SUBJECT: FLUID MECHANICS

INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} = 0$$

where ∂V = Change of velocity

∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t = \text{constant}} \neq 0.$$

Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number,

where D = Diameter of pipe

V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant.}$$

Problem *A 25 cm diameter pipe carries oil of sp. gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20 cm. Find the velocity at this section and also mass rate of flow of oil.*

Solution. Given :

at section 1, $D_1 = 25 \text{ cm} = 0.25 \text{ m}$

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049 \text{ m}^3$$

$V_1 = 3 \text{ m/s}$

at section 2, $D_2 = 20 \text{ cm} = 0.2 \text{ m}$

$$A_2 = \frac{\pi}{4} (0.2)^2 = 0.0314 \text{ m}^2$$

$V_2 = ?$

Mass rate of flow of oil = ?

Applying continuity equation at sections 1 and 2,

$$A_1 V_1 = A_2 V_2$$

or $$0.049 \times 3.0 = 0.0314 \times V_2$$

$\therefore$ $$V_2 = \frac{0.049 \times 3.0}{0.0314} = \textbf{4.68 m/s. Ans.}$$

Mass rate of flow of oil $= \text{Mass density} \times Q = \rho \times A_1 \times V_1$

Sp. gr. of oil $= \dfrac{\text{Density of oil}}{\text{Density of water}}$

$\therefore$ Density of oil = Sp. gr. of oil × Density of water

$$= 0.9 \times 1000 \text{ kg/m}^3 = \frac{900 \text{ kg}}{\text{m}^3}$$

$\therefore$ Mass rate of flow $= 900 \times 0.049 \times 3.0 \text{ kg/s} = \textbf{132.23 kg/s. Ans.}$

CONTINUITY EQUATION IN THREE-DIMENSIONS

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$

$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u \, dydz + \dfrac{\partial}{\partial x} (\rho u \, dydz) \, dx$

$\therefore$ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$

$$= \rho u \, dydz - \rho u \, dydz - \frac{\partial}{\partial x} (\rho u \, dydz) dx$$

$$= -\frac{\partial}{\partial x} (\rho u \, dydz) \, dx$$

$$= -\frac{\partial}{\partial x} (\rho u) \, dx \, dydz \qquad \{\because dydz \text{ is constant}\}$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y} (\rho v) \, dxdydz$$

and in z-direction

$$= -\frac{\partial}{\partial z} (\rho w) \, dxdydz$$

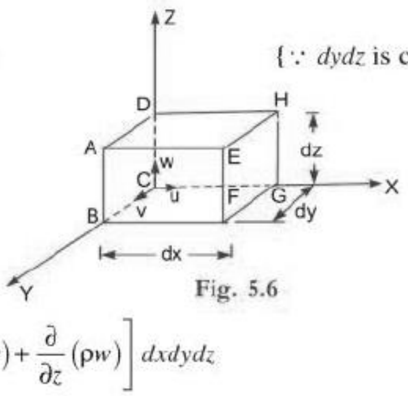


Fig. 5.6

$\therefore$ $$\text{Net gain of masses} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho.\ dx.\ dy.\ dz$ and its rate of increase with time is $\frac{\partial}{\partial t}$ ($\rho\ dx.\ dy.\ dz$) or

$\frac{\partial \rho}{\partial t}\ .\ dx\ dy\ dz.$

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\ dxdydz$$

or $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0$$ [Cancelling $dx.dy.dz$ from both sides] .

Equation , is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation becomes as

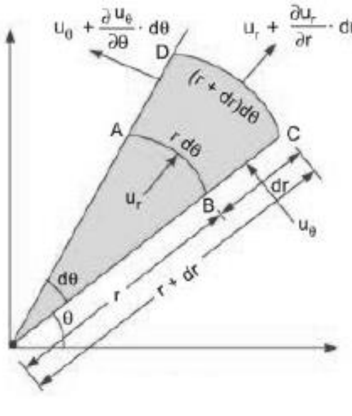
$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Continuity Equation in Cylindrical Polar Co-ordinates. The continuity equation in cylindrical polar co-ordinates (*i.e.*, r, θ, z co-ordinates) is derived by the procedure given below.

