

Unit-1 & Unit-2

CALCULUS –MA0111 PARTIAL DIFFERENTIATION AND APPLICATIONS OF PARTIAL DIFFERENTIATION

❖ **Introduction to topic :**

Here, we are going to discuss continuity & derivatives of functions having more than one independent variables.

❖ **Definition of function having more than one independent variables:**

If three variables x, y, z are related in such a way that z depends upon the values of x & y , then z is called function of two variables x & y .

It is denoted $z = f(x, y)$.

Similarly, $w = g(x, y, z)$ is a function of three variables x, y, z .

$U = f(x_1, x_2, x_3, \dots, x_n)$ is a function of n variables $x_1, x_2, x_3, \dots, x_n$.

❖ **Definition of continuity of two variable functions:**

A function $f(x, y)$ is said to be continuous at (a, b) if

(i) f is defined at (a, b)

(ii) $\lim_{(x, y) \rightarrow (a, b)} f(x, y)$ exists and

(iii) $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$ in whatever manner x & y approach a & b respectively.

❖ **Definition of partial derivatives of first order:**

Let $z = f(x, y)$ be a function of two variables .

We can make change in z by three ways.

First: keep y constant and make change in value of x

Second : keep x constant and make change in value of y

Third : make change in value of x & y both simultaneously.

If we make change in value of x , keeping y constant then the relative change in value of z is called partial derivative of z with respect to x . and it is denoted by

$\frac{\partial f}{\partial x}$ or f_x or $\frac{\partial z}{\partial x}$. Similarly, partial derivative of z with respect to y is denoted by

$$\frac{\partial f}{\partial x} \text{ or } f_x \text{ or } \frac{\partial z}{\partial x}. \text{ Also, } \frac{\partial f}{\partial x} = f_x = \frac{\partial z}{\partial x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

$$\frac{\partial f}{\partial y} = f_y = \frac{\partial z}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

❖ **PARTIAL DIFFERENTIATION OF HIGHER ORDER:**

Let $z = f(x, y)$ then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are first order derivatives of z with respect to x & y .

Second order partial derivatives of z with respect to x & y are denoted by

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = f_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = f_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = f_{yx}, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = f_{xy}.$$

Third order partial derivatives of z with respect to x & y are denoted by

$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x^2} \right) = f_{xxx}, \quad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial y^2} \right) = f_{yyy},$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x \partial y} \right) = f_{yxx}, \quad \frac{\partial^3 z}{\partial y^2 \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial y \partial x} \right) = f_{xyy}$$

$$\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial y^2} \right) = f_{yyx}, \quad \frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x^2} \right) = f_{xxy}.$$

➤ **SOME OTHER NOTATIONS TO DENOTE P.D. :**

let $x = f(r, \theta)$ & $y = g(r, \theta)$ (i.e. x & y both are the functions of r & θ)
then indirectly x is a function of y and vice versa

In this case,

$\left(\frac{\partial x}{\partial r} \right)_\theta$ denotes derivative of x w.r.t. r keeping θ constant

$\left(\frac{\partial y}{\partial r} \right)_\theta$ denotes derivative of y w.r.t. r keeping θ constant

$\left(\frac{\partial x}{\partial \theta} \right)_r$ denotes derivative of x w.r.t. θ keeping r constant

$\left(\frac{\partial y}{\partial \theta} \right)_r$ denotes derivative of y w.r.t. θ keeping r constant

if we eliminate θ from above function we will get x as a function of y & r
or y as a function of x & r .

$\left(\frac{\partial x}{\partial y} \right)_r$ denotes derivative of x w.r.t. y keeping r constant

$\left(\frac{\partial x}{\partial r} \right)_y$ denotes derivative of x w.r.t. r keeping y constant

And similarly other derivatives can be defined.

