Newton Raphson Method : (N-R Method)

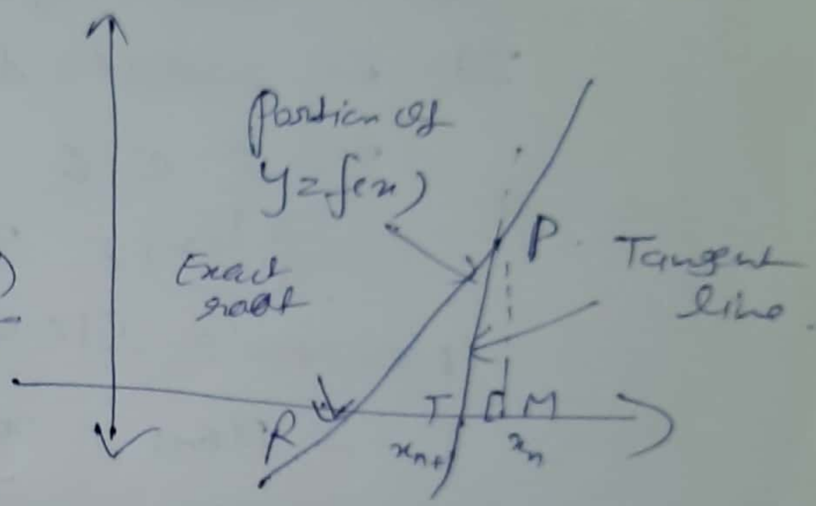
This method is faster due to the use of local behaviour of the fun $f(x)$ like derivative of the fun. Moreover; it requires only one (intial guess) for the root.

The equation

$f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$n = 0, 1, 2 \ldots$



Newton's formula converges provided the initial guess x_0 is chosen sufficiently close to the root.

Newton's method converges conditionally while Regula falsi method always converges, When once N·R method converges, it converges faster and is preferred.

Find by Newton's method, the real root of the equation $3x = \cos x + 1$

$f(x) = 3x - \cos x - 1$

$f(0) = -2 = -ve,$

$f(1) = 1.4597 = +ve.$

So a root of $f(x) = 0$ lies b/w 0 and 1. It is nearer to 1. Let us take

$x_0 = 0.6$

$f'(x) = 3 + \sin x$

Newton's Iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n - \cos x_n + 1}{3 + \sin x_n}$$

Putting $n=0$, the first approximation x_1 is given by.

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$$= \frac{0.6 \sin(0.6) + \cos(0.6) + 1}{3 + \sin 0.6}$$

$$= 0.6071$$

$n=1$, $x_2 = 0.6071. =$

14. $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$; $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$.

Solve the following equations by Gauss-Jordan method :

15. $10x + y + z = 12$; $x + 10y + z = 12$; $x + y + 10z = 12$.
16. $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + z = 2$. *(Kerala, B. Tech., 2003)*
17. $x_1 + 2x_2 + x_3 = 8$; $2x_1 + 3x_2 + 4x_3 = 20$; $4x_1 + 3x_2 + 2x_3 = 16$.
18. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$; $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$.

Solve the following equations by factorization method :

19. $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.
20. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 14$. *(Andhra B.E., 2004)*
21. $10x + y + 2z = 13$; $3x + 10y + z = 14$; $2x + 3y + 10z = 15$.
22. $2x - 6y + 8z = 24$; $5x + 4y - 3z = 2$; $3x + y + 2z = 16$.
23. $2x_1 - x_2 + x_3 = -1$; $2x_2 - x_3 + x_4 = 1$; $x_1 + 2x_3 - x_4 = -1$; $x_1 + x_2 + 2x_4 = 3$.

3.5. ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods*, as these methods yield the solution after a certain amount of fixed computation. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy. Thus in an iterative method, the amount of computation depends on the degree of accuracy required.

For large systems, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a self correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

Simple iterative methods can be devised for systems in which the coefficients of the leading diagonal are large as compared to others. We now describe three such methods :

(1) Jacobi's iteration method. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \text{...(1)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, solve for x, y, z respectively. Then the system can be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}\right\} \quad \text{...(2)}$$

Let us start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively. Substituting these on the right sides of (2), the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Substituting the values x_1, y_1, z_1 on the right sides of (2), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is repeated till the difference between two consecutive approximations is negligible.

