INDUS INSTITUTE OF ENGINEERING & TECHNOLOGY

Semester: IV **Subject: COMPLEX ANALYSIS(MA0411)**

UNIT-I
INTRODUCTION TO COMPLEX VARIABLES

8 hours

1. Introduction to Complex numbers, geometrical representation of complex numbers, polar form
2. Eulers formula, De Moivres theorem and its application in finding roots of complex numbers
3. Elementary complex functions: exponential functions, trigonometric functions, hyperbolic functions and logarithm of a complex number, relation between circular and hyperbolic functions
4. Complex planes, Concepts on sets in the complex plane: domain, circles, discs, half planes, neighborhood of a point, Different curves and their regions in complex planes.

1 Introduction to Complex Number

- A complex number is an ordered pair (x, y) of real numbers x and y.
- We write it as $x + iy$ where $i = \sqrt{-1}$ is called the imaginary number or imaginary unit. Complex numbers are usually denoted by $z, w, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots$.
- If z is a complex number, then $z = x + iy$, where x and y are real numbers.
- x is called the real part of z and is denoted by $Re(z)$.
- y is called the imaginary part of z and is denoted by $Im(z)$.
- For $z = x + iy$,
  (i) If $x = 0, y \neq 0$, then $z = iy$ is called purely imaginary number.
  (ii) If $x \neq 0, y = 0$, then $z = x$ is called purely real number.
- The conjugate of a complex number $z = x + iy$ is $x - iy$ and is denoted by $\bar{z}$.
- $Re(z) = x = \dfrac{z + \bar{z}}{2}$ and $Im(z) = y = \dfrac{z - \bar{z}}{2i}$
- $z = \bar{z}$ iff z is a real number or $y = 0$.

2 Algebra of Complex Numbers

For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

(1) $z_1 = z_2 \Leftrightarrow x_1 = x_2 \ \&\ y_1 = y_2$

(2) $z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$

(3) $z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

(4) $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \cdot \dfrac{x_2 - iy_2}{x_2 - iy_2} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$, if $z_2 \neq 0$

3 Geometrical Representation of a complex number

- For each complex number $z = x + iy$, we get a point in the plane in the ordered pair form (x, y).
- The plane is also called complex plane or Argand diagram or Gaussian plane.
- The horizontal axis is called the real axis and vertical axis is called the imaginary axis.
- The absolute value or the modulus of a complex number $z = x+iy$ is denoted by $|z| = |x+iy|$ and is defined by $r = \sqrt{x^2 + y^2}$.
- Argument or amplitude of $z = x + iy$ denoted by $argz$ or $ampz$ is defined by $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, if $x \neq 0$.
- If $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ lies between $-\pi$ and π, it is called the principal value of $argz$.
  General value of $argz$ is $2n\pi + \theta$.
  Unless or otherwise stated we shall consider only the principal values of the argument.

4 Polar form of a complex number

Let $z = x + iy$ be a complex number and $P(x, y)$, the corresponding point on the complex plane. Let r be the distance of P from the origin O and θ be the angle between $\overrightarrow{OP}$ and real axis as show in figure below.
Figure
Then $x = r\cos\theta$ and $y = r\sin\theta$ with $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$

- The complex number, thus can be written as $z = r\cos\theta + ir\sin\theta$. This form of complex number is called Polar form.
- In short it can also be written as $z = rcis(\theta)$.
- The conjugate of z is given by $\bar{z} = rcis(-\theta)$.
- cis $\theta +$ cis $(-\theta) = 2\cos\theta$ and cis $\theta -$ cis $(-\theta) = 2i\sin\theta$
- In polar form if $z_1 = r_1\cos\theta_1$ and $z_2 = r_2\cos\theta_2$,
  (a) $z_1 \cdot z_2 = r_1r_2\tan(\theta_1 + \theta_2)$
  (b) $\frac{z_1}{z_2} = \frac{r_1}{r_2}\tan(\theta_1 - \theta_2)$

5 Exponential form of complex number and Euler's Formula

- For any given x, $e^{ix} = \cos x + i\sin x$. This is called Euler's formula.
- Thus, for any complex number, $z = r\cos\theta+ir\sin\theta = r(\cos\theta+i\sin\theta) = re^{i\theta}$ is the exponential form of a complex number.
- The conjugate of z is given by $\bar{z} = r(\cos\theta - i\sin\theta) = re^{-i\theta}$.

