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INDUS UNIVERSITY

INDUS INSTITUTE OF ENGINEERING & TECHNOLOGY

Semester: IV **Subject: COMPLEX ANALYSIS(MA0411)**

UNIT-II
THEORY OF COMPLEX FUNCTIONS

7 hours

1. Limit, continuity and differentiability of complex functions
2. Analytic function, Cauchy-Riemann equations in Cartesian and polar forms (without proof)
3. Harmonic functions, conformal mappings, some standard conformal transformations: translation, magnification and rotation, inversion.

1 Functions of Complex Variable

- Let $z = x + iy$ be a complex variable, where x and y are real numbers and $i^2 = -1$. Then the function $f(z)$ is called a complex function and is denoted by $w = f(z)$
- $w = f(z)$ is also complex in general and so we have $w = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real valued functions and respectively the real and imaginary parts of the w.

2 Limits and Continuity

- A function $w = f(z)$ is said to have a limit l as z approaches to a point z_0 if for a given small positive number ϵ, we can find a positive number δ such that for all $z \neq z_0$ in the (circular)disk $|z - z_0| < \delta$, we have $|f(z) - l| < \epsilon$. Symbolically,

$$\lim_{z \to z_0} f(z) = l$$

- A function $w = f(z)$ is said to be continuous at $z = z_0$, if:
  (1) $f(z_0)$ is defined;
  (2) $\lim_{z \to z_0} f(z)$ exists and
  (3) $\lim_{z \to z_0} f(z) = f(z_0)$
- If f is continuous at each point of its domain D, then we say that f is a continuous function.
- **Algebra of Limits:** Let $f(z)$ and $g(z)$ be two functions for which $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$ exists. Then we have:
  1. $\lim_{z \to z_0} [f(z) \pm g(z)] = l \pm m$
  2. $\lim_{z \to z_0} [f(z) \cdot g(z)] = lm$
  3. $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{l}{m}$, provided $m \neq 0$

- Let $f(z) = u(x,y) + iv(x,y)$, then the limit of $f(z)$ at $z_0 = x_0 + iy_0$ can be written as: $\lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$
- $w = f(z)$ is continuous at $z = z_0$ when $u(x,y)$ and $v(x,y)$ both are continuous at (x_0, y_0).
- A polynomial function is a continuous function on the whole of $\mathbb{C}$
- A rational function is a continuous function at every point of its domain of definition.

3 Differentiability of a complex function

- Let $z_0 + \Delta z$ be a point in the neighbourhood of z_0. A function $f(z)$ is said to be differentiable at a point z_0, if the limit $\lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ or $\lim_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists. The limit is denoted by $f'(z_0)$ and is known as the derivative of $f(z)$ at z_0.
- **Algebra of differentiable functions:** Let $f(z)$ and $g(z)$ be two functions defined in the neighbourhood of z_0 and assume that f and g are differentiable at z_0. Then we have:
  - (i) $f \pm g$ is differentiable at z_0 and $(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$
  - (ii) fg is differentiable at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$
  - (iii) If $g(z_0) \neq 0$, then $\dfrac{f}{g}$ is differentiable at z_0 and $\left(\dfrac{f}{g}\right)'(z_0) = \dfrac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$
- A polynomial function is differentiable on the whole of $\mathbb{C}$.
- A rational function is differentiable at every point of its domain of definition.