❖ **DERIVATIVES OF FUNCTION OF FUNCTION :**

If $z = f(t)$ and $t = g(x,y)$ then $z = f(g(x,y))$ is a composite function of f & g
 If we consider z as a function of t alone then we have ordinary derivative of z
 w.r.t. t which is $\frac{dz}{dt}$, but if we consider z as a composite function of f & g then z
 will be function of two variables .in this case we can find partial derivatives of z
 w.r.t. x & y which are $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.And relation between above derivatives is

$$\frac{\partial z}{\partial x} = \frac{dz}{dt} \cdot \frac{\partial t}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = \frac{dz}{dt} \cdot \frac{\partial t}{\partial y} .$$

➤ **CHAIN RULE FOR PARTIAL DIFFERENTIATION :**

(A) If $u = f(x, y)$ and if $x = g(t)$ & $y = h(t)$ then u is indirectly function of t alone

Hence,
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \dots\dots\dots(1)$$

$\frac{du}{dt}$ is called total differential coefficient

(i) If we eliminate dt from both sides we get Total Derivative of u

That can be given by
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \dots\dots\dots(2)$$

(ii) If we replace t by x in above (1) then we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} \dots\dots\dots(3)$$

In this case u is a function of x & y and y is function of x . so ultimately u becomes function of x alone.

If $u = f(x_1, x_2, x_3, x_4, \dots, x_n)$ and all $x_1, x_2, x_3, x_4, \dots, x_n$ are functions of a single variable t .

Then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dt} + \frac{\partial u}{\partial x_3} \frac{dx_3}{dt} + \dots\dots\dots + \frac{\partial u}{\partial x_n} \frac{dx_n}{dt} \dots\dots(4)$$

And
$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \dots\dots\dots + \frac{\partial u}{\partial x_n} dx_n \dots\dots(5)$$

(B) If $z = f(x, y)$ and if $x = g(u, v)$ & $y = h(u, v)$ then z is indirectly function of two variables u & v .

Hence,
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \dots\dots\dots(6)$$

❖ **Solved Problems Based On Above Discussion :**

1). If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then verify: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

2). If $u = x^2 y + y^2 z + z^2 x$, provethat $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$

3) If $v = (1 - 2xy + y^2)^{\frac{-1}{2}}$,

provethat, (i) $x \frac{\partial v}{\partial x} - y \frac{\partial v}{\partial y} = y^2 v^3$

(ii) $\frac{\partial}{\partial x} \left[(1 - x^2) \frac{\partial v}{\partial x} \right] + \frac{\partial u}{\partial y} \left[y^2 \frac{\partial v}{\partial y} \right] = 0$

4). If $z = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$,

prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

5). If $z(x + y) = x^2 + y^2$,

prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$.

6). If $f(x, y) = \frac{1}{\sqrt{y}} e^{\frac{-(x-a)^2}{4y}}$,

prove that (i) $\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial y} = 0$ (ii) $\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0$.

7). If $V = r^m$, where $r^2 = x^2 + y^2 + z^2$,

provethat $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)r^{m-2}$

8). provethat at a point $x = y = z$ of the surface $x^x y^y z^z = c$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log x)^{-1}$

9). If $u = e^{xyz}$ show that $\frac{\partial^3 u}{\partial y \partial x \partial z} = -(1 + 3xyz + x^2 y^2 z^2) \cdot u$

10). If $\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1$

where u is a function of x, y, z .

prove that, $\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 = 2 \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right]$

11). If $u = \frac{x + y + z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