Consider a two-dimensional incompressible flow field. The two-dimensional polar co-ordinates are r and θ. Consider a fluid element $ABCD$ between the radii r and $r + dr$ as shown in Fig. . The angle subtended by the element at the centre is $d\theta$. The components of the velocity V are u_r in the radial direction and u_θ in the tangential direction. The sides of the element are having the lengths as



Side $AB = rd\theta$, $BC = dr$, $DC = (r + dr)\ d\theta$, $AD = dr$.

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

Consider the flow in radial direction

Mass of fluid entering the face AB per unit time

$= \rho \times$ Velocity in r-direction $\times$ Area

$= \rho \times u_r \times (AB \times 1)$ ($\because$ Area $= AB \times$ Thickness $= rd\theta \times 1$)

$= \rho \times u_r \times (rd\theta \times 1) = \rho\ .\ u_r.\ rd\theta$

Mass of fluid leaving the face CD per unit time

$= \rho \times$ Velocity $\times$ Area

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}.dr\right) \times (CD \times 1) \qquad (\because \text{Area} = CD \times 1)$$

$$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}\,dr\right) \times (r + dr)d\theta \qquad [\because CD = (r + dr)\ d\theta]$$

$$= \rho \times \left[u_r \times r + u_r dr + r\frac{\partial u_r}{\partial r}\,dr + \frac{\partial u_r}{\partial r}(dr)^2\right] d\theta$$

$$= \rho\left[u_r \times r + u_r \times dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

[The term containing $(dr)^2$ is very small and has been neglected]

∴ Gain of mass in r-direction per unit time

$$= (\text{Mass through } AB - \text{Mass through } CD) \text{ per unit time}$$

$$= \rho.\ u_r.\ rd\theta - \rho\left[u_r.r + u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= \rho.\ u_r.\ rd\theta - \rho.\ u_r.\ r.\ d\theta - \rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right] d\theta$$

$$= -\rho\left[u_r.dr + r\frac{\partial u_r}{\partial r}.dr\right].\ d\theta$$

$$= -\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right] r.\ dr.\ d\theta$$

[This is written in this form because ($r.\ d\theta.\ dr.\ 1$) is equal to volume of element]

Now consider the flow in θ-direction

Gain in mass in θ-direction per unit time

$$= (\text{Mass through } BC - \text{Mass through } AD) \text{ per unit time}$$

$$= [\rho \times \text{Velocity through } BC \times \text{Area} - \rho \times \text{Velocity through } AD \times \text{Area}]$$

$$= \left[\rho.u_\theta.dr \times 1 - \rho\left(u_\theta + \frac{\partial u_\theta}{\partial \theta}.d\theta\right) \times dr \times 1\right]$$

$$= -\rho\left(\frac{\partial u_\theta}{\partial \theta}.d\theta\right) dr \times 1 \qquad (\because \text{Area} = dr \times 1)$$

$$= -\rho\frac{\partial u_\theta}{\partial \theta}.\frac{r.d\theta.dr}{r} \qquad [\text{Multiplying and dividing by } r]$$

∴ Total gain in fluid mass per unit time

$$= -\rho\left[\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right].\ r.\ dr.\ d\theta - \rho\frac{\partial u_\theta}{\partial \theta}.\frac{rd\theta.\ dr}{r}$$

VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are functions of space-co-ordinates and time. Mathematically, the velocity components are given as

$$u = f_1(x, y, z, t)$$
$$v = f_2(x, y, z, t)$$
$$w = f_3(x, y, z, t)$$

and Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the **total acceleration** in x, y and z directions respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \frac{dy}{dt} = v \text{ and } \frac{dz}{dt} = w$

$\therefore$
$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly,
$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$
$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

or
$$\frac{\partial u}{\partial t} = 0, \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$
$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$
$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Acceleration vector
$$A = a_x i + a_y j + a_z k$$
$$= \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

Ref: R.K. bansal

Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by

$$\left.\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} \end{aligned}\right\}$$

where u_r = velocity component in radial direction (*i.e.*, in r direction)
and u_θ = velocity component in tangential direction (*i.e.*, in θ direction)

