Numerical Differentiation Unit-2

It is a process of calculating the value of derivative of a function at some assigned value of x from the given set of values (x_i, y_i).

If the values of x are equispaced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula.

Formulae and derivative

Consider the function $y = f(x)$ which is tabulated for the values $x (= x_0 + ih)$ $i = 0, 1, 2$.

(i) Derivatives using forward difference formula:—

We know that $1 + \Delta = E = e^{hD}$

$$hD = \log(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} - \cdots$$

or, $$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} - \cdots\right]$$

and $$D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots\right]^2$$

$$= \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \cdots\right]$$

$$D^3 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \cdots\right]$$

Dy_0 :- $\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \cdots\right]$$

and $\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$

2. <u>Derivatives using backward difference formula</u> :-

We know that

$$1 - \nabla = E^{-1} = e^{-hD}$$

$$-hD = \log(1-\nabla)$$

$$= -\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} \cdots\right]$$

$$D = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} \cdots\right]$$

$$D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^4 + \cdots\right]^2$$

$$= \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \cdots\right]$$

$$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \cdots\right]$$

Applying these identities to y_n -

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n \cdots\right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Ex Given that

x :	1	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find dy/dx and $\frac{d^2y}{dx^2}$ at

(a) $x = 1.1$ (b) $x = 1.6$

Sol The Difference table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.002	
1.3	9.129		-0.026		0		-0
		0.322		0.004		-0.001	
1.4	9.451		-0.023		-0.001		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

12. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t second.

t :	0	0.2	0.4	0.6	0.8	1.0	1.2
θ :	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when $t = 0.6$ second.

13. Find dy/dx at $x = 1$ from the following table by constructing a central difference table : (*V.T.U., B.E., 2004*)

x :	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y :	0.644218	0.717356	0.783327	0.841471	0.891207	0.932039	0.963558

14. Find the value of $f'(x)$ at $x = 0.04$ from the following table using Bessel's formula.

x :	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

15. If $y = f(x)$ and y_n denotes $f(x_0 + nh)$, prove that, if powers of h above h^6 be neglected.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{3}{4h}\left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right].$$

16. Given the following pairs of values of x and y :

x :	1	2	4	8	10
y :	0	1	5	21	27

Determine numerically dy/dx at $x = 4$.

17. Using the following data, find x for which y is minimum and find this value of y.

x :	0.60	0.65	0.70	0.75
y :	0.6221	0.6155	0.6138	0.6170

18. From the following table, find the value of x for which y is maximum and find this value of y.

x :	1.2	1.3	1.4	1.5	1.6
y :	0.9320	0.9636	0.9855	0.9975	0.9996

8.4. NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.

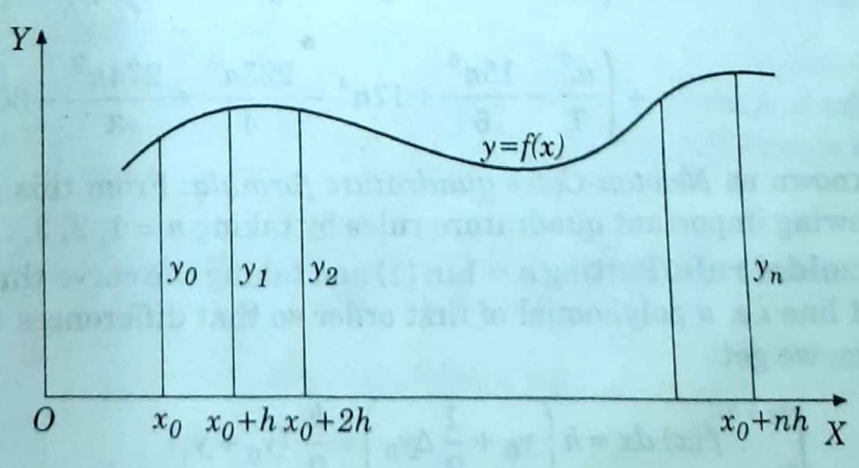


Fig. 8.1

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values only.