- 12) If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$, then provethat $\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{y^2 + x^2}{y^2 - x^2}$.
- 13) If $u = ax + by$ and $v = bx - ay$, then provethat $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial v}{\partial y}\right)_u \left(\frac{\partial y}{\partial v}\right)_x = 1$.
- 14) If $x^2 = au + bv$ and $y^2 = au - bv$, then provethat $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$.
- 15) If $x = u \tan v$ and $y = u \sec v$, then provethat $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$.
- 16) If $x = r \cos \theta$ and $y = r \sin \theta$, then provethat (i) $\frac{1}{r} \left(\frac{\partial x}{\partial \theta}\right)_r = r \left(\frac{\partial \theta}{\partial x}\right)_y$.
- 17) If $xyz = a^3$, then provethat $\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial x}{\partial z}\right)_y = -1$.
- 18) Find $\frac{du}{dt}$ when $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$
- 19) If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, provethat $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.
- 20) If z be a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$
provethat $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$.
- 21) If $z = e^{ax+by} f(ax-by)$, prove that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2baz$.
- 22) If $u = f\left(\frac{x}{z}, \frac{y}{z}\right)$, provethat $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.
- 23) If $z = f(x, y)$, $x = u \cosh v$ and $y = u \sinh v$, provethat
 $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2} \left(\frac{\partial z}{\partial v}\right)^2$.
- 24) find the total differential of $f(x, y, z) = xe^{y^2-z^2}$
- 25) If z is a function of x and y and u & v be two variables such that
 $u = ix + my$, $v = ly - mx$, then provethat
 $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2) \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$
- 26) If z is a function of x and y and u & v be two variables such that
 $x = e^u \sin v$, $y = e^u \cos v$, then provethat
 $(x^2 + y^2) \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$
- 27) If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and ϕ is a function of x, y, z then
show that $x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} + z \frac{\partial \phi}{\partial z} = u \frac{\partial \phi}{\partial u} + v \frac{\partial \phi}{\partial v} + w \frac{\partial \phi}{\partial w}$.

❖ **IMPLICIT FUNCTION :**

If we can not express x as a function of y or y as a function x even though we have relationship $f(x, y) = 0$, then the function $f(x, y) = 0$ is called an implicit function.

for e.g. $x^3 + y^3 + 3axy = 0$.

➤ **(A) DIFFERENTIATION OF IMPLICIT FUNCTION :**

To find derivative of implicit function we can use following method.

If $f(x, y) = 0$ is a given implicit function

then assume,

(i) $u = f(x, y)$ and $y = g(x)$

Hence, by chain rule $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$ but $f(x, y) = 0 = u$

$$\therefore 0 = f_x + f_y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y}$$

(ii) $u = f(x, y)$ and $x = h(y)$

Hence, by chain rule $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial x} \frac{dx}{dy} + \frac{\partial f}{\partial y}$ but $f(x, y) = 0 = u$

$$\therefore 0 = f_x \frac{dx}{dy} + f_y$$

$$\therefore \frac{dx}{dy} = -\frac{f_y}{f_x}$$

➤ **(B) PARTIAL DIFFERENTIATION OF IMPLICIT FUNCTION :**

$f(x, y, z) = 0$ represents an implicit function .

To find partial derivative of implicit function we can use following method.

If $f(x, y, z) = 0$ is a given implicit function

then assume,

(i) $u = f(x, y, z)$ and $z = g(x, y)$

Hence, by chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x}$$

And

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y}$$

But $f(x, y, z) = 0 = u$

$$\therefore 0 = f_x + f_z \frac{\partial z}{\partial x}$$

And

$$\therefore 0 = f_y + f_z \frac{\partial z}{\partial y}$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

(ii) $u = f(x, y, z)$ and $y = h(x, z)$

Hence, by chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x}$$

And

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z}$$

But $f(x, y, z) = 0 = u$

$$\therefore 0 = f_x + f_y \frac{\partial y}{\partial x}$$

And

$$\therefore 0 = f_z + f_y \frac{\partial y}{\partial z}$$

$$\therefore \frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$$

$$\therefore \frac{\partial y}{\partial z} = -\frac{f_z}{f_y}$$

(iii) $u = f(x, y, z)$ and $x = t(y, z)$

Hence, by chain rule

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial y}$$

And

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial z}$$

But $f(x, y, z) = 0 = u$

$$\therefore 0 = f_y + f_x \frac{\partial x}{\partial y}$$

$$\therefore 0 = f_z + f_x \frac{\partial x}{\partial z}$$

$$\therefore \frac{\partial x}{\partial y} = -\frac{f_y}{f_x}$$

$$\therefore \frac{\partial x}{\partial z} = -\frac{f_z}{f_x}$$

❖ **Solved Problems Based On Above Discussion**

1) Find $\frac{dy}{dx}$ by implicit differentiation.