Obs. In the absence of any better estimates for x_0, y_0, z_0, these may each be taken as zero.

Example 3.22. *Solve, by Jacobi's iteration method, the equations*

$20x + y - 2z = 17 ; 3x + 20y - z = -18 ; 2x - 3y + 20z = 25.$

Sol. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (*i*), we get

$$x_1 = \frac{17}{20} = 0.85, \quad y_1 = -\frac{18}{20} = -0.9, \quad z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right sides of the equations (*i*), we obtain.

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.03$$

Substituting these values on the right sides of the equations (i), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.00125$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -1.0015$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.00325$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.000025$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9965$$

Putting these values, we have

$$x_5 = \frac{1}{20}(-17 - y_4 + 2z_4) = 0.999966$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.000078$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.999956$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(-17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -0.999997$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 0.999992$$

The values in the 5th and 6th iterations being practically the same, we can stop. Hence the solution is

$$x = 1, y = -1, z = 1.$$

(2) Gauss-Seidal iteration method. This is a modification of Jacobi's method. As before the system of equations :

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad ...(1)$$

is written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y) \end{aligned}\right\} \quad ...(2)$$

Here also we start with the initial approximations x_0, y_0, z_0 for x, y, z respectively which may each be taken as zero. Substituting $y = y_0, z = z_0$ in the first of the equations (2), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2), we have

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2), we obtain

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

and so on *i.e.* as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeated till the values of x, y, z are obtained to desired degree of accuracy.

> **Obs. 1.** Since the most recent approximations of the unknowns are used while proceeding to the next step, the *convergence in the Gauss-Seidal method is twice as fast as in Jacobi's method.*
>
> **Obs. 2.** *Jacobi and Gauss-Seidal methods converge for any choice of the initial approximations if in each equation of the system,* **the absolute value of the largest co-efficient is almost equal to or in atleast one equation greater than the sum of the absolute values of all the remaining coefficients.**

Example 3.23. *Apply Gauss-Seidal iteration method to solve the equations of Ex. 3.22.*

(*Madras, B. Tech.*, 2003)

Sol. We write the given equations in the form

$$x = \frac{1}{20}(17 - y + 2z) \qquad \text{...}(i)$$

$$y = \frac{1}{20}(-18 - 3x + z) \qquad \text{...}(ii)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \qquad \text{...}(iii)$$

First iteration

Putting $y = y_0, z = z_0$ in (*i*), we get $\quad x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$

Putting $x = x_1, z = z_0$ in (*ii*), we have $\quad y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$

Putting $x = x_1, y = y_1$ in (*iii*), we obtain $\quad z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$

Second iteration

Putting $y = y_1, z = z_1$ in (*i*), we get

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

Putting $x = x_2$, $z = z_1$ in (*ii*), we obtain $y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$

Putting $x = x_2$, $y = y_2$ in (*iii*), we get $z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$

Third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

Example 3.24. *Solve the equations :*

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Gauss-Seidal iteration method. (*Manipal, B.E., 2000*)

Sol. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad ...(i)$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad ...(ii)$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad ...(iii)$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad ...(iv)$$

First iteration

Putting $x_2 = 0, x_3 = 0, x_4 = 0$ in (*i*), we get $x_1 = 0.3$

Putting $x_1 = 0.3, x_3 = 0, x_4 = 0$ in (*ii*), we obtain $x_2 = 1.56$

Putting $x_1 = 0.3, x_2 = 1.56, x_4 = 0$ in (*iii*), we obtain $x_3 = 2.886$

Putting $x_1 = 0.3, x_2 = 1.56, x_3 = 2.886$ in (*iv*), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56, x_3 = 2.886, x_4 = -0.1368$ in (*i*), we obtain $x_1 = 0.8869$

Putting $x_1 = 0.8869, x_3 = 2.886, x_4 = -0.1368$ in (*ii*), we obtain $x_2 = 1.9523$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_4 = -0.1368$ in (*iii*), we have $x_3 = 2.9566$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_3 = 2.9566$ in (*iv*), we get $x_4 = -0.0248$.