4 Cauchy-Reimann(C-R) Equations in Cartesian co-ordinates

- **Necessary conditions for a function f(z) to be differentiable at $\mathbf{z_0}$:**
  If $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 \in \mathbb{C}$, than the first order partial derivates of u and v, i.e. u_x, u_y, v_x, v_y exists at this point and satisfy the Cauchy Reimann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial y}$$

  Note: If $f(z)$ is differentiable at a point $z_0 = (x_0, y_0)$, then

$$f'(z_0) = \left(\frac{\partial u}{\partial x}\right)_{(x_0,y_0)} + i\left(\frac{\partial v}{\partial x}\right)_{(x_0,y_0)} = \left(\frac{\partial u}{\partial x}\right)_{(x_0,y_0)} - i\left(\frac{\partial u}{\partial y}\right)_{(x_0,y_0)} = \left(\frac{\partial v}{\partial y}\right)_{(x_0,y_0)} + i\left(\frac{\partial v}{\partial x}\right)_{(x_0,y_0)}$$

- **Sufficient conditions for a function f(z) to be differentiable at $\mathbf{z_0}$:**
  For $f(z) = u(x,y) + iv(x,y)$ if
  - (i) u_x, u_y, v_x, v_y are continuous functions and
  - (ii) $f(z)$ satisfies Cauchy Reimann Equations

  at $z_0 = (x_0, y_0)$ then $f(z)$ is said to be differentiable at z_0.

5 Analytic Functions

- A single-valued function $f(z)$ is said to be **analytic at a point** z_0 in the domain D of a $z-$ plane, if:
  (i) $f(z)$ is differentiable at z_0
  (ii) $f(z)$ is differentiable in some neighbourhood of z_0.
- A function $f(z)$ is said to be **analytic** in a domain D, if it is analytic everywhere in D.
- An analytic function is also known as holomorphic or regular or monogenic function.
- A function which is analytic everywhere in the complex plane is known as **entire function**.
- The **necessary and sufficient conditions** for the function $f(z) = u(x, y) + iv(x, y)$ to be analytic in a domain D of $z-$ plane are:
  (i) u_x, u_y, v_x, v_y are continiuous functions of x and y in D.
  (ii) CR equations, i.e. $u_x = v_y$ and $u_y = -v_x$ are satisfied everywhere in D.
- If $f(z) = u(x, y) + iv(x, y)$ is analytic everywhere in a domain D, then it is differentiable everywhere in D and hence it satisfies $CR-$ equations everywhere in D. Thus,

$$f'(z) = u_x - iu_y = v_y + iv_x$$

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- A polynomial functions with complex co-effecients is analytic everywhere in the complex plane.
- A rational function is analytic everywhere in its domain.
- The real and imaginary part of an analytic function are called conjugate functions.
- **Determination of an Analytic function whose either real or imaginary part is known**
  1. **Method: 1** For a given analytic function $f(z)$, assume that Re $[f(z)] = u(x, y)$ is given. This method will find Im $[f(z)] = v(x, y)$ and hence $f(z)$ can be evaluated.
     (i) For $u(x, y)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
     (ii) As $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, integrating w.r.t y keeping x constant, we obtain
     $v = \int \left(\frac{\partial u}{\partial x}\right) dy + \phi(x)$, where $\phi(x)$ is to be determined.
     (iii) Differentiate v obtained in (1ii) w.r.t. x and use $\frac{\partial u}{\partial y}$ in (1i) to compare $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and obtain $\phi'(x)$.
     (iv) Integrate $\phi'(x)$ w.r.t. x and substitute in (1ii) to get $v(x, y)$.
     (v) Thus, $f(z) = u(x, y) + iv(x, y)$ is obtained. Use both functions to obtain $f(z)$ in terms of z.
  2. **Method: 1** If Im $[f(z)] = v(x, y)$ is given, then $u(x, y)$ and hence $f(z)$ can be found using the following steps:
     (i) For $v(x, y)$, find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

(ii) As $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, integrating w.r.t x keeping y constant, we obtain

$u = \int \left(\frac{\partial v}{\partial y}\right) dy + \phi(y)$, where $\phi(y)$ is to be determined.

(iii) Differentiate u obtained in (2ii) w.r.t. y and use $\frac{\partial v}{\partial x}$ in (2i) to compare $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and obtain $\phi'(y)$.

(iv) Integrate $\phi'(y)$ w.r.t. y and substitute in (2ii) to get $u(x, y)$.