(i) $x^3 + y^3 + 3xy = 0$.

(ii) $(\cos x)^y = (\sin y)^x$

2) If $f(x, y, z) = 0$, prove that $\left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial x}{\partial y}\right)_z = -1$

3) If $f(x, y) = 0$ and $\phi(x, y) = 0$, show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$.

4) If $u(x, y) = c_1$ and $v(x, y) = c_2$ show that the curves $u = \text{constants}$

and $v = \text{constants}$ intersect orthogonally if $\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0$.

❖ **HOMOGENEOUS FUNCTIONS:**

A function f of two variables x & y is said to be homogeneous function of degree n if,

$$f(tx,ty)=t^n f(x, y) \text{ or } f(x, y) = x^n g\left(\frac{y}{x}\right) \text{ or } f(x, y) = y^n h\left(\frac{x}{y}\right).$$

For e.g. $u = \frac{x^2 y^2}{x + y}$ is a homogeneous function of degree 3.

$$\text{Since } u = \frac{x^4 \left(\frac{y}{x}\right)^2}{x \left(1 + \left(\frac{y}{x}\right)\right)} = x^3 g\left(\frac{y}{x}\right) \text{ or } u(tx,ty) = t^3 \frac{x^2 y^2}{x + y}.$$

➤ **EULER'S THEOREM :**

Statement :

If u is a homogeneous function of x & y of degree n then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu.$$

Proof:

Since $u = f(x, y)$ is a homogeneous function of degree n , it can be written as

$$u(x, y) = x^n g\left(\frac{y}{x}\right) \dots\dots\dots(i)$$

differentiating (i) partially w.r.t x

$$\begin{aligned} \frac{\partial u}{\partial x} &= nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right) \left(\frac{-y}{x^2}\right) \\ &= nx^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} y g'\left(\frac{y}{x}\right) \end{aligned}$$

hence,

$$x \frac{\partial u}{\partial x} = nx^n g\left(\frac{y}{x}\right) - x^{n-1} y g'\left(\frac{y}{x}\right) \dots\dots\dots(ii)$$

differentiating (i) partially w.r.t y

$$\begin{aligned} \frac{\partial u}{\partial y} &= x^n g'\left(\frac{y}{x}\right) \left(\frac{1}{x}\right) \\ &= x^{n-1} g'\left(\frac{y}{x}\right) \end{aligned}$$

hence,

$$y \frac{\partial u}{\partial y} = x^{n-1} y g'\left(\frac{y}{x}\right) \dots\dots\dots(iii)$$

Now, adding (ii) & (iii) we get,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nx^n g\left(\frac{y}{x}\right) \\ = nu.$$

➤ **Cor-1:** If u is a homogeneous function of x, y of degree n , then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

➤ **Cor-2:** If z is a homogeneous function of x, y of degree n and $z = f(u)$,

$$\text{Then, } x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}.$$

➤ **Cor-3:** If z is a homogeneous function of x, y of degree n and then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1].$$

$$\text{where } g(u) = n \frac{f(u)}{f'(u)}.$$

❖ **Solved Problems Based On Above Discussion**

1) verify Euler's theorem for (i) $u = e^{\frac{x}{y}}$

(ii) $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$.

2) If $u = f\left(\frac{x}{y}\right)$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.

3) If $u = \sec^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.

4) If $z = xy f\left(\frac{y}{x}\right)$, prove that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$.