- Thus for any complex number z, we have:

Cartesian form:	$x + iy$
Polar form:	$r(\cos\theta + i\sin\theta) = rcis(\theta)$
Exponential form:	$re^{i\theta}$

- For any real or complex number x, $\cos x + i \sin x = e^{ix}$ and $\cos x - i \sin x = e^{-ix}$. Thus,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- **De Moivre's Theorem:** If n is any rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i \sin n\theta$.

6 Hyperbolic Functions

If x is a real or complex number, we define:

- The hyperbolic sine of x as $\sinh x = \dfrac{e^x - e^{-x}}{2}$.
- The hyperbolic cosine of x as $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Other hyperbolic functions are defined as:

- $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
- $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
- $\text{cosech} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$
- $\text{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

6.1 Few results on hyperbolic functions

- $\sinh 0 = 0$
- $\cosh 0 = 1$
- $\dfrac{d}{dx}(\sinh x) = \cosh x$
- $\dfrac{d}{dx}(\cosh x) = \sinh x$

6.2 Relation between Hyperbolic and Trigonometric Functions

These relations are useful in seperating the real and imaginary parts of a complex number.

1.	$\sin(ix) = i \sinh x$	7.	$\sinh(ix) = i \sin x$
2.	$\cos(ix) = \cosh x$	8.	$\cosh(ix) = \cos x$
3.	$\tan(ix) = i \tanh x$	9.	$\tanh(ix) = i \tan x$
4.	$\cot(ix) = -i \coth x$	10.	$\coth(ix) = -i \cot x$
5.	$\sec(ix) = \text{sech} x$	11.	$\text{sech}(ix) = \sec x$
6.	$\text{cosec}(ix) = -i \text{ cosech } x$	12.	$\text{cosech}(ix) = -i \text{ cosec } x$

6.3 Hyperbolic Identities

Using the relations obtained in 6.2, we have the following identities:

1. Pythaogarean Identities
   - $\cosh^2 x - \sinh^2 x = 1$
   - $1 - \tanh^2 x = \text{sech}^2 x$
   - $\coth^2 x - 1 = \text{cosech}^2 x$
2. Double angle identities
   - $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2 \tanh x}{1 - tanh^2 x}$
   - $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$
   - $\tanh 2x = \dfrac{2 \tanh x}{1 + tanh^2 x}$
3. Triple angle identities
   - $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$
   - $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$

6.4 Inverse Hyperbolic Functions

For any real or complex number x,

1. $\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$
2. $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$
3. $\tanh^{-1} x = \dfrac{1}{2} \log \left(\dfrac{1 + x}{1 - x} \right)$

7 Real and Imaginary parts of basic complex functions

Let $z = x + iy = rcis(\theta) = re^{i\theta}$

1. $e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x \cos y + i\, e^x \sin y$
2. $\sin z = \sin(x + iy) = \sin x \cosh y + i\ \cos x \sinh y$
3. $\cos z = \cos(x + iy) = \cos x \cosh y - i\ \sin x \sinh y$
4. $\tan z = \tan(x + iy) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y} + i \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$
5. $\sinh z = \sinh(x + iy) = \sinh x \cos y + i\ \cosh x \sin y$
6. $\cosh z = \cosh(x + iy) = \cosh x \cos y - i\ \sinh x \sin y$
7. $\tanh z = \tanh(x + iy) = \dfrac{\sinh 2x}{\cosh 2x + \cos 2y} + i \dfrac{\sin 2y}{\cosh 2x + \cos 2y}$
8. $\log\ z = \log(x + iy) = \log(re^{i\theta}) = \log r + i\, \theta = \dfrac{1}{2} \log(x^2 + y^2) + i\ \tan^{-1} \left(\dfrac{y}{x} \right)$

8 Curves and Regions in Complex Plane

- **Circle:** $|z - \alpha| = \rho$, where $\alpha \in C$ is the centre of the circle and $\rho > 0$ is a real number and radius of the circle.