8.5. NEWTON-COTES QUADRATURE FORMULA

Let $$I = \int_a^b f(x)\, dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots x_n$. (Fig. 8.1)

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ... $x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)\, dx = h \int_0^n f(x_0 + rh)\, dr, \quad \text{Putting } x = x_0 + rh,\ dx = h\,dr$$

$$= h \int_0^n \Big[y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!} \Delta^5 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!} \Delta^6 y_0 + \ldots\ldots \Big] dr$$

[by Newton's forward interpolation formula

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\, dx = nh \Big[y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0$$

$$+ \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n \right) \frac{\Delta^4 y_0}{4!}$$

$$+ \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n \right) \frac{\Delta^5 y_0}{5!}$$

$$+ \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n \right) \frac{\Delta^6 y_0}{6!} + \ldots \Big] \quad \ldots(1)$$

his is known as *Newton-Cotes quadrature formula*. From this general formula, we he following important quadrature rules by taking $n = 1, 2, 3, \ldots\ldots$

Trapezoidal rule. Putting $n = 1$ in (1) and taking the curve through (x_0, y_0) and $(x_1,$ traight line *i.e.* a polynomial of first order so that differences of order higher than me zero, we get

$$\int_{x_0}^{x_0+h} f(x)\, dx = h\left(y_0 + \frac{1}{2} \Delta y_0 \right) = \frac{h}{2} (y_0 + y_1)$$

Similarly $\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} \mathbf{f(x)\,dx} = \frac{\mathbf{h}}{\mathbf{2}}\,[(\mathbf{y_0 + y_n}) + \mathbf{2(y_1 + y_2 + \dots + y_{n-1})}] \quad \dots(2)$$

This is known as the *trapezoidal rule*.

> **Obs.** The area of each strip (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and $x_0 + nh$ is approximately equal to the sum of the areas of the n trapeziums.

II. Simpson's one-third rule. Putting $n = 2$ in (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola *i.e.* a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h(y_0 + \Delta y_0 + \tfrac{1}{6}\Delta^2 y_0) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly $\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n),\ n \text{ being even.}$$

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_0+nh} \mathbf{f(x)\,dx} = \frac{\mathbf{h}}{\mathbf{3}}\,[(\mathbf{y_0 + y_n}) + \mathbf{4(y_1 + y_3 + \dots + y_{n-1})}$$

$$+ \mathbf{2(y_2 + y_4 + \dots + y_{n-2})}] \quad \dots(3)$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

> **Obs.** While applying (3), *the given interval must be divided into even number of equal sub-intervals*, since we find the area of two strips at a time.

III. Simpson's three-eighth rule. Putting $n = 3$ in (1) above and taking the curve through $(x_i, y_i) : i = 0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]$$

which is known as *Simpson's three-eighth rule*. ...(4)

Obs. While applying (4), *the number of sub-intervals should be taken as multiple of 3.*

IV. Boole's rule. Putting $n = 4$ in (1) above and neglecting all differences above the fourth, we obtain

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots$$

This is known as *Boole's rule*. ...(5)

Obs. While applying (5), the number of sub-intervals should be taken as a multiple of 4.

V. Weddle's rule. Putting $n = 6$ in (1) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = 6h\left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \quad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots)$$

...(6)

Obs. *While applying (6), the number of sub-intervals should be taken as a multiple of 6. Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.*

Example 8.6. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,* → 1.4108

(ii) Simpson's 1/3 rule, → ~~1.4108~~ 1.3662 *(Madras, B.E., 2003)*

(iii) Simpson's 3/8 rule, → 1.3571

(iv) Weddle's rule and compare the results with its actual value. → 1.3735 *(Rohtak, B.E., 2003)*

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$ $= y$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \tfrac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)] = 1.357$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= 0.3[1 + 5(0.5) + 0.2 + 6(0.1) + 0.0588 + 5(0.0385) + 0.027] = 1.$$

Also $$\int_0^6 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^6 = \tan^{-1} 6 = 1.4056$$

This shows that the value of the integral found by Weddle's rule is the nearest tual value followed by its value given by Simpson's 1/3 rule.