5) If $f(x, y) = \sqrt{x^2 - y^2} \sin^{-1}\left(\frac{y}{x}\right)$,

prove that $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f(x, y)$.

6) If $u = \frac{x^2 y^2}{x + y}$

prove that $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = 2 \frac{\partial u}{\partial x}$.

7) show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$.

when (i) $u = \frac{x - y}{x + y}$ (ii) $u = \frac{xy}{x - y}$ (iii) $u = (x - y)f\left(\frac{y}{x}\right)$

8) If $u = \cos \sqrt{\frac{x - y}{x + y}}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.

9) If $u = \sin^{-1} \sqrt{\frac{\frac{1}{x^4} + \frac{1}{y^4}}{\frac{1}{x^5} + \frac{1}{y^5}}}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{20} \tan u$.

10) If $u = \tan^{-1} \sqrt{\frac{x^3 + y^3}{x - y}}$, show that (i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$.

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \sin u \cos 3u$.

11) If $u = \operatorname{cosec}^{-1} \sqrt{\frac{\frac{1}{x^2} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}}}$, show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left(\frac{13}{12} + \frac{1}{12} \tan^2 u \right)$$

12) If $u = f\left(\frac{y}{x}\right) + x \phi\left(\frac{x}{y}\right) + 4xy$, show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 8xy$.

❖ **Jacobians:**

- (1) If $u = f(x, y)$ and $v = g(x, y)$ be differentiable functions then the Jacobian of u & v with respect to x & y is defined and denoted by

$$\frac{\partial(u, v)}{\partial(x, y)} \text{ is given by } J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

- (2) If $u = f(x, y, z)$, $v = g(x, y, z)$ & $w = h(x, y, z)$ be differentiable functions then the Jacobian of u, v & w with respect to x, y & z is defined and denoted by

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} \text{ is given by } J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

❖ **Chain rule for Jacobians:**

- (1) If $u = f(x, y)$ & $v = g(x, y)$ and $x = h(r, s)$ & $y = t(r, s)$ be differentiable functions then

$$\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)}.$$

- (2) $\frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(r, s, t)} = \frac{\partial(u, v, w)}{\partial(r, s, t)}.$

❖ **Note:**

- (1) If $J = \frac{\partial(u, v)}{\partial(x, y)}$ and $J' = \frac{\partial(x, y)}{\partial(u, v)}$ then $J.J' = 1$.
- (2) If $J = \frac{\partial(u, v, w)}{\partial(x, y, z)}$ and $J' = \frac{\partial(x, y, z)}{\partial(u, v, w)}$ then $J.J' = 1$.

❖ **Solved Problems Based On Above Discussion :**

- (1) If $x = r \cos \theta$, $y = r \sin \theta$, then show that $\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}$
- (2) If $x = e^v \sec u$, $y = e^v \tan u$, then show that $J.J' = 1$.
- (3) If $x + y = 2e^\theta \cos \phi$, $x - y = 2ie^\theta \sin \phi$, then prove that $J.J' = 1$.
- (4) If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, show that $\frac{\partial(u, v)}{\partial(x, y)} = 0$, if $x.y \neq 1$ also show that $u = \tan v$.
- (5) If $u_1 = f_1(x_1)$, $u_2 = f_2(x_1, x_2)$ & $u_3 = f_3(x_1, x_2, x_3)$.
Show that $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3}$

❖ **POWER SERIES EXPANSIONS :**

➤ **Maclaurin's Expansions for functions of two variables:**

Maclaurin series Expansions for the function $f(x, y)$ can be given by

$$f(x, y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!} [x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \dots \text{(i)}$$

➤ **Taylor's Expansions for functions of two variables:**

Taylor series Expansions for the function $f(x, y)$ can be given by

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)] + \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \dots \text{(ii)}$$

➤ **Cor-1:**

Putting $x = a + h$ & $y = b + k$ in above equation (ii) we get,

$$f(a + h, b + k) = f(a, b) + [h f_x(a, b) + k f_y(a, b)] + \frac{1}{2!} [h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)] + \dots \text{(iii)}$$