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

is a Laplace equation.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\left.\begin{aligned} \frac{\partial \psi}{\partial x} &= v \\ \text{and} \quad \frac{\partial \psi}{\partial y} &= -u \end{aligned}\right\}$$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } u_\theta = -\frac{\partial \psi}{\partial r}$$

where u_r= radial velocity and u_θ = tangential velocity

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

Substituting the values of u and v from equation

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Substituting the values of u and v from equation the above rotational component, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2} = 0$

EULER'S EQUATION OF MOTION

This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as :

Consider a stream-line in which flow is taking place in s-direction as shown in Fig. Consider a cylindrical element of cross-section dA and length ds. The forces acting on the cylindrical element are:

1. Pressure force pdA in the direction of flow.
2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow.
3. Weight of element $\rho g dA ds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element × acceleration in the direction s.

$$\therefore \quad pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos\theta$$

$$= \rho dA ds \times a_s \qquad \text{.............(1)}$$

where a_s is the acceleration in the direction of s.

Now $\quad a_s = \frac{dv}{dt}$, where v is a function of s and t.

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \quad a_s = \frac{v\partial v}{\partial s}$$

Substituting the value of a_s in equation 1 and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} ds dA - \rho g\, dA ds \cos\theta = \rho dA ds \times \frac{\partial v}{\partial s}$$

Dividing by $\rho ds dA$, $-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$

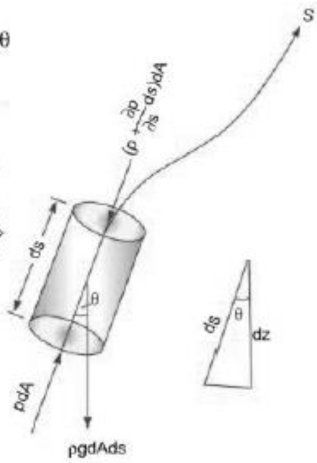
or
$$\frac{\partial p}{\rho \partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

But from Fig. 6.1 (b), we have $\cos\theta = \frac{dz}{ds}$

$$\therefore \quad \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0 \quad \text{or} \quad \frac{dp}{\rho} + gdz + vdv = 0$$

or
$$\frac{dp}{\rho} + gdz + vdv = 0$$

is known as Euler's equation of motion.



(a) (b)
Forces on a fluid element.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion

$$\int \frac{dp}{\rho} + \int g dz + \int v dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\therefore \quad \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

or
$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

or
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant}$$

is a Bernoulli's equation in which

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

$v^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.

Ref: R.K. bansal

Venturimeter. A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It consists of three parts :

(*i*) A short converging part, (*ii*) Throat, and (*iii*) Diverging part. It is based on the Principle of Bernoulli's equation.

Expression for rate of flow through venturimeter

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing (say water), as shown in Fig.

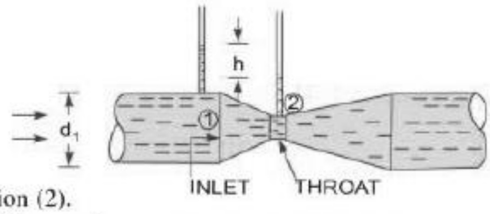
Let d_1 = diameter at inlet or at section (1),

p_1 = pressure at section (1)

v_1 = velocity of fluid at section (1),

a = area at section (1) = $\frac{\pi}{4} d_1^2$

and d_2, p_2, v_2, a_2 are corresponding values at section (2).



Venturimeter.

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

As pipe is horizontal, hence $z_1 = z_2$

$$\therefore \quad \frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad \text{or} \quad \frac{p_1 - p_2}{\rho g} = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

But $\frac{p_1 - p_2}{\rho g}$ is the difference of pressure heads at sections 1 and 2 and it is equal to h or $\frac{p_1 - p_2}{\rho g} = h$