(v) Thus, $f(z) = u(x, y) + iv(x, y)$ is obtained. Use both functions to obtain $f(z)$ in terms of z.

3. **Method: 2 (Milne-Thomson Method / Short cut method)** For a given analytic function $f(z)$, assume that Re $[f(z)] = u(x, y)$ is given.
   This method will find $f(z)$ in terms of z and Im $[f(z)] = v(x, y)$ can be evaluated from $f(z)$ by substituting $z = x + iy$.

   (i) For $u(x, y)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

   (ii) Substitute $x = z$ and $y = 0$ in (3i) to obtain $u_x(z, 0)$ and $u_y(z, 0)$

   (iii) Evaluate $f(z)$ by integrating $u_x(z, 0) - iu_y(z, 0)$ w. r. t. z.

4. **Method: 2** If Im $[f(z)] = v(x, y)$ is given, then $u(x, y)$ and hence $f(z)$ can be found using the following steps:

   (i) For $v(x, y)$, find $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

   (ii) Substitute $x = z$ and $y = 0$ in (4i) to obtain $v_x(z, 0)$ and $v_y(z, 0)$

   (iii) Evaluate $f(z)$ by integrating $v_y(z, 0) + iv_x(z, 0)$ w. r. t. z.

- **Determination of an Analytic function whose sum or difference of real and imaginary parts are known:**
  Let $f(z) = u + iv$ be a complex function. Then $if(z) = iu - v = -v + iu$
  Thus, $(1 + i)f(z) = (u - v) + i(u + v)$. Let $F(z) = (1 + i)f(z) = U + iV$
  So, $U = u - v$ and $V = u + v$

  1. If $f(z) = u + iv$ is an analytic function, where $u - v$ is given, then Re $[F(z)]$ is known and methods in (1) or (3) can be used to find $F(z)$ and hence $f(z)$.
  2. If $f(z) = u + iv$ is an analytic function, where $u + v$ is given, then Im $[F(z)]$ is known and methods in (2) or (4) can be used to find $F(z)$ and hence $f(z)$.

- A function $f(x, y)$ is said to be harmonic in a domain D, if

  (i) $f(x, y)$ satisfies Laplace's equation. i.e. $f_{xx} + f_{yy} = 0$

  (ii) f_{xx}, f_{xy}, f_{yy} are continuous functions of x and y in D.

- The real and imaginary part of an analytic functions are harmonic functions. They are called conjugate harmonic functions of each other.

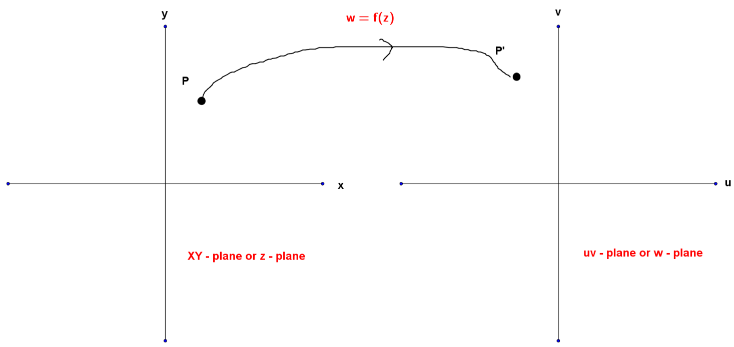
- If u and v are random harmonic functions, then it is not necessary that $u + iv$ is an analytic function.