Example 8.7. *Use Simpson's $1/3^{rd}$ rule to find $\int_0^{0.6} e^{-x^2}\,dx$ by taking seven ordi*

(V.T.U., 20

Sol. Divide the interval (0, 0.6) into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $1/3^{rd}$ rule, we have

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}[(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3}[1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3}(16.0543) = 0.5351.$$

Obs. Applications of Simpson's rule. If the various ordinates in § 8.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similarly if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following examples illustrate these applications.

Example 8.8. *The velocity v(km/min) of a moped which starts from rest, is given a fixed intervals of time t (min) as follows :*

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Sol. If s (km) be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\therefore \quad \left| s \right|_{t=0}^{20} = \int_0^{20} v\,dt = \frac{h}{3}[X + 4.O + 2.E], \text{ by Simpson's rule}$$

Here $h = 2$, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$$\therefore \quad X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$

$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance $= \left| s \right|_{t=0}^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72) = 309.33$ km.

Example 8.9. *A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates :*

x :	0.00	0.25	0.50	0.75	1.00
y :	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule. (*Manipal, B.E., 2001*)

Sol. Here $h = 0.25$, $y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$ etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2 \, dx = \pi \cdot \frac{h}{3} [(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]$$

$$= \frac{0.25\pi}{3} [\{1 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.9589)^2]$$

$$= \frac{0.25 \times 3.1416}{3} [1.7081 + 7.2216 + 1.839]$$

$$= 0.2618(10.7687) = 2.8192.$$

PROBLEMS 8.2

1. Use trapezoidal rule to evaluate $\int_0^1 x^3 \, dx$ considering five sub-intervals.

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (*i*) Trapezoidal rule taking $h = 1/4$.
   (*ii*) Simpson's 1/3 rule taking $h = 1/4$. (*Nagarjuna, B.E., 2001*)
   (*iii*) Simpson's 3/8 rule taking $h = 1/6$. (*Delhi, B.E., 2002*)
   (*iv*) Weddle's rule taking $h = 1/6$. (V.T.U., B.E., 2004)
   Hence compute an approximate value of π in each case.

3. Evaluate $\int_0^1 \frac{dx}{1+x}$ taking 7 ordinates by applying Simpson's 3/8th rule. Deduce the value of $\log_e 2$. (V.T.U., B.E., 2001)

4. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts. (*Madras, B.E., 2000*)

5. Evaluate $\int_0^4 e^x \, dx$ by Simpson's rule, given that

$$e = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.6$$

   and compare it with the actual value. (*Nagarjuna, B. Tech., 2003*)

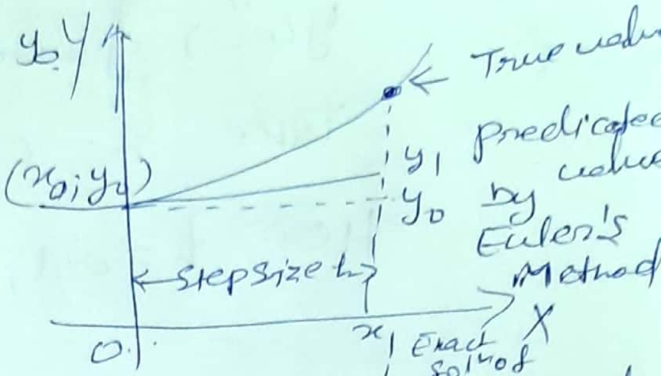
6. Calculate the value of $\int_0^{\pi/2} \sin x \, dx$ by Simpson's $\frac{1}{3}$-rule, using 11 ordinates.