Substituting this value of $\frac{p_1 - p_2}{\rho g}$ in the above equation, we get

$$h = \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

Now applying continuity equation at sections 1 and 2

$$a_1 v_1 = a_2 v_2 \quad \text{or} \quad v_1 = \frac{a_2 v_2}{a_1}$$

Substituting this value of v_1 in equation (6.6)

$$h = \frac{v_2^2}{2g} - \frac{\left(\frac{a_2 v_2}{a_1}\right)^2}{2g} = \frac{v_2^2}{2g}\left[1 - \frac{a_2^2}{a_1^2}\right] = \frac{v_2^2}{2g}\left[\frac{a_1^2 - a_2^2}{a_1^2}\right]$$

or $$v_2^2 = 2gh \frac{a_1^2}{a_1^2 - a_2^2}$$

$$\therefore \quad v_2 = \sqrt{2gh \frac{a_1^2}{a_1^2 - a_2^2}} = \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gh}$$

$\therefore$ Discharge, $$Q = a_2 v_2$$

$$= a_2 \frac{a_1}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh} = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

Value of 'h' given by differential U-tube manometer

Case I. Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

S_h = Sp. gravity of the heavier liquid

S_o = Sp. gravity of the liquid flowing through pipe

x = Difference of the heavier liquid column in U-tube

Then $$h = x\left[\frac{S_h}{S_o} - 1\right]$$

Case II. If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$

where S_l = Sp. gr. of lighter liquid in U-tube

S_o = Sp. gr. of fluid flowing through pipe

x = Difference of the lighter liquid columns in U-tube.

Case III. Inclined Venturimeter with Differential U-tube manometer. The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right]$$

Case IV. Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right]$$

Problem *A 30 cm × 15 cm venturimeter is inserted in a vertical pipe carrying water, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 20 cm. Find the discharge. Take $C_d = 0.98$.*

Solution. Given :

Dia. at inlet, $d_1 = 30$ cm

$\therefore$ $a_1 = \frac{\pi}{4}(30)^2 = 706.85\ \text{cm}^2$

Dia. at throat, $d_2 = 15$ cm

$\therefore$ $a_2 = \frac{\pi}{4}(15)^2 = 176.7\ \text{cm}^2$

$$h = x\left[\frac{S_h}{S_o} - 1\right] = 20\left[\frac{13.6}{1.0} - 1.0\right] = 20 \times 12.6 = 252.0 \text{ cm of water}$$

$C_d = 0.98$

Discharge,

$$Q = C_d \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 \times 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 252}$$

$$= \frac{86067593.36}{\sqrt{499636.3 - 31222.9}} = \frac{86067593.36}{684.4}$$

$= 125756\ \text{cm}^3/\text{s} =$ **125.756 lit/s. Ans.**

Orifice Meter or Orifice Plate.

Let p_1 = pressure at section (1),
v_1 = velocity at section (1),
a_1 = area of pipe at section (1), and

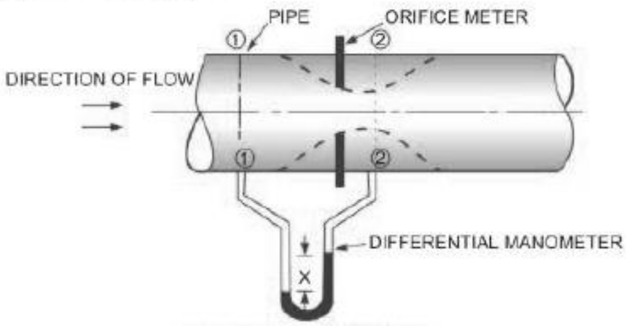


Fig. 6.12. *Orifice meter.*

p_2, v_2, a_2 are corresponding values at section (2). Applying Bernoulli's equation at sections (1) and 2), we get

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2$$

r $$\left(\frac{p_1}{\rho g}+z_1\right)-\left(\frac{p_2}{\rho g}+z_2\right)=\frac{v_2^2}{2g}-\frac{v_1^2}{2g}$$

But $$\left(\frac{p_1}{\rho g}+z_1\right)-\left(\frac{p_2}{\rho g}+z_2\right)=h=\text{Differential head}$$