Third iteration

Putting $x_2 = 1.9523, x_3 = 2.9566, x_4 = -0.0248$ in (*i*), we obtain $x_1 = 0.9836$

Putting $x_1 = 0.9836, x_3 = 2.9566, x_4 = -0.0248$ in (*ii*), we obtain $x_2 = 1.9899$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_4 = -0.0248$ in (*iii*), we get $x_3 = 2.9924$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_3 = 2.9924$ in (*iv*), we get $x_4 = -0.0042$.

Fourth iteration. Proceeding as above

$x_1 = 0.9968, x_2 = 1.9982, x_3 = 2.9987, x_4 = -0.0008$

$x_1 = 0.9994, x_2 = 1.9997, x_3 = 2.9997, x_4 = -0.0001.$

Fifth iteration is $x_1 = 0.9994, x_2 = 1.9999, x_3 = 2.9999, x_4 = -0.0001$

Sixth iteration is $x_1 = 0.9999, x_2 = 2, x_3 = 3, x_4 = 0.$

Hence the solution is $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0.$

(3) Relaxation method*. Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations

$$\left.\begin{aligned} R_x &= d_1 - a_1x - b_1y - c_1z \\ R_y &= d_2 - a_2x - b_2y - c_2z \\ R_z &= d_3 - a_3x - b_3y - c_3z \end{aligned}\right\}$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step, by giving increments to the variables. For this purpose we construct the following *operation table* :

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments. (The table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed ; *e.g.* to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Obs. 1. As a check, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.

Obs. 2. *Relaxation method can be applied successfully only if the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row i.e.* if in the equations (1)

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
$$|c_3| \geq |a_3| + |b_3|$$

where > sign should be valid for at least one row.

Example 3.25. *Solve, by Relaxation method, the equations :*

$$9x - 2y + z = 50 \; ; \; x + 5y - 3z = 18 \; ; \; -2x + 2y + 7z = 19.$$

(Madras, B.E., 2000 S)

*This method was originally developed by R.V. Southwell in 1935, for application to structural engineering problems.

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Power Method ⇒ In many engineering problems, it is required to compute the numerically largest eigen value and the corresponding eigen vector.

If $X_1, X_2, \ldots X_n$ be the eigen vectors corresponding to the eigen value $\lambda_1, \lambda_2, \ldots \lambda_n$, then an arbitrary column vector can be written as

$$X = k_1 X_1 + k_2 X_2 + \cdots + k_n X_n$$

Then $AX = k_1 A X_1 + k_2 A X_2 + \cdots + k_n A X_n$

$$AX = k_1 \lambda_1 X_1 + k_2 \lambda_2 X_2 + \cdots + k_n \lambda_n X_n$$

Simplary. $A^2 X = k_1 \lambda_1^2 X_1 + k_2 \lambda_2^2 X_2 + \cdots + k_n \lambda_n^2 X_n$

$$A^r X = k_1 \lambda_1^r X_1 + k_2 \lambda_2^r X_2 + \cdots + k_n \lambda_n^r X_n$$

If $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ then λ_1 is the highest root and corresponding to the eigen value is eigen vector.

Determine the largest eigen value and corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solⁿ Let the initial eigen vector be $[1, 0, 0]'$.

then

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

So first approximation to the eigen value is 2 and corresponding eigen vector $X^{(1)} = [1, -0.5, 0]'$

Hence

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$AX^{(2)} = 2.8 \begin{bmatrix} -1 \\ 1 \\ 0.43 \end{bmatrix}$$ ~~$AX^{(3)} = 3.43 [0$~~

$$Ax^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix}$$

$$Ax^{(4)} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix}$$

$$Ax^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix}.$$

7

INTERPOLATION

7.1. INTRODUCTION

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	x_2 .... x_n
y :	y_0	y_1	y_2 .... y_n.

