INDUS INSTITUTE OF ENGINEERING & TECHNOLOGY

Semester: IV **Subject: COMPLEX ANALYSIS(MA0411)**

UNIT-IV
COMPLEX SEQUENCE & SERIES AND RESIDUES

7 hours

1. Power series
2. Taylor series
3. Laurent series
4. Singularities, poles and zeros
5. Theory of residues
6. Definite real integrals

1 Complex Sequence and Series

- A sequence of complex numbers $\{z_1, z_2, \dots, z_n, \dots\}$ or $\{z_n\}$ is obtained by an assignment to each positive integer n, a complex number z_n.
- A sequence $\{z_n\}$ is said to be convergent if $\lim\limits_{n\to\infty} z_n$ is finite and unique.
- A sequence $\{z_n\}$ is said to be divergnet, if $\lim\limits_{n\to\infty} z_n$ is infinite.
- An infinite complex series is defined as the sum of terms of a given sequence $\{z_n\}$ of complex numbers. It is written as $\sum\limits_{n=1}^{n=\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$
- The n^{th} partial sum of the series given above is denoted by S_n and is defined by $S_k = \sum\limits_{k=1}^{k=n} z_k$
- A complex series is said to be convergent to a sum S, if $\lim\limits_{n\to\infty} S_n = S$ otherwise it is divergent.
- **Power Series** is of the form

$$\sum_{n=0}^{n=\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots$$

  where $a_0, a_1, a_2, \dots$ are real or complex co-effecients and z_0 is a fixed (complex) point called the center. It is called a power series in powers of $z - z_0$ or about z_0 or a power series centered at z_0.
- Whe $z_0 = 0$, the power series reduces to $\sum\limits_{n=0}^{n=\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$ which is a power series centered at origin.

- Region of convergence is a set of all points z for which the series converges.
- Regarding the convergence of the power series, there are three possibilites:
  (a) The series converges on at the point z_0.
  (b) The series converges for all z, i.e. in entire complex plane or z - plane.
  (c) The series converges everywhere inside a circular disk $|z - z_0| < R$ and diverges every where in $|z - z_0| > R$.
  Here R is called the radius of convergence and the circle $|z - z_0| = R$ is called the circle of convergence.
- The case when $|z - z_0| = R$ needs to be investigated seperately and the series may converge or diverge on it.
- If the disk is centered at infinity, the power series takes the form

$$\sum_{n=0}^{n=\infty} a_n z^{-n} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n} + \cdots$$

- **Taylor Series** If $f(z)$ is a complex function, analytic inside and on a simple closed curve C in the z - plane, then higher derivatives of $f(z)$ exists inside C. Thus, for fixed points z_0 and $z_0 + h$,

$$f(z_0 + h) = f(z_0) + hf'(z_0) + \frac{h^2}{2!}f''(z_0) + \frac{h^3}{3!}f'''(z_0) + \cdots$$

  Substituting $z_0 + h = z$,

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \frac{(z - z_0)^3}{3!}f'''(z_0) + \cdots$$

  This series is known as Taylor's series expansion of $f(z)$ about $z = z_0$.
  The region of convergence of this series is $|z - z_0| < R$, a disk centered at z_0 with radius R.
- Taylor's series reduces to Maclaurin's series when $z_0 = 0$. It is given by:

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \cdots$$

- **Laurent Series** If $f(z)$ is a complex function analytic on two concentric circles C_1 and C_2 with center z_0 and radii R_1 and R_2 and in the annulus region $R_1 < |z - z_0| < R_2$, then for each point within the annulus, $f(z)$ can be represented uniquely by the series of the form:

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n(z - z_0)^n = \sum_{n=0}^{n=\infty} a_n(z - z_0)^n + \sum_{n=1}^{n=\infty} \frac{b_n}{(z - z_0)^n}$$

  where the co-effecients are given by: $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$ and

  $b_n = a_{-n} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{-n+1}}\,dz$ taken counterclockwise around any simple closed path $C : |z - z_0| = \rho$ which lies in between the annulus $R_1 < |z - z_0| < R_2$.
- In Laurent series $\sum_{n=1}^{n=\infty} \frac{b_n}{(z - z_0)^n}$ is called the Principal part and $\sum_{n=0}^{n=\infty} a_n(z - z_0)^n$ is called the regular part of the series.

