \frac{y^3}{3}\right]_0^t = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{t^3}{2} - \frac{t^3}{3}\right]$$

$$= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{t^3}{6} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^3$$

$$\therefore \qquad \bar{u} = \frac{Q}{\text{Area}} = -\frac{\frac{1}{12\mu}\frac{\partial p}{\partial x}.t^3}{t \times 1} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2 \qquad \text{..(E)}$$

Dividing equation **(D)** by equation **(E)**, we get

$$\frac{U_{max}}{\bar{u}} = \frac{-\frac{1}{8\mu}\frac{\partial p}{\partial x}t^2}{-\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2} = \frac{12}{8} = \frac{3}{2} \qquad \text{...(F)}$$

(*iii*) **Drop of Pressure head for a given Length.** From equation **(E)**, we have

$$\bar{u} = -\frac{1}{12\mu}\frac{\partial p}{\partial x}t^2 \quad \text{or} \quad \frac{\partial p}{\partial x} = -\frac{12\mu\bar{u}}{t^2}$$

Integrating this equation w.r.t. x, we get

$$\int_2^1 dp = \int_2^1 -\frac{12\mu\bar{u}}{t^2}\,dx$$

or

$$p_1 - p_2 = -\frac{12\mu\bar{u}}{t^2}[x_1 - x_2] = \frac{12\mu\bar{u}}{t^2}[x_2 - x_1]$$

or

$$p_1 - p_2 = \frac{12\mu\bar{u}L}{t^2} \qquad [\because x_1 - x_2 = L]$$

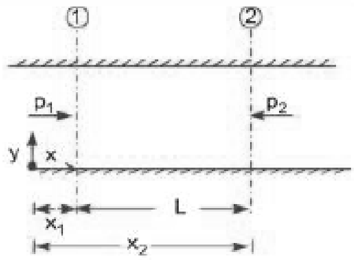


Fig. 9.8

If h_f is the drop of pressure head, then

$$h_f = \frac{p_1 - p_2}{\rho g} = \frac{12\mu\bar{u}L}{\rho g t^2}$$

(*iv*) **Shear Stress Distribution.** It is obtained by substituting the value of u from equation **(C)** into

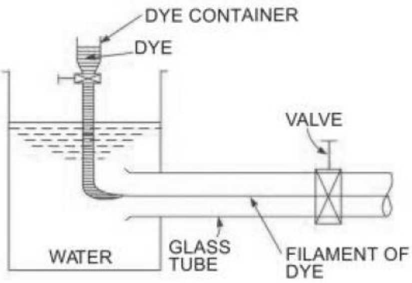
$$\tau = \mu\frac{\partial u}{\partial y}$$

$$\therefore \qquad \tau = \mu\frac{\partial u}{\partial y} = \mu\frac{\partial}{\partial y}\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(ty - y^2\right)\right] = \mu\left[-\frac{1}{2\mu}\frac{\partial p}{\partial x}(t - 2y)\right]$$

$$\tau = -\frac{1}{2}\frac{\partial p}{\partial x}[t - 2y]$$

REYNOLDS EXPERIMENT

The type of flow is determined from the Reynolds number *i.e.*, $\frac{\rho V \times d}{\mu}$. This was demonstrated by O. Reynold in 1883. His apparatus is shown in Fig. 10.1.



Reynold apparatus.

The apparatus consists of :

(*i*) A tank containing water at constant head,

(*ii*) A small tank containing some dye,

(*iii*) A glass tube having a bell-mouthed entrance at one end and a regulating value at other ends.

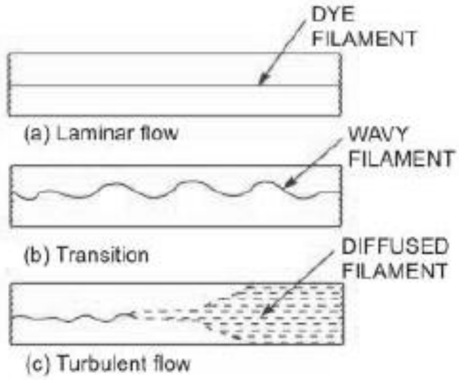
The water from the tank was allowed to flow through the glass tube. The velocity of flow was varied by the regulating valve. A liquid dye having same specific weight as water was introduced into the glass tube as shown in Fig.

The following observations were made by Reynold :

(*i*) When the velocity of flow was low, the dye filament in the glass tube was in the form of a straight line. This straight line of dye filament was parallel to the glass tube, which was the case of laminar flow as shown in Fig. (*a*).

(*ii*) With the increase of velocity of flow, the dye-filament was no longer a straight-line but it became a wavy one as shown in Fig. (*b*). This shows that flow is no longer laminar.

(*iii*) With further increase of velocity of flow, the wavy dye-filament broke-up and finally diffused in water as shown in Fig. (*c*). This means that the fluid particles of the dye at this higher velocity are moving in random fashion, which shows the case of turbulent flow. Thus in case of turbulent flow the mixing of dye-filament and water is intense and flow is irregular, random and disorderly.



Different stages of filament.