7. Integrate numerically $\int_0^{\pi/2} \sqrt{\cos \theta} \, d\theta$. (*P.T.U., B.E., 2001*)

Euler's Method ⇒

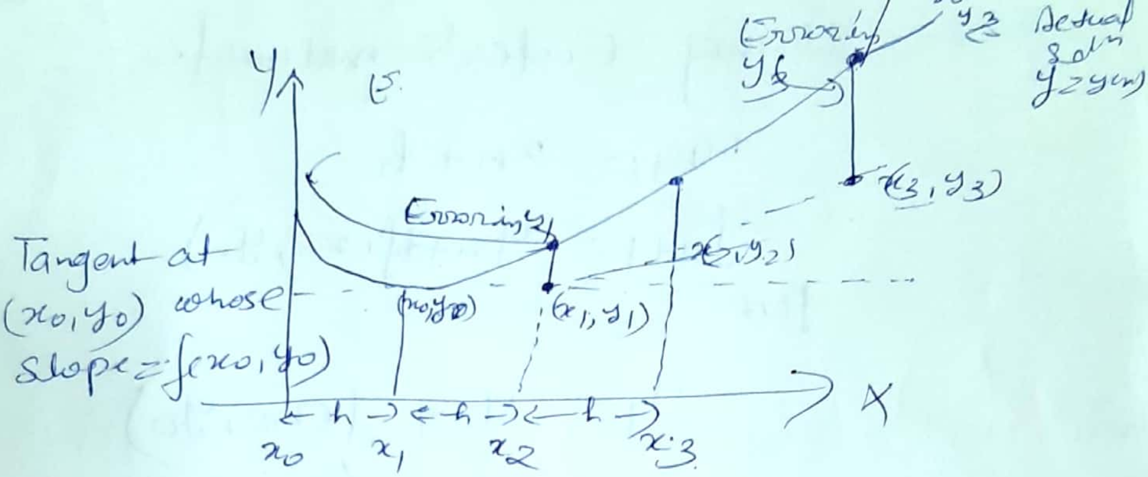
The O.D.E $\frac{dy}{dx} = f(x, y)$

$y(x_0) = y_0$



$x_{n+1} = x_n + h$

$y_{n+1} = y_n + h f(x_n, y_n)$



Formula:- $x_n = x_{n-1} + h$

$y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$

$\frac{dy}{dx} = x + 2y, \quad y(0) = 0,$

$h = 0.1, \quad y(0.4) = ?$ $\quad$ 0.0684

$x_1 = x_0 + h = 0.1$	$y_1 = 0 + 0.1(0+0) = 0$
$x_2 = x_1 + h = 0.2$	$y_2 = 0 + 0.1(0.1 + 0) = 0.01$
$x_3 = x_2 + h = 0.3$	$y_3 = 0.01 + 0.1(0.2 + 0.02) = 0.01 + .022$
$x_4 = x_3 + h = 0.4$	$y_3 = .032$
	$y_4 = 0.032 + 0.1(0.3 + 0.064)$
	$= 0.032 + 0.1(0.364)$
	$= 0.032 + 0.0364$

Using Euler's Method, find $y(0.2)$ given $\frac{dy}{dx} = y - \frac{2x}{y}$ $y(0) = 1$

Take $h = 0.1$

Here $h = 0.1$, $x_0 = 0$, $y_0 = 1$, $f(x, y) = y - \frac{2x}{y}$

Using Euler's method.

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + hf(x_n, y_n)$

Put $n = 0$,

$y_1 = y_0 + f(x_0, y_0)$

$= 1 + 0.1\left(y_0 - \frac{2x_0}{y_0}\right) = 1 + 0.1 = 1.1$

Therefore $y_1 = y(x_1) = y(0.1) = 1.1$

Again using the formula, for $n = 1$.