➤ **Cor-2:**

Putting $a = x$ & $b = y$ in above equation (iii) we get,

$$f(x + h, y + k) = f(x, y) + [h f_x(x, y) + k f_y(x, y)] + \frac{1}{2!} [h^2 f_{xx}(x, y) + 2hk f_{xy}(x, y) + k^2 f_{yy}(x, y)] + \dots \text{(iv)}$$

Solved Problems Based On Above Discussion :

- (1) Expand e^{xy} in powers of $(x-1)$ & $(y-1)$.
- (2) Expand $f(x, y) = \sin xy$ in powers of $(x-1)$ & $\left(y - \frac{\pi}{2}\right)$ up to second degree terms.
- (3) Expand $e^x \log(1 + y)$ in powers of x & y up to third degree terms.
- (4) Expand $e^{ax} \sin by$ in powers of x & y .
- (5) Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x-1)$ & $(y-1)$.

❖ **MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES :**

Working rules to find maximum and minimum values of a function $f(x, y)$:

- (1) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
- (2) Solve simultaneous equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$
 $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ is necessary condition for function $f(x, y)$ to be maximum or minimum.
- (3) let $(a_1, b_1), (a_2, b_2), \dots$ be solutions of simultaneous equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. These points are called optimal points.
- (4) Evaluate $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at above points.
- (5) Use following table to decide that function is either maxima or minima at optimal points.

Sr no.	$r t - s^2$	r	conclusion
1	> 0	< 0	Point of maxima
2	> 0	> 0	Point of minima
3	> 0	$= 0$	Further investigation required
4	$= 0$	any	Further investigation required
5	< 0	any	Point of neither Maxima nor Minima Saddle point

- (6) points :

Sr no.	Points	Definition
1	Optimal point	A point where function satisfies $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$
2	Extreme point	A point where function takes its Maximum or Minimum value.
3	Point of maxima	A point where function takes its Maximum value.
4	Point of minima	A point where function takes its Minimum value.
5	Saddle point	A point where function is neither Maxima nor Minima

- (7) values :

Sr no.	Points	Definition
1	Optimal value	Value of function at optimal point
2	Maximum value	Value of function at Point of maxima
3	Minimum value	Value of function at Point of minima

❖ **Problems Based On Above Discussion :**

Find Maximum and Minimum values for the following.

- (1) $x^2 + y^2 + 6x + 12$
- (2) $x^2 + 2xy + 2y^2 + 2x + y$
- (3) $x^3 + y^3 - 3axy$
- (4) $2(x^2 - y^2) - x^4 + y^4$
- (5) $\sin x + \sin y + \sin(x + y) \quad x \geq 0, y \leq \frac{\pi}{2}$.
- (6) $x^2y + xy^2 - axy$
- (7) $x^3 + y^3 - 63(x + y) + 12xy$
- (8) $\sin x \sin y \sin(x + y) \quad x > 0, y < \pi$.

❖ **LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS :**

This method is useful to find extreme values of a function

$$u = f(x, y, z)$$

subject to the condition $\phi(x, y, z) = 0$

Working rules to find maximum and minimum values of a function

$$u = f(x, y, z)$$

Subject to the condition $\phi(x, y, z) = 0$

- (1) Form the Lagrangian equation
 $u = f(x, y, z) + \lambda \phi(x, y, z)$.
- (2) Solve equations $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$ & $\frac{\partial u}{\partial z} = 0$ along with $\phi(x, y, z) = 0$
and find the values of x, y, z & λ .
The values of x, y, z so obtained will give the extreme values of $f(x, y, z)$.