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The term interpolation however, is taken to include extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found. Conversely, if the form of $f(x)$ is not known (as is the case in most of the applications), it is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i). In such cases, $f(x)$ is replaced by a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Any other value may be calculated from $\phi(x)$ which is known as the *interpolating function* or *smoothing function*. If $\phi(x)$ is a polynomial, then it called the *interpolating polynomial* and the process is called the *polynomial interpolation*. Similarly when $\phi(x)$ is a finite trigonometric series, we have trigonometric interpolation. But we shall confine ourselves to polynomial interpolation only.

The study of interpolation is based on the calculus of finite differences. We begin by deriving two important *interpolation formulae* by means of forward and backward differences of a function. These formulae are often employed in engineering and scientific investigations.

7.2. NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ of x. Let these values of x be equi-spaced such that $x_i = x_0 + ih$ $(i = 0, 1, \ldots)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \ldots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad \ldots(1)$$

Putting $x = x_0, x_1, \ldots, x_n$ successively in (1), we get

$$y_0 = a_0, y_1 = a_0 + a_1(x_1 - x_0), y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0$, $\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore \quad a_1 = \frac{1}{h}\Delta y_0$$

Also $\quad \Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1) = a_1 h + a_2 . 2h . h = \Delta y_0 + 2h^2 a_2$

$$\therefore \quad a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!\,h^2}\Delta^2 y_0$$

Similarly $\quad a_3 = \frac{1}{3!\,h^3}\Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots \quad \ldots(2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$$x - x_0 = ph, \; x - x_1 = x - x_0 - (x_1 - x_0) = ph - h = (p - 1)h,$$

$$x - x_2 = x - x_1 - (x_2 - x_1) = (p - 1)h - h = (p - 2)h \text{ etc.}$$

Hence, writing $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$\mathbf{y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots(p-\overline{n-1})}{n!}\Delta^n y_0} \quad \ldots(3)$$

It is called *Newton's forward interpolation formula* as (3) contains y_0 and the forward differences of y_0.

Otherwise : Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0 \qquad [\because E = 1 + \Delta$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\ldots\right\} y_0 \quad \ldots(4)$$

[using Binomial theorem

i.e.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1}y_0$ and higher differences will be zero. Hence (4) will become

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots + \frac{p(p-1)(p-\overline{n-1})}{n!}\Delta^n y_0$$

which is same as (3).

Obs. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of* y_0.

7.3. NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values x_0, $x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \quad [\because E^{-1} = 1 - \nabla$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots\ldots\right] y_n$$

[using Binomial theorem

i.e.

$$\mathbf{y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots} \quad \ldots(1)$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of* y_n.

Example 7.1. *The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

(*i*) $x = 218$ *ft*

(*ii*) $x = 410$.

(*Madras B.E.*, 2003 S)

(V.T.U., B.E., 2002)

Sol. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		– 0.39		
				0.15	
		2.01			– 0.07
200	**15.04**		– 0.24		
				0.08	
		1.77			– 0.05
250	16.81		**– 0.16**		
				0.03	
		1.61			**– 0.01**
300	18.42		– 0.13		
				0.02	
		1.48			
350	19.90		**– 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 = 0.03$ etc.

Since $x = 218$ and $h = 50$, $\therefore \quad p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{0.36(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

$$= 15.04 + 0.637 + 0.018 + 0.002 + 0.0004$$

$= 15.697$ *i.e.* 15.7 nautical miles.

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $\quad x_n = 400, p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward difference

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{40}$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11)$$

$$+ \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53 \text{ nautical miles.}$$

Example 7.2. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of students :	*31*	*42*	*51*	*35*	*31*

(*Manipal, B.E., 2000*)

Sol. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of students (y_x) :	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	**73**		**9**		
		51		**– 25**	
60	**124**		**– 16**		**37**
		35		**12**	
70	**159**		**– 4**		
		31			
80	**190**				

We shall find y_{45} *i.e.* number of students with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because\ h = 10$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\,\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40}$$

$$+ \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25)$$

$$+ \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.87, \text{ on simplification.}$$