$x_1 = x_0 + h = 0.1$

$y_2 = y_1 + h\,(f(x_1, y_1))$

$= 1.1 + 0.1\left(y_1 - \frac{2x_1}{y_1}\right)$

$= 1.1 + 0.1\left(1.1 - \frac{0.2}{1.1}\right)$

$= 1.1 + 0.1\left(\frac{1.21 - 0.2}{1.1}\right)$

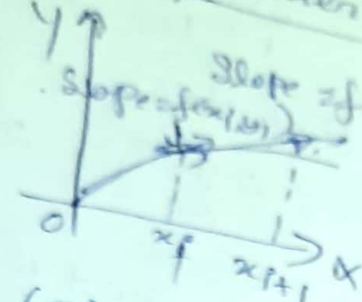
$= 1.1 + 0.1 \times 1.01 = 1.1 + .101$

$= 1.201 = 1.1908$

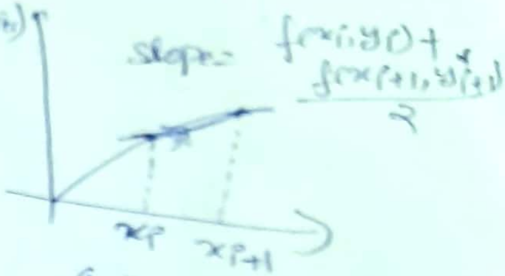
Therefore $y_2 = y(x_2) = y(0.2) = 1.1918$

Modified Euler's Method ⇒

The O.D.E $\frac{dy}{dx} = f(x,y)$



(a) Predictor



(b) Corrector.

$$y^*_{i+1} = y_n + h f(x_n, y_n)$$

Predictor eqn

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^*_{i+1}) \right]$$

or, $y_{i+1} = y_n + hk$.

which is corrector eqn.

* Given the eqn $y' = 2y/x$, $y(1) = 2$

Estimate $y(2)$ using Euler's method

(b) Heun's Method using $h = 0.25$

Given, $x_0 = 1$, $y_0 = 2$, $h = 0.25$

$$f(x,y) = 2y/x$$

using Euler's formula

$y_1 = 3 \quad y = 4.00 \quad y = 5.6$

$0.25, \; 0.25$

For $n=0$, $\quad x_0 \; x_1 \; x_2 \; x_3 \; x_4$

$1 \quad 1.25 \quad 1.5 \quad 1.75 \quad 2$

$y_1 = y_0 + h f(x_0, y_0)$

$= 2 + 0.25\left(\frac{2y_0}{x_0}\right) = 2 + 0.25 \times \frac{2\times 2}{1}$

$y_1 = 3$

For $n=1$ $\quad x_1 = x_0 + h = 1.25$

$y_1 = y(x_1) = y(1.25) = 3$

$y_2 = y_1 + h f(x_1, y_1)$

$= 3 + 0.25 \frac{2y_1}{x_1}$

$= 3 + 0.25 \frac{2 \times 3}{1.25}$

$\frac{6}{5} = 1.2$

$= 3 + 1.2 = 4.20$

For $n=2$, $\quad x_2 = x_1 + h = 1.5$

$y_3 = y_2 + h f(x_2, y_2)$

$= 4.20 + 0.25 \times \frac{2 \times 4.20}{1.50}$

$= 4.20 + 1.40 = 5.60$

$x_0 = 1$
$x_1 = 1.25$
$x_2 = 1.5$
$x_3 = 1.75$

For $n=3$ $\quad x_3 = 1.75$

$y_4 = y_3 + h f(x_3, y_3)$

$= 5.60 + 0.25 \times \frac{2 \times 5.60}{1.75}$

$= 5.60 + \frac{11.20}{7} \quad 1.60 \quad = 7.20$

$y_4 = 7.20$

Modified Euler's method:-

$y^{*}_{n+1} = y_n + h f(x_n, y_n)$ predictor

$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y^{*}_{n+1})]$

$x_0 = 1,\ x_1 = 1.25,\ x_2 = 1.5,\ x_3 = 1.75,\ x_4 = 2.$

$y_0 = 2 \quad y(x_1) = y^{*}_1 = 3,\ y(x_2) = y^{*}_2 = 4.20,\ y(x_3) = y^{*}_3 = 5.$

$y^{*}_4 = y(x_4) = 7.20.$

24
600
1255

$y_1 = 2 + \frac{0.25}{2}\left[\frac{2y_0}{x_0} + \frac{2y^{*}_1}{x_1}\right]$

$= 2 + 0.125\left[2\times 2 + \frac{2\times 3}{1.25}\right]$

$= 2 + 0.125[4 + 4.8]$

$= 2 + 0.125 \times 8.8 = 3.10.$

Therefore $y_1 = y(x_1) = y(1.25) = 3.10$

$y_2 = y_1 + \frac{h}{2}\left[\frac{2y_1}{x_1} + \frac{2y^{*}_2}{x_2}\right]$

$= 3.10 + \frac{0.25}{2}\left[4.8 + \frac{2\times 4.200}{1.500}\right]$ 2.8 4.96 5.79

