1. Complex line integral: Properties and evaluation

- Concept of integration as an inverse process of differentiation is also applicable in case of complex functions provided, the function is analytic.
  i.e. If \( F'(z) = f(z) \), then \( \int f(z) \, dz = F(z) + c \), where \( c \) is a complex constant.
  This is called indefinite integral

- The definite integral of a complex variable \( \int_{z_1}^{z_2} f(z) \, dz \) depends upon the path from \( z_1 \) to \( z_2 \) in the complex plane.

- Complex definite integral can also be written as \( \int_C f(z) \, dz \), where \( C \) is the path of integration.

- Complex definite integral is also called Complex Line Integral or simply line integral.

- When \( C \) is a closed path, i.e. when \( z_1 \) and \( z_2 \) co-incides, the integral denoted by \( \oint_C f(z) \, dz \).

- Properties of Line Integral:
  1. Let \( f(z) = u + iv \) and \( dz = dx + idy \), then \( \int_C f(z) \, dz = \int_C (u + iv)(dx + idy) \)
     \[ = \int_C (udx - vdy) + i \int_C (vdx + udy) \]
  2. Linearity Property: Let \( k_1, k_2 \) be complex constants and \( f(z) \) and \( g(z) \) be complex functions, then \( \int_C [k_1 f(z) + k_2 g(z)] \, dz = k_1 \int_C f(z) \, dz + k_2 \int_C g(z) \, dz \)
  3. \( \int_{z_1}^{z_2} f(z) \, dz = -\int_{z_2}^{z_1} f(z) \, dz \)
4. Given that $C$ is an arc with end points $A$ and $B$, $$\int_C |dz| = \int_C ds = L,$$ where $L$ is the length of the arc $C$ from $A$ to $B$.

5. $$\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz,$$ where $C$ is the curve consisting of two curves $C_1$ and $C_2$.

- If $f(z)$ is not analytic, value of $\int_{z_1}^{z_2} f(z) \, dz$ is different for different paths from $z_1$ to $z_2$. i.e. the value of the integral depends upon the path, when the function is not an analytic function.

- When $f(z)$ is an analytic function, $\int_{z_1}^{z_2} f(z) \, dz$ is independent of the path from $z_1$ to $z_2$.

2 Simply and Multiply connected domains

- A path with coincident end points, not intersecting or touching itself is called a simple closed path or contour. Integral the simple closed path is called contour integral.

- A domain $D$ is called a simply connected domain, if every simple closed paths lying inside $D$ can be contracted to a point in $D$ without leaving $D$. A domain $D$ which is not simply connected is called multiply connected domain.

![Simply-connected domain](image1) ![Multiply-connected domains](image2)

- **Fundamental Theorem of Complex Integration:** If $f(z)$ is analytic function in a simply connected domain $D$, then for every simple closed path $C$ in $D$, $$\oint_C f(z) \, dz = 0.$$

- **Evaluation of line integral:** If $f(z)$ be an analytic function in a simply connected domain $D$, then there exists an analytic function $F(z)$ with $F'(z) = f(z)$ in $D$ then along any path joining $z_1$ and $z_2$ in $D$, $$\int_{z_1}^{z_2} f(z) \, dz = F(z_2) - F(z_1).$$

- **Cauchy’s Integral Theorem for Multiply Connected Domain:** Let $f(z)$ be analytic between two simple closed paths $C_1$ and $C_2$, where $C_2$ lies entirely inside the curve $C_1$, then $$\oint_{C_1} f(z) \, dz = \oint_{C_2} f(z) \, dz.$$

- Let $C_1, C_2, \ldots, C_n$ be finite simple closed paths inside a simple closed path $C$ and $f(z)$ is analytic within the domain between the paths $C_1, C_2, \ldots, C_n$ then $$\oint_C f(z) \, dz = \oint_{C_1} f(z) \, dz + \oint_{C_2} f(z) \, dz + \cdots + \oint_{C_n} f(z) \, dz$$
3 Cauchy’s Integral Formula

- Let \( f(z) \) be an analytic function \textbf{within and on a simple closed path} \( C \). If \( z_0 \) is any point in \( C \) where \( \frac{f(z)}{z-z_0} \) is not analytic, then
  \[
  \oint_C \frac{f(z)}{z-z_0} \, dz = 2\pi if(z_0)
  \]

- \textbf{Generalized Cauchy’s integral formula:} Let \( f(z) \) be an analytic function \textbf{within and on a simple closed path} \( C \). If \( z_0 \) is any point in \( C \) where \( \frac{f(z)}{z-z_0} \) is not analytic, then
  \[
  \oint_C f(z) \left( \frac{1}{z-z_0} \right)^{n+1} \, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)
  \]

- If \( f(z) \) is analytic on \( C_1 \) and \( C_2 \) and in the ring-shaped domain bounded by \( C_1 \) and \( C_2 \) and \( z_0 \) is any point in that domain where \( \frac{f(z)}{z-z_0} \) is not analytic, then
  \[
  \oint_{C_1} \frac{f(z)}{z-z_0} \, dz + \oint_{C_2} \frac{f(z)}{z-z_0} \, dz = 2\pi if(z_0)
  \]

where the outer integral over \( C_1 \) is taken counter clockwise and the inner integral over \( C_2 \) is taken clockwise.