INDUS INSTITUTE OF ENGINEERING & TECHNOLOGY

Semester: IV **Subject: COMPLEX ANALYSIS(MA0411)**

UNIT-III
COMPLEX INTEGRATION

8 hours

1. Complex line integral: Properties and evaluation
2. Simply and multiply connected domains
3. Cauchys integral theorem, Cauchys integral theorem for multiply connected domains
4. Cauchys integral formula (without proof)

1 Complex line integral: Properties and evaluation

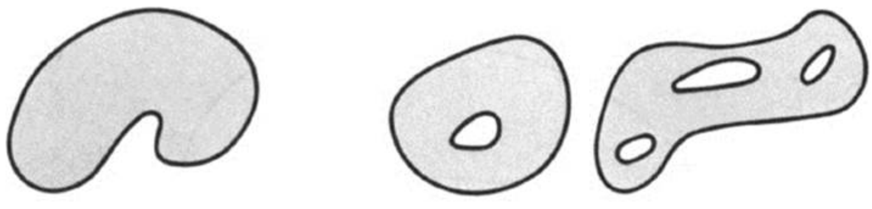
- Concept of integration as an inverse process of differentiation is also applicable in case of complex functions provided, the function is analytic.
  i.e. If $F'(z) = f(z)$, then $\int f(z)\,dz = F(z) + c$, where c is a complex constant.
  This is called indefinite integral
- The definite integral of a complex variable $\int\limits_{z_1}^{z_2} f(z)\,dz$ depends upon the path from z_1 to z_2 in the complex plane.
- Complex definite integral can also be written as $\int\limits_C f(z)\,dz$, where C is the path of integration.
- Complex definite integral is also called Complex Line Integral or simply line integral.
- When C is a closed path, i.e. when z_1 and z_2 co-incides, the integral denoted by $\oint\limits_C f(z)\,dz$.
- Properties of Line Integral:
  1. Let $f(z) = u + iv$ and $dz = dx + idy$, then $\int\limits_C f(z)\,dz = \int\limits_C (u + iv)(dx + idy)$
     $= \int\limits_C (udx - vdy) + i\int\limits_C (vdx + udy)$
  2. Linearity Property: Let k_1, k_2 be complex constants and $f(z)$ and $g(z)$ be complex functions, then $\int\limits_C [k_1 f(z) + k_2 g(z)]\,dz = k_1\int\limits_C f(z)\,dz + k_2\int\limits_C g(z)\,dz$
  3. $\int\limits_{z_1}^{z_2} f(z)\,dz = -\int\limits_{z_2}^{z_1} f(z)\,dz$

4. Given that C is an arc with end points A and B, $\int_C |dz| = \int_C ds = L$, where L is the length of the arc C from A to B.
5. $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C is the curve consisting of two curves C_1 and C_2.

- If $f(z)$ is not analytic, value of $\int_{z_1}^{z_2} f(z)\,dz$ is different for different paths from z_1 to z_2. i.e. the value of the integral depends upon the path, when the function is not an analytic function.
- When $f(z)$ is an analytic function, $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the path from z_1 to z_2.

2 Simply and Multiply connected domains

- A path with coincident end points, not intersecting or touching itself is called a **simple closed path** or **contour**. Integral the simple closed path is called **contour integral**.
- A domain D is called a **simply connected domain**, if every simple closed paths lying inside D can be contracted to a point in D without leaving D.
  A domain D which is not simply connected is called **multiply connected domian**.



- **Fundamental Theorem of Complex Integration:** If $f(z)$ is analytic function in a simply connected domain D, then for every simple closed path C in D, $\oint_C f(z)\,dz = 0$.
- **Evaluation of line integral:** If $f(z)$ be an analytic function in a simply connected domain D, then there exists an analytic function $F(z)$ with $F'(z) = f(z)$ in D then along any path joining z_1 and z_2 in D, $\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$.
- **Cauchy's Integral Theorem for Multiply Connected Domain:** Let $f(z)$ be analytic between two simple closed paths C_1 and C_2, where C_2 lies entirely inside the curve C_1, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

- Let $C_1, C_2, \ldots, C_n$ be finite simple closed paths inside a simple closed path C and $f(z)$ is analytic within the domain between the paths $C_1, C_2, \ldots, C_n$ then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

3 Cauchy's Integral Formula

- Let $f(z)$ be an analytic function **within and on a simple closed path** C. If z_0 is any point in C where $\frac{f(z)}{z-z_0}$ is not analytic, then $\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$

- **Generalized Cauchy's integral formula:** Let $f(z)$ be an analytic function **within and on a simple closed path** C. If z_0 is any point in C where $\frac{f(z)}{z-z_0}$ is not analytic, then

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(z_0)$$

- If $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 and z_0 is any point in that domain where $\frac{f(z)}{z-z_0}$ is not analytic, then

$$\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \oint_{C_2} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

where the outer integral over C_1 is taken counter clockwise and the inner integral over C_2 is taken clockwise.