$\therefore$ $$h=\frac{v_2^2}{2g}-\frac{v_1^2}{2g}\quad\text{or}\quad 2gh={v_2}^2-{v_1}^2$$

r $$v_2=\sqrt{2gh+v_1^2}\qquad ...(i)$$

Now section (2) is at the vena-contracta and a_2 represents the area at the vena-contracta. If a_0 is ıe area of orifice then, we have

$$C_c=\frac{a_2}{a_0}$$

/here C_c = Co-efficient of contraction

$\therefore$ $$a_2=a_0\times C_c\qquad ...(ii)$$

By continuity equation, we have

$$a_1v_1=a_2v_2\quad\text{or}\quad v_1=\frac{a_2}{a_1}v_2=\frac{a_0C_c}{a_1}v_2\qquad ...(iii)$$

Substituting the value of v_1 in equation (*i*), we get

$$v_2=\sqrt{2gh+\frac{a_0^2C_c^2v_2^2}{a_1^2}}$$

or $${v_2}^2=2gh+\left(\frac{a_0}{a_1}\right)^2{C_c}^2{v_2}^2\text{ or }{v_2}^2\left[1-\left(\frac{a_0}{a_1}\right)^2C_c^2\right]=2gh$$

$\therefore$ $$v_2=\frac{\sqrt{2gh}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2C_c^2}}$$

$\therefore$ The discharge $Q=v_2\times a_2=v_2\times a_0\,C_c$ $\{\because a_2=a_0C_c$ from (*ii*)$\}$

$$=\frac{a_0C_c\sqrt{2gh}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2C_c^2}}\qquad ...(iv)$$

Ref: R.K. bansal

The above expression is simplified by using

$$C_d = C_c \frac{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$$\therefore \qquad C_c = C_d \frac{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2}}$$

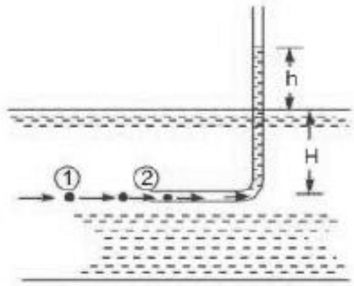
Substituting this value of C_c in equation (*iv*), we get

$$Q = a_0 \times C_d \frac{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2}} \times \frac{\sqrt{2gh}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$$= \frac{C_d a_0 \sqrt{2gh}}{\sqrt{1-\left(\frac{a_0}{a_1}\right)^2}} = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}.$$

Pitot-tube. It is a device used for measuring the velocity of flow at any point in a pipe or a channel. It is based on the principle that if the velocity of flow at a point becomes zero, the pressure there is increased due to the conversion of the kinetic energy into pressure energy. In its simplest form, the pitot-tube consists of a glass tube, bent at right angles as shown in Fig. 6.13.

The lower end, which is bent through 90° is directed in the up-stream direction as shown in Fig. The liquid rises up in the tube due to the conversion of kinetic energy into pressure energy. The velocity is determined by measuring the rise of liquid in the tube.



Pitot-tube.

Consider two points (1) and (2) at the same level in such a way that point (2) is just as the inlet of the pitot-tube and point (1) is far away from the tube.

Let
p_1 = intensity of pressure at point (1)
v_1 = velocity of flow at (1)
p_2 = pressure at point (2)
v_2 = velocity at point (2), which is zero
H = depth of tube in the liquid
h = rise of liquid in the tube above the free surface.

Applying Bernoulli's equation at points (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $z_1 = z_2$ as points (1) and (2) are on the same line and $v_2 = 0$.

$$\frac{p_1}{\rho g} = \text{pressure head at (1)} = H$$

$$\frac{p_2}{\rho g} = \text{pressure head at (2)} = (h + H)$$

Substituting these values, we get

$$\therefore \qquad H + \frac{v_1^2}{2g} = (h + H) \quad \therefore \quad h = \frac{v_1^2}{2g} \quad \text{or} \quad v_1 = \sqrt{2gh}$$

Ref: R.K. bansal

Problem *Find the velocity of the flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 100 mm. Take co-efficient of pitot-tube 0.98 and sp. gr. of oil = 0.8.*

Solution. Given :

Diff. of mercury level, $x = 100 \text{ mm} = 0.1 \text{ m}$

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_g = 13.6$

$C_v = 0.98$

Diff. of pressure head, $h = x\left[\frac{S_g}{S_o} - 1\right] = .1\left[\frac{13.6}{0.8} - 1\right] = 1.6 \text{ m of oil}$