    1. $|z| = \rho$ is the equation of a standard circle, where origin is the centre of the circle.
    2. $|z| = 1$ is the equation of a unit circle, where the centre is origin and radius is 1.

- **Ciruclar Disk**

    1. $|z - \alpha| < \rho$ is an open circular disk. It is the set of all points in the interior of the circle $|z - \alpha| = \rho$.
    2. $|z - \alpha| \leq \rho$ is a closed circular disk. It is the set of all points in the interior of the circle and on the circle $|z - \alpha| = \rho$.
    3. $|z| < 1$ and $|z| \leq 1$ are called open unit disk and closed unit disk respectively.

- **Exterior of a circle:** $|z - \alpha| > \rho$ is the exterior of the circle, which is the set of all points outside the circle.
  $|z| > 1$ and $|z| \geq 1$ are open and closed exterior of unit circle respectively.

- **Neighbourhood of a point** z_0 is a set of all points within but not on a circle of any radius with centre z_0.
  i.e. Neighbourhood of z_0 denoted by $N(z_0)$ is an open circular disk with centre z_0 and any radius.

- **Deleted neighbourhood or punctured neighbourhood of a point** z_0 contains all points of neighbourhood of z_0 except z_0 and is denoted by $N^*(z_0)$.
  i.e. $N^*(z_0) = \{z \in C \,|\, 0 < |z - z_0| < \rho\}$.

  Note: Any point z_0 has infinitely many such neighbourhoods.

- **Annulus:** Region between two cocentric circles is called annulus.

    1. $\rho_1 < |z - \alpha| < \rho_2$ is an open annulus.
    2. $\rho_1 \leq |z - \alpha| \leq \rho_2$ is a closed annulus.
       Two more types of annulus are:
    3. $\rho_1 \leq |z - \alpha| < \rho_2$
    4. $\rho_1 < |z - \alpha| \leq \rho_2$

- **Half planes**

    1. Set of all points $z = x + iy$, where $y > 0$ is called an upper half plane.
    2. Set of all points $z = x + iy$, where $y < 0$ is called an lower half plane.
    3. Set of all points $z = x + iy$, where $x > 0$ is called a right half plane.
    4. Set of all points $z = x + iy$, where $x < 0$ is called a left half plane.
       Other types of half planes are:
    5. Upper and lower half planes are respectively $y \geq 0$ and $y \leq 0$.
    6. Right and left half planes are respectively $x \geq 0$ and $x \leq 0$.

9 Sets in the complex plane

- **Set** of points in the complex plane is any collection of finitely or infinitely many points.
- A set of points S in the complex plane is called an **open set** if every point of S has a neighbourhood which is entirely contained in set S.
- The **complement of a set** S in the complex plane is the set of all points of a complex plane that do not belong to S.
- If complement of a set S in a complex plane is open then the set S is said to be a **closed set**.
- **Interior point:** A point $z_0 \in S$ is said to be an interior point of S, if there exists a neighbourhood of z_0 such that $N(z_0) \subset S$.
- **Boundary point:** A point $z_0 \in S$ is said to be a boundary point of S, if for every neighbourhood of z_0, $N(z_0) \nsubseteq S$.
- **Open set:** A complex set S is said to be open if every point $z_0 \in S$ is an interior point.
- A set S is said to be **closed set** if its complement is an open set.
- A set S is said to be **connected set**, if any two points in S can be joined by finitely many line segments such that each point on the line segmente is a point of S.
- An open connected set is called **domain**.
- A domain containing some or all of its boundary points is called a **Region**.
- A region containing all of its boundary points is called a **closed region**.
- A region is said to be **bounded**, if it can be enclosed in a circle of finite radius otherwise it is called **unbounded**.
- A closed and bounded region is called a **compact set**.