❖ **Problems Based On Above Discussion :**

- (1) Find the maximum value of xyz subject to the restriction $x + y + z = a$.
- (2) Divide 30 into three parts such that the continued product of the first, the square of the second and the cube third be a maximum.
- (3) Divide 120 into three parts so that the sum of the products taken two at a time shall be a maximum.
- (4) Find the point in the space such that the sum of whose e coordinates is 48 and whose distance from the origin is minimum.
- (5) If $u = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}$, where $x + y + z = 1$ prove that the stationary

$$\text{values of } u \text{ is given by } x = \frac{a}{a+b+c}, y = \frac{b}{a+b+c}, z = \frac{c}{a+b+c}.$$

❖ **TANGENT PLANE TO THE SURFACE:**

Let the equation of the surface be $f(x, y, z) = 0$.

Then the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point

$$P(x_1, y_1, z_1) \text{ is } (x - x_1) \left(\frac{\partial f}{\partial x} \right)_P + (y - y_1) \left(\frac{\partial f}{\partial y} \right)_P + (z - z_1) \left(\frac{\partial f}{\partial z} \right)_P = 0.$$

❖ **NORMAL LINE TO THE SURFACE:**

Let the equation of the surface be $f(x, y, z) = 0$.

Then the equation of the normal line to the surface $f(x, y, z) = 0$ at the point

$$P(x_1, y_1, z_1) \text{ is } \frac{(x - x_1)}{\left(\frac{\partial f}{\partial x} \right)_P} = \frac{(y - y_1)}{\left(\frac{\partial f}{\partial y} \right)_P} = \frac{(z - z_1)}{\left(\frac{\partial f}{\partial z} \right)_P} .$$

❖ **Problems Based On Above Discussion :**

- (1) Find the equations of the tangent plane and the normal line to the surface
 - (I) $z = 3x^2 + 2y^2$ at $(1, 2, 11)$.
 - (II) $x^2 + 2y^2 + 3z^2 = 12$ at $(1, 2, -1)$.
- (2) show that the surfaces $z = xy - 2$ and $x^2 + y^2 + z^2 = 3$ have the same tangent plane at $(1, 1, -1)$.
- (3) Show that the tangent plane to the surface $x^2 = y(x + z)$ at any point passes through origin.
- (4) Show that the plane $lx + my + nz = p$ touches the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if $l^2 a^2 + m^2 b^2 + n^2 c^2 = p^2$.

❖ **ERRORS AND APPROXIMATIONS :**

➤ **ERRORS:**

- (1) δx is an error in x .
- (2) $|\delta x|$ is called an absolute error in x .
- (3) $\frac{\delta x}{x}$ is called the relative error in x .
- (4) $\frac{\delta x}{x} \times 100$ is called the percentage error in x .

➤ Let $f(\mathbf{x}, \mathbf{y})$ be a continuous function of x and y .

Let δx & δy be the change in x & y respectively then related change in

$f(\mathbf{x}, \mathbf{y})$ is given by $\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$.approximately.

❖ **Problems Based On Above Discussion :**

- (1) Find the percentage error in the area of the rectangle when an error of One percent is measuring its length and breadth.
- (2) If the kinetic energy $T = \frac{1}{2}mv^2$, find approximately the change in T as m changes from 49 to 49.5 and v changes from 1600 to 1590.
- (3) If measurements of radius of base and height of a right circular cone are in error by -1 % and 2%, prove that there will be no error in volume.
- (4) The acceleration of a piston is calculated by the formulae $f = w^2 r \left(\cos\theta + \frac{r}{l} \cos 2\theta \right)$, w, θ being variables. The acceleration is calculated for $\theta = 30^\circ$ and $\frac{r}{l} = \frac{1}{4}$. If the values of w, θ are each 1% small, show that the calculated value of f is about 1.5% too small.
- (5) find the approximate value of
 - (I) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at point (3.01, 4.02, 11.98) .
 - (II) $\sqrt[4]{[(1.9)^3 + (2.1)^3]}$
 - (III) $\sqrt[5]{[(3.82)^2 + 2(2.1)^3]}$