$\therefore$ Velocity of flow $= C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1.6} =$ **5.49 m/s. Ans.**

Problem *A pitot-static tube is used to measure the velocity of water in a pipe. The stagnation pressure head is 6 m and static pressure head is 5 m. Calculate the velocity of flow assuming the co-efficient of tube equal to 0.98.*

Solution. Given :

Stagnation pressure head, $h_s = 6 \text{ m}$

Static pressure head, $h_t = 5 \text{ m}$

$\therefore$ $h = 6 - 5 = 1 \text{ m}$

Velocity of flow, $V = C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1} =$ **4.34 m/s. Ans.**

Problem *A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.*

Solution. Given :

Diff. of mercury level, $x = 170 \text{ mm} = 0.17 \text{ m}$

Sp. gr. of mercury, $S_g = 13.6$

Sp. gr. of sea-water, $S_o = 1.026$

$\therefore$ $h = x\left[\frac{S_g}{S_o} - 1\right] = 0.17\left[\frac{13.6}{1.026} - 1\right] = 2.0834 \text{ m}$

$\therefore$ $V = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2.0834} = 6.393 \text{ m/s}$

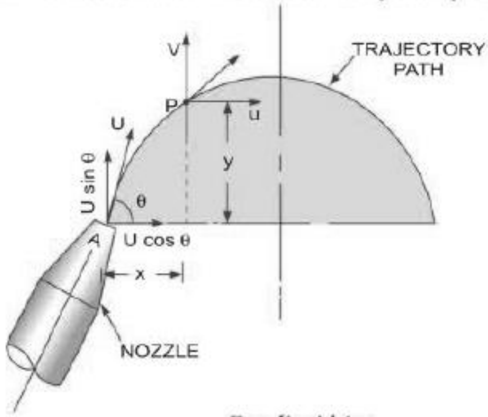
$= \frac{6.393 \times 60 \times 60}{1000}$ km/hr = **23.01 km/hr. Ans.**

FREE LIQUID JETS

Free liquid jet is defined as the jet of water coming out from the nozzle in atmosphere. The path travelled by the free jet is parabolic.

Consider a jet coming from the nozzle as shown in Fig Let the jet at A, makes an angle θ with the horizontal direction. If U is the velocity of jet of water, then the horizontal component and vertical component of this velocity at A are $U \cos\theta$ and $U \sin\theta$.

Consider another point $P(x, y)$ on the centre line of the jet. The co-ordinates of P from A are x and y. Let the velocity of jet at P in the x- and y-directions are u and v. Let a liquid particle takes time 't' to reach from A to P. Then the horizontal and vertical distances travelled by the liquid particle in time 't' are :



Free liquid jet.

$$x = \text{velocity component in } x\text{-direction} \times t$$
$$= U \cos\theta \times t \quad \text{...}(i)$$

and
$$y = (\text{vertical component in } y\text{-direction} \times \text{time} - \frac{1}{2} g t^2)$$
$$= U \sin\theta \times t - \frac{1}{2} g t^2 \quad \text{...}(ii)$$

{$\because$ Horizontal component of velocity is constant while the vertical distance is affected by gravity}

From equation (i), the value of t is given as $t = \dfrac{x}{U \cos\theta}$

Substituting this value in equation (ii)

$$y = U \sin\theta \times \frac{x}{U\cos\theta} - \frac{1}{2} \times g \times \left(\frac{x}{U\cos\theta}\right)^2 = x\frac{\sin\theta}{\cos\theta} - \frac{gx^2}{2U^2\cos^2\theta}$$
$$= x\tan\theta - \frac{gx^2}{2U^2}\sec^2\theta \qquad \left\{\because \frac{1}{\cos^2\theta} = \sec^2\theta\right\}$$

(i) **Maximum height attained by the jet.** Using the relation $V_2^2 - V_1^2 = -2gS$, we get in this case $V_1 = 0$ at the highest point

$$V_1 = \text{Initial vertical component}$$
$$= U \sin\theta$$

–ve sign on right hand side is taken as g is acting in the downward direction but particles is moving up.