- **CR equations in Polar form** Let $f(z)$ be analytic function in its domain.
  In polar form, $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$.
  Then the CR equations in polar form are given by: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.
  The derivative f' can be calculated using: $f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{-i}{re^{i\theta}}(u_\theta + iv_\theta)$

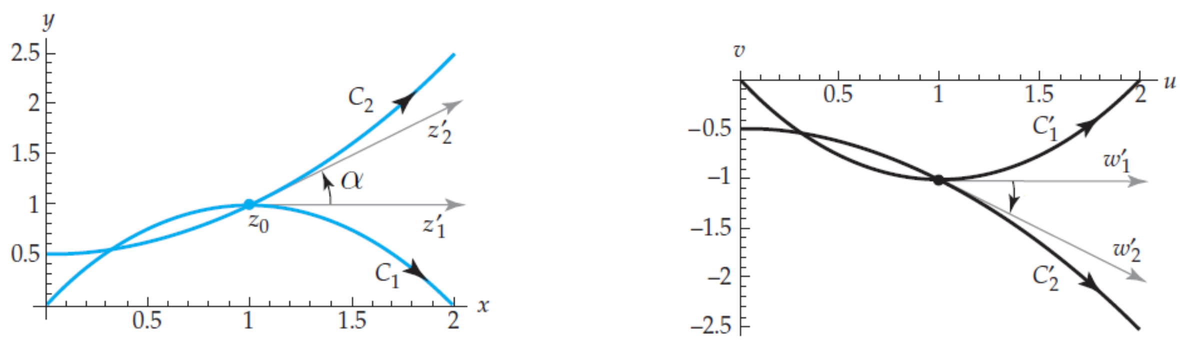
- **Laplace equation in polar form** is given by: $\nabla^2\phi = \dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$

6 Conformal Mapping

- For a complex function $w = f(z)$, $z = x + iy$ is an independent variable and $w = u + iv$ is a dependent variable.
- Thus, z and $f(z) = w$ cannot be plotted on the same set of axes as four variables x, y, u, v are involved.
- We plot z and $f(z)$ in two different planes.
- The plane containing the independent variable $z = x + iy$ is called the $z-$ plane or $xy-$ plane.
- The plane contianing the dependent variable $w = u + iv$ is called the $w-$ plane or $uv-$ plane.
- Thus, corresponding to each point (x, y) in $z-$ plane, we have a point (u, v) in $w-$ plane which is the image of (x, y) under a given function f.
- We say that $w = f(z)$ is a mapping or transformation of a point P within a region in $z-$ plane called the domain to a point P' within a region in $w-$ plane called the range.



- Let C_1 and C_2 be two smooth curves in $z-$ plane intersecting at a point z_0 at an angle α with $0 < \alpha < \pi$. Let C_1' and C_2' be the corresponding curves in $w-$ plane under $w = f(z)$ intersecting at w_0 as shown in figure below.



- If C_1' and C_2' intersect at angle α, when $w = f(z)$ is called isogonal mapping.
  Example of an isogonal mapping is $f(z) = \bar{z}$ as arg $(\bar{z}) = -$arg (z)
- If C_1' and C_2' intersect at angle α preserving the sense of rotation, then $w = f(z)$ is called conformal mapping.

- If $f(z)$ is an analytic function then $w = f(z)$ defines a conformal mapping except at point where $f'(z) = 0$. These points are called critical points.

- **Conformal mapping by some elementary functions:**

  1. Identity Transformation: $w = z$
  2. Translation: $w = z + \alpha$, $\alpha \in \mathbb{C}$
  3. Rotation: $w = e^{i\theta} z$, where θ is a real constant.

     Note: If $\theta > 0$, the rotation is counter clockwise and if $\theta < 0$, the rotation is clockwise
  4. Stretching or Magnification: $w = az$, where a is a positive real constant.

     Note: If $0 < a < 1$, the graph is contracted if $a > 1$, the graph is stretched.
  5. Linear Transformation: $w = \alpha z + \beta$, where α and β are complex constants.
  6. Inversion: $w = \dfrac{1}{z}$

- Under the transformation $w = \dfrac{1}{z}$:

  1. A circle or a line that passes through the origin transforms into a line.
  2. A circle or a line that does not pass through the origin transforms into a circle.