$= 3.10 + 0.25\left[\frac{4.96 + 5.6}{2}\right]$

$= 3.10 + 0.25\left[\frac{10.4}{2}\right]$ 5.2

$= 3.10 + 1.300 = 4.400$

3.10

Modified Euler's :-

$$y^*_{n+1} = y_n + h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right]$$

$$f(x,y) = \frac{2y}{x}, \quad y(1) = 2, \quad h = 0.5$$

$$y(2) = ?$$

$$y^*_1 = y_0 + 0.5 \times \frac{2y_0}{x_0}$$

$$= 2 + 0.5 \times \frac{2 \times 2}{1}$$

$$y^*_1 = 2 + 2 = 4, \quad y^*_1 = y(1.5) = 4$$

$$y_1 = y_0 + \frac{h}{2}\left[\frac{2y_0}{x_0} + \frac{2y^*_1}{x_1}\right]$$

$$= 2 + \frac{0.5}{2}\left[2 \times 2 + \frac{2 \times 4}{1.5}\right]$$

$\frac{8}{1.5}$

$$= 2 + 0.5\left[4 + 5.33\right]$$

$$= 2 + 0.5 \times 9.33 = 4.665$$

$$y_1 = 6.665$$

$$y^*_2 = y_1 + 0.5 \times \frac{2y_1}{x_1}$$

$$= 6.665 + 0.25 \times \frac{2 \times 6.6650}{1.5000} \quad 4.44$$

$$= 6.665 + 0.25 \times 8.88$$

$$= 6.665 + \cancel{0.25 \times} 2.200 \quad 20 \quad 22$$

$$= \begin{matrix} 2.220 \\ 6.665 \end{matrix} = 8.885$$

$x_0 = 1,\ x_1 = 1.5,\ x_2 = 2$

$y_2 = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2...)\right]$

$= 6.665 + \frac{0.5}{2}\left[\frac{2y_1}{x_1} + \frac{2y_2}{x_2}\right]$

$= 6.665 + \frac{0.5}{2}\left[\frac{2\times 6.665}{1.5} + \frac{\cancel{2}\times 6.885}{\cancel{2}}\right]$ 4.4425

$= 6.665 + \frac{0.5}{2}\left[\begin{matrix}13.330\\ 8.885\end{matrix}\right]$

$= 6.665 + 0.125 \times 22.935$

$= 6.665 + 5.55375$ 12 3

$= \begin{matrix}6.665\\ 5.554\end{matrix}$ 12.219

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Runge kutta Fourth order ⇒

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$k_1 = h f(x_0, y_0)$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = h f(x_0 + h, y_0 + k_3)$

Finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k = \frac{1}{6}(0.2 + 0.48 + 0.4880 + 0.288)$$

$$= \frac{1}{6} \times 1.4568 = 0.2428$$

$\cancel{x_0 = 0,}$ $y_1 = y_0 + k$

Apply Runge kutta fourth order method to find an approximate value of y when $x = 0.2$, given $\frac{dy}{dx} = x + y$, and $y = 1$ when $x = 0$

$x_0 = 0,\ y_0 = 1,\ h = 0.2$

$f(x_0, y_0) = 1$

$k_1 = h\, f(x_0, y_0) = 0.2 \times 1 = 0.2$

$k_2 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$

$\frac{0.20}{2.0}$ $.1$

$= 0.2\, f(0.1, 1.1)$

$= 0.2 \times 1.2 = 0.24$

$k_3 = h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$

$= 0.2\, f(0.1, 1.12)$.

$= 0.2440$

$k_4 = h\, f(x_0 + h,\ y_0 + k_3)$

$= 0.2\, f(0.2, 1.244)$

$= 0.2888$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}\ 0.2 + 0.48 + 0.4880 + 0.2$

$= 0.2428$

$y_1 = 1 + 0.2428$

$y_1 = 1.2428.$

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