$\therefore$ $$0 - (U \sin\theta)^2 = -2g \times S$$

where S is the maximum vertical height attained by the particle.

or $$-U^2 \sin^2\theta = -2gS$$

$\therefore$ $$S = \frac{U^2 \sin^2\theta}{2g}$$

(ii) **Time of flight.** It is the time taken by the fluid particle in reaching from A to B as shown in Fig. Let T is the time of flight.

Using equation (ii), we have $y = U \sin\theta \times t - \frac{1}{2} g t^2$

when the particle reaches at B, $y = 0$ and $t = T$

$\therefore$ Above equation becomes as $0 = U \sin\theta \times T - \frac{1}{2} g \times T^2$

or $$0 = U \sin\theta - \frac{1}{2} gT \quad \{\text{Cancelling } T\}$$

or $$T = \frac{2U \sin\theta}{g}$$

Ref: R.K. bansal

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or $$T = \frac{2U \sin\theta}{g}$$

(*iii*) **Time to reach highest point.** The time to reach highest point is half the time of flight. Let T^* is the time to reach highest point, then

$$T^* = \frac{T}{2} = \frac{2U \sin\theta}{g \times 2} = \frac{U \sin\theta}{g}$$

(*iv*) **Horizontal range of the jet.** The total horizontal distance travelled by the fluid particle is called horizontal range of the jet, *i.e.*, the horizontal distance AB in Fig. is called horizontal range of the jet. Let this range is denoted by x^*.

Then x^* = velocity component in x-direction × time taken by the particle to reach from A to B

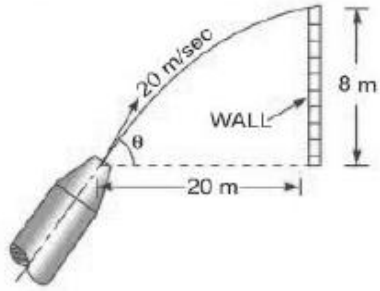
$$= U \cos\theta \times \text{Time of flight}$$

$$= U \cos\theta \times \frac{2U \sin\theta}{g} \qquad \left\{\because T = \frac{2U \sin\theta}{g}\right\}$$

$$= \frac{U^2}{g} 2 \cos\theta \sin\theta = \frac{U^2}{g} \sin 2\theta$$

Problem *A vertical wall is of 8 m in height. A jet of water is coming out from a nozzle with a velocity of 20 m/s. The nozzle is situated at a distance of 20 m from the vertical wall. Find the angle of projection of the nozzle to the horizontal so that the jet of water just clears the top of the wall.*

Solution. Given :



Height of wall = 8 m

Velocity of jet, $U = 20$ m/s

Distance of jet from wall, $x = 20$ m

Let the required angle $= \theta$

Using equation (6.24), we have

$$y = x \tan\theta - \frac{gx^2}{2U^2} \sec^2\theta$$

where $y = 8$ m, $x = 20$ m, $U = 20$ m/s

$$8 = 20 \tan\theta - \frac{9.81 \times 20^2}{2 \times 20^2} \sec^2\theta$$

$$= 20 \tan\theta - 4.905 \sec^2\theta$$

$$= 20 \tan\theta - 4.905\,[1 + \tan^2\theta] \qquad \{\because \sec^2\theta = 1 + \tan^2\theta\}$$

$$= 20 \tan\theta - 4.905 - 4.905 \tan^2\theta$$

or $$4.905 \tan^2\theta - 20 \tan\theta + 8 + 4.905 = 0$$

or $$4.905 \tan^2\theta - 20 \tan\theta + 12.905 = 0$$

$\therefore$ $$\tan\theta = \frac{20 \pm \sqrt{20^2 - 4 \times 12.905 \times 4.905}}{2 \times 4.905} = \frac{20 \pm \sqrt{400 - 253.19}}{9.81}$$

$$= \frac{20 \pm \sqrt{146.81}}{9.81} = \frac{20 \pm 12.116}{9.81} = \frac{32.116}{9.81} \text{ or } \frac{7.889}{9.81}$$

$\therefore$ $$= 3.273 \text{ or } 0.8036$$

$\therefore$ $$\theta = \mathbf{73^\circ\ 0.8' \text{ or } 38^\circ\ 37'}. \textbf{ Ans.}$$

Ref: R.K. bansal