UNIT-3-VECTOR DIFFERENTIAL CALCULUS

Vector Algebra :

1. Unit Vector : If $\vec{a}$ is any vector with $|\vec{a}| \neq 0$ and $\hat{a}$ is a unit vector in the direction of $\vec{a}$, then $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

2. Scalar or Dot or Inner product of two vector $\vec{a}$ and $\vec{b}$ is defined and denoted by

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta \quad \text{if} \quad \vec{a} \neq 0, \vec{b} \neq 0$$
$$= 0 \quad \text{if} \quad \vec{a} = 0 \text{ or } \vec{b} = 0$$

where θ, $(0 \leq \theta \leq \pi)$ is the angle between $\vec{a}$ and $\vec{b}$.

3. For mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$
$$\text{and} \quad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

4. If $\vec{a} = (a_1, a_2, a_3) = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

and $\vec{b} = (b_1, b_2, b_3) = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

5. Vector or Cross or Exterior product of two vectors $\vec{a}$ and $\vec{b}$ is defined and denoted by

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$$

where θ, $(0 \le \theta \le \pi)$ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

6. $\hat{i}, \hat{j}, \hat{k}$ form a right–handed triple of mutually perpendicular vectors

$$\hat{i} \times \hat{i} = 0, \quad \hat{j} \times \hat{j} = 0, \quad \hat{k} \times \hat{k} = \vec{0}$$
$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

7. If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\,(a_2b_3 - a_3b_2) + \hat{j}\,(a_3b_1 - a_1b_3) + \hat{k}\,(a_1b_2 - a_2b_1)$$

8. Scalar Triple Product : If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors the scalar triple product is defined and denoted by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = [\vec{a}\ \vec{b}\ \vec{c}]$$

9. If $\vec{a} = [a_1, a_2, a_3] = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = [b_1, b_2, b_3] = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = [c_1, c_2, c_3] = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

then $$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

10. Vector Triple Product : If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then the vector triple product is defined and denoted by

(a) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{b} \cdot \vec{c})\,\vec{a}$

(b) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Vector Calculus :

11. Let $\vec{r} = \vec{f}(t)$ be a vector function of t, then

$$\frac{d\vec{r}}{dt} = \frac{d\vec{f}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t}$$

(a) If $\vec{r} = \vec{f}(t)$ be the vector equation of a curve then $\frac{d\vec{r}}{dt}$ is a tangent vector to the curve at any point.

(b) If t denotes the time and $\vec{r}$ the position vector of a moving particle P relative to O and $\vec{V}$ represent the velocity vector of the particle at P then $\vec{V} = \frac{d\vec{r}}{dt}$ and its direction is along the trangent at P.

If $\vec{V}$ is the vector velocity, then $\frac{d\vec{V}}{dt}$ represents the acceleration

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

12. The **vector differential operator** is written as ∇ (read del or nabla) and defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

13. The **gradient** of a scalar function $\phi(x, y, z)$ is defined and denoted by

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

14. The gradient of a scalar field ϕ is a vector normal to the surface $\phi(x, y, z)$ = constant and has magnitude equal to the rate of change of ϕ along this normal.

15. $\nabla\phi$ gives the maximum rate of change of ϕ and the magnitude of this is $|\nabla\phi|$.

16. The **directional derivative** of $\phi(x, y, z)$ at any point $P(x, y, z)$ in any direction $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is the dot product of $\nabla\phi$ at P and the unit vector in the direction of a vector $\vec{a}$.

Thus, Directional derivative $= (\nabla\phi) \cdot \hat{a}$.

17. The **divergence** of a differentiable vector function

$\vec{V}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined and denoted by

$$\text{Div } \vec{V} = \nabla \cdot \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

18. The divergence of $\vec{V}$ gives the rate of outward flow per unit volume at a point of the fluid.

19. If there is no gain of the fluid anywhere, then div $\vec{V} = 0$. This is called the **equation of continuity** for an incompressible fluid.

20. If the flux entering at any element of the space is the same as the leaving it i.e. if div $\vec{V} = 0$ everywhere, then such a vector point function is called a **Solenoidal vector function** or **Solenoidal**.

21. The curl of a differentiable vector function

$\vec{V}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined and denoted by

$$\text{Curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

22. The curl of a vector field has something to do with rotational properties of the field. In short curl of a vector field gives the rotation.