- If $f(z)$ is analytic at all points inside C_1, then by Cauchy's theorem $b_n = 0$ for each $n \geq 1$
- The region of convergence of Laurent's series is the annulus $R_1 < |z - z_0| < R_2$.
- If z_0 is the only singular point inside C_1, then the series is convergent in the deleted neighbourhood $0 < |z - z_0| < R$

2 Singularities, Zeros and Poles

- A point z_0 is said to be a **singular point** or singularity of a function $f(z)$, if $f(z)$ is not analytic (or not defined) at z_0 but analytic at some points of each neighbourhood of z_0.
- A singular point z_0 of $f(z)$ is said to be **isolated**, if there is a neighbourhood of z_0 which contains no other singular points of $f(z)$ excpet z_0.
  i.e. z_0 is said to be an **isolated singular point** if it is analytic in some deleted neighbourhood of z_0, $0 < |z - z_0| < \rho$.
- Let z_0 be an isolated singular point of $f(z)$, then the Laurent series exists of the form $f(z) = \sum_{n=0}^{n=\infty} a_n (z - z_0)^n + \sum_{n=1}^{n=\infty} \frac{b_n}{(z - z_0)^n}$ which is valid in some annulus $0 < |z - z_0| < R$
- If $f(z)$ has only Taylor series expansion about $z = z_0$, i.e. the Laurent series expansion has sero principal part, then z_0 is called the regular point of $f(z)$.
- If the principal part of Laurent series contains only finite number of terms, say m, then the singularity $z = z_0$ is said to be a pole of order m.
- In the principal part of the Laurent series, if $b_1 \neq 0$ and $b_2, b_3, \ldots$ are all zeros, then the singularity $z = z_0$ is said to be a simple pole or pole of order 1.
- Let $f(z)$ be analytic function in a domain D and $z = z_0$ in D. If $f(z_0) = 0$, then z_0 is said to be a zero of $f(z)$.
- If $z = z_0$ is a zero of $f(z)$ then $a_0 = 0$.
- If $z = z_0$ is a zero of $f(z)$ with $a_0 = 0$ and $a_1 \neq 0$, then z_0 is called a simple zero.
- If $z = z_0$ is a zero of $f(z)$ with $a_0 = a_1 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$, then z_0 is called a zero of order m.
- The zeros of an analytic function $f(z)$ are isolated.
- If $f(z)$ is an analytic function at $z = z_0$ and have n^{th} order zero at z_0, then $\frac{1}{f(z)}$ has a pole of n^{th} order at z_0.

3 Theory of Residues

- If a complex function $f(z)$ has a pole at the point $z = z_0$, then the co-effecient b_1 of $\frac{1}{z - z_0}$ in Laurent series expansion of $f(z)$ about z_0 is called the **residue** of $f(z)$ at the point $z = z_0$.
- Thus, we have: $\underset{z = z_0}{\text{Res}} \, f(z) = b_1 = \oint_C f(z)\, dz$
- Calculation of Residues:

(a) Let $z = z_0$ be a simple pole of $f(z)$, then

$$\underset{z = z_0}{\text{Res}} \; f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

(b) Let $z = z_0$ be a pole of order $m > 1$ for $f(z)$, then

$$\underset{z = z_0}{\text{Res}} \; f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]$$

- **Cauchy's Residue Theorem:** If $f(z)$ is an analytic funcion inside and on a simple closed path C except at a finite number of singularities inside C. Then

$$\oint_C f(z)\, dz = 2\pi i \times \text{Sum of residues of } f(z) \text{ at all finite number of singularities inside } C$$

where the integral is taken counter clockwise around C.

4 Definite Real Integrals

In this section definite real integral of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$ is evaluated using complex integration.

Steps to solve the integral:

(i) Substitute $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + z^{-1}}{2} = \dfrac{z^2+1}{2z}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i} = \dfrac{z^2-1}{2zi}$

(ii) $dz = iz d\theta$

(iii) Evaluate the integral so obtained over $|z| = 1$ using Residue theorem.