In case of laminar flow, the loss of pressure head was found to be proportional to the velocity but in case of turbulent flow, Reynold observed that loss of head is approximately proportional to the square of velocity. More exactly the loss of head, $h_f \propto V^n$, where n varies from 1.75 to 2.0

FRICTIONAL LOSS IN PIPE FLOW

When a liquid is flowing through a pipe, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity. This viscous action causes loss of energy which is usually known as frictional loss.

On the basis of his experiments, William Froude gave the following laws of fluid fraction for turbulent flow.

The frictional resistance for turbulent flow is :

(*i*) proportional to V^n, where n varies from 1.5 to 2.0,

(*ii*) proportional to the density of fluid,

(*iii*) proportional to the area of surface in contact,

(*iv*) independent of pressure,

(*v*) dependent on the nature of the surface in contact.

Expression for Loss of Head Due to Friction in Pipes.

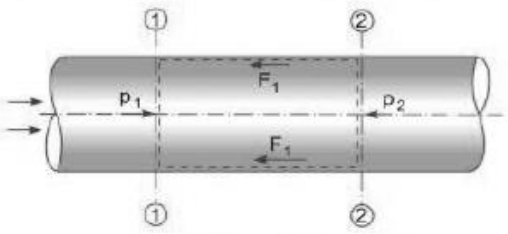
L = length of the pipe between sections 1-1 and 2-2,

d = diameter of pipe,

f' = frictional resistance per unit wetted area per unit velocity,

h_f = loss of head due to friction,

and p_2, V_2 = are values of pressure intensity and velocity at section 2-2.



Uniform horizontal pipe.

Applying Bernoulli's equations between sections 1-1 and 2-2,

Total head at 1-1 = Total head at 2-2 + loss of head due to friction between 1-1 and 2-2

or $$\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2+h_f$$

But $z_1 = z_2$ as pipe is horizontal

$V_1 = V_2$ as dia. of pipe is same at 1-1 and 2-2

$$\therefore \quad \frac{p_1}{\rho g}=\frac{p_2}{\rho g}+h_f \text{ or } h_f=\frac{p_1}{\rho g}-\frac{p_2}{\rho g} \quad ...(i)$$

But h_f is the head lost due to friction and hence intensity of pressure will be reduced in the direction of flow by frictional resistance.

Now frictional resistance = frictional resistance per unit wetted area per unit velocity × wetted area × velocity^2

or $$F_1 = f' \times \pi dL \times V^2 \quad [\because \text{ wetted area} = \pi d \times L,\ \text{velocity} = V = V_1 = V_2]$$

$$= f' \times P \times L \times V^2 \quad [\because \pi d = \text{Perimeter} = P] \ ...(ii)$$

The forces acting on the fluid between sections 1-1 and 2-2 are :

1. pressure force at section 1-1 = $p_1 \times A$

where A = Area of pipe

2. pressure force at section 2-2 = $p_2 \times A$

3. frictional force F_1 as shown in Fig. 10.3.

Resolving all forces in the horizontal direction, we have

$$p_1A - p_2A - F_1 = 0$$

or $$(p_1 - p_2)A = F_1 = f' \times P \times L \times V^2 \quad [\because \text{ From } (ii),\ F_1 = f'PLV^2]$$

or $$p_1 - p_2 = \frac{f' \times P \times L \times V^2}{A}$$

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Equating the value of $(p_1 - p_2)$, we get

$$\rho g h_f = \frac{f' \times P \times L \times V^2}{A}$$

or

$$h_f = \frac{f'}{\rho g} \times \frac{P}{A} \times L \times V^2 \quad ...(iii)$$

In equation (*iii*), $\frac{P}{A} = \frac{\text{Wetted perimeter}}{\text{Area}} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$

$\therefore$

$$h_f = \frac{f'}{\rho g} \times \frac{4}{d} \times L \times V^2 = \frac{f'}{\rho g} \times \frac{4LV^2}{d} \quad ...(iv)$$

Putting $\frac{f'}{\rho} = \frac{f}{2}$, where f is known as co-efficient of friction.

Equation (*iv*), becomes as $h_f = \frac{4 \,.\, f}{2g} \cdot \frac{LV^2}{d} = \frac{4f \,.\, L \,.\, V^2}{d \times 2g}$

Expression for Co-efficient of Friction in Terms of Shear Stress. The equation gives the forces acting on a fluid between sections 1-1 and 2-2 of Fig. in horizontal direction as

$$p_1 A - p_2 A - F_1 = 0$$

or

$$(p_1 - p_2)A = F_1 = \text{force due to shear stress } \tau_0$$
$$= \text{shear stress} \times \text{surface area}$$
$$= \tau_0 \times \pi d \times L$$

or

$$(p_1 - p_2)\,\frac{\pi}{4}\,d^2 = \tau_0 \times \pi d \times L \quad \left\{\because A = \frac{\pi}{4} d^2\right\}$$

Cancelling πd from both sides, we have

$$(p_1 - p_2)\,\frac{d}{4} = \tau_0 \times L$$

or

$$(p_1 - p_2) = \frac{4\tau_0 \times L}{d}$$

can be written as $h_f = \frac{p_1 - p_2}{\rho g} = \frac{4f \,.\, L \,.\, V^2}{d \times 2g}$