23. If curl $\vec{V} = \vec{0}$, then the field $\vec{V}$ is called **irrotational**.

24. A field which is not irrotational is sometimes called a **Vortex field**.

25. A vector field $\vec{F}$ which can be derived from a scalar field ϕ so that $\vec{F} = \nabla\phi$ is called a **conservative vector field** and ϕ is called the **scalar potential**.

26. (a) If f and g are scalar point functions then

(i) $\nabla(f + g) = \nabla f + \nabla g$ (ii) $\nabla(fg) = f\nabla g + (\nabla f)\,g$

(b) If $\vec{u}$ and $\vec{v}$ are vector point functions and ϕ, a scalar point function, then

(i) $\nabla \cdot (\vec{u} + \vec{v}) = \nabla \cdot \vec{u} + \nabla \cdot \vec{v}$

(ii) $\nabla \cdot (\phi \vec{u}) = \phi(\nabla \cdot \vec{u}) + (\nabla\phi) \cdot \vec{u}$

(iii) $\nabla \cdot (\vec{u} \times \vec{v}) = (\nabla \times \vec{u}) \cdot \vec{v} - \vec{u} \cdot (\nabla \times \vec{v})$

(c) (i) $\nabla \times (\vec{u} + \vec{v}) = \nabla \times \vec{u} + \nabla \times \vec{v}$

(ii) $\nabla \times (\phi \vec{u}) = (\nabla\phi) \times \vec{u} + \phi(\nabla \times \vec{u})$

(iii) $\nabla \times (\vec{u} \times \vec{v}) = (\nabla \cdot \vec{v})\,\vec{u} - (\nabla \cdot \vec{u})\,\vec{v} + (\vec{v} \cdot \nabla)\,\vec{u} - (\vec{u} \cdot \nabla)\,\vec{v}$

(d) $\nabla(\vec{u} \cdot \vec{v}) = (\vec{v} \cdot \nabla)\,\vec{u} + (\vec{u} \cdot \nabla)\,\vec{v} + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \times (\nabla \times \vec{v})$

27. (i) Div (grad ϕ) $= \nabla \cdot \nabla\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

(ii) Curl (grad ϕ) $= \nabla \times \nabla\phi = \vec{0}$

(iii) div curl $\vec{V} = \nabla \cdot (\nabla \times \vec{V}) = 0$

(iv) Curl curl $\vec{V} = \nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$

28. $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called Laplace's operator or Laplacian and $\nabla^2\phi = 0$ is called Laplace's (wave) equation.

UNIT-4-**VECTOR INTEGRAL CALCULUS**

1. Line Integrals : $\int_C \vec{F} \cdot d\vec{r}$, $\int_C \vec{F} \times d\vec{r}$ where $\vec{F}(x, y, z)$ is a vector point function

   $\oint_C \vec{F} \cdot d\vec{r}$, where C is closed path, $\int_C \phi \, d\vec{r}$, where ϕ is a scalar point function.

2. Work done by a force : $W = \int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}$ is the force acting along a curve described by the path $\vec{r}(t)$.

3. Green's Theorem in the plane :

   If $M(x, y)$, $N(x, y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ be continuous everywhere in a region R of xy–plane bounded by a closed curve C, then

   $$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

4. Surface Integrals : $\iint_S \vec{F} \cdot \hat{n}\, ds$, where $\vec{F}$ = vector point function and $\hat{n}$ = unit outward drawn normal.

   $\iint_S \phi \, d\vec{s}$, $\iint_S \vec{F} \times d\vec{s}$, $\iint_S \vec{u} \cdot \vec{v} \, d\vec{s}$, $\iint (\vec{u} \times \vec{v})\, ds$

   where ϕ is a scalar point function.

5. Stokes's Theorem

   If S be an open two sided surface bounded by a closed, non intersecting curve (simple closed curve) and if a vector function $\vec{F}(x, y, z)$ has continuous first partial derivatives in a domain in a space containing S. Then

   $$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, ds = \iint_S (\nabla \times \vec{F}) \cdot d\vec{s}$$

   where C is described in positive (anticlockwise) direction, and $\hat{n}$ is unit positive (outward drawn) normal to S.