* ANGULAR MOMENTUM OPERATORS IN SPHERICAL VARIABLES

The motto of this topic is to find the eigenvalues and the eigenfunctions of L_z and $\vec{L}^2$.

$$L_z Y_{lm} = m\hbar Y_{lm}$$
$$\vec{L}^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm} \quad \cdots (1)$$

m and $l(l+1)$ are real numbers.

To write out the operators in spherical coordinates we have

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \cdots (2)$$

So that

$$\left.\begin{aligned} dx &= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi \\ dy &= \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta . \end{aligned}\right\} (3)$$

Solving these, we get

$$\left.\begin{aligned} dr &= \sin\theta\cos\phi\, dx + \sin\theta\sin\phi\, dy + \cos\theta\, dz \\ d\theta &= \frac{1}{r}(\cos\theta\cos\phi\, dx + \cos\theta\sin\phi\, dy - \sin\theta\, dz) \\ d\phi &= \frac{1}{r\sin\theta}(-\sin\phi\, dx + \cos\phi\, dy) \end{aligned}\right\} \cdots (4)$$

With the help of this equation we get

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \quad (5)$$

Finally obtaining

$$L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\frac{\partial}{\partial \phi} \quad \text{--- } (6)$$

Introducing

$$L_\pm = L_x \pm iL_y \quad \cdots \quad (7)$$

$$L_\pm = \frac{\hbar}{i}\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \pm i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right]$$

$$= \frac{\hbar}{i}\left[\pm iz\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) \mp (x \pm iy)\frac{\partial}{\partial z}\right]$$

$$= \pm\hbar\, r\cos\theta\left(\sin\theta \exp \pm i\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \exp \pm i\phi\frac{\partial}{\partial \theta} \pm \frac{i\exp \pm i\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \mp$$

$$\hbar\, r\sin\theta \exp \pm i\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$\text{---} \quad (8)$$

Thus $L_\pm = \hbar \exp \pm i\phi \left(\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right)$ ... (9)

$$L_+L_- = (L_x + iL_y)(L_x - iL_y)$$

$$= L_x^2 + L_y^2 - i[L_x, L_y] \quad \dots (10)$$

$$\therefore \boxed{\begin{aligned} \vec{L}^2 &= L_z^2 + L_+L_- + i[L_x, L_y] \\ &= L_+L_- + L_z^2 - \hbar L_z \end{aligned}} \quad \dots (11)$$

This is a second order differential operator involving θ & ϕ is obtained.

* EIGEN FUNCTIONS AND EIGEN VALUES OF L_z:

Consider $L_z Y_{lm} = m\hbar Y_{lm}$ ... (12)

using equation (6), $\frac{\partial}{\partial\phi} Y_{lm}(\theta, \phi) = im Y_{lm}(\theta, \phi)$ ... (13)

The solution is of the form $Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta) \Phi_m(\phi)$

where $\frac{d\Phi_m(\phi)}{d\phi} = im\Phi_m(\phi)$ ... (14)

The solution to this is normalised such that

$$\int_0^{2\pi} d\phi |\Phi_m|^2 = 1 \quad \dots (15)$$ is

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \dots (16)$$

A transformation $\phi \to \phi + 2\pi$ leaves the system invariant, so it is necessary that

$$e^{2\pi i m} = 1 \quad \ldots (17)$$

where m is an integer. It is not always true as the quantities that enter it into physical observables are of the type $\int_0^{2\pi} d\phi\, \psi_1^*(\phi) A \psi_2(\phi)$ with wave function $\psi(\phi)$ of the form

$$\psi(\phi) = \sum_{m=-\infty}^{\infty} C_m \frac{\exp im\phi}{\sqrt{2\pi}} \quad \ldots (18)$$

The eigenvalue equation for L_z appears in another context. Consider a classical rotator, rotating in the $x-y$ plane. If the moment of inertia is I, then energy is

$$E = \frac{L_z^2}{2I} \quad \ldots (19)$$

Hamiltonian is $H = \frac{L_z^2}{2I}$ $\ldots (20)$

The eigenfunctions are $\exp \pm im\phi$. for $E_m = \frac{\hbar^2 m^2}{2I}$ (21)
There is a degeneracy since H commutes with L_z and the two eigen functions for a given E_m correspond to the 2 senses of rotation.

For a given N particles rigidly fixed on a circle with equal angles $2\pi/N$ between neighbouring particles and if the particles are identical then the solution of the energy eigenvalue equation is

$$H\Phi_E(\phi) = E\Phi_E(\phi). \quad \ldots (22)$$

will be again $\exp \pm i\lambda\phi$. $\ldots$
The physical system is unaltered under a rotation of $2\pi/N$ radians. For $\lambda = N \times$ (an integer), the energy is i.e.

$$\boxed{E = \frac{\hbar^2 (Nm)^2}{2I}} \quad \ldots (23)$$

* RAISING AND LOWERING OPERATORS FOR ANGULAR MOMENTUM.

The eigenfunctions of the hermitian operators L_z & $\vec{L}^2$ will be orthogonal, if the eigenvalues are different and with proper normalisation we can write

$$\langle Y_{l'm'} | Y_{lm} \rangle = \delta_{ll'} \delta_{mm'} \quad \cdots \quad (24)$$

since $\langle Y_{lm} | (L_x^2 + L_y^2 + L_z^2) Y_{lm} \rangle =$
$$\langle L_x Y_{lm} | L_x Y_{lm} \rangle + \langle L_y Y_{lm} | L_y Y_{lm} \rangle + m^2 \hbar^2 \geq 0 \cdots (25)$$

it follows that
$$l(l+1) \geq 0 \quad \cdots \quad (26)$$

It is known that $\vec{L}^2 = L_+ L_- + L_z^2 - \hbar^2 L_z \cdots (27)$.
Also $\quad \vec{L}^2 = L_- L_+ + L_z^2 + \hbar L_z \quad \cdots \quad (28)$

$$\therefore \quad [L_+, L_-] = 2\hbar L_z \quad \cdots \quad (29)$$

The remaining commutation relations are
$$[L_+, L_z] = [L_x + iL_y, L_z] = -i\hbar L_y - \hbar L_x$$
$$= -\hbar L_+ \quad \cdots \quad (30)$$

and
$$[L_-, L_z] = \hbar L_- \quad \cdots \quad (31)$$

From $[\vec{L}^2, \vec{L}] = 0$, it follows that

$$[\vec{L}^2, L_\pm] = 0, \quad [\vec{L}^2, L_z] = 0 \quad \cdots \quad (32)$$

This implies that
$$\vec{L}^2 L_\pm Y_{lm} = L_\pm \vec{L}^2 Y_{lm} = l(l+1)\hbar^2 L_\pm Y_{lm} \cdots (33)$$

i.e.
$L_\pm Y_{lm}$ are also eigenfunctions of $\vec{L}^2$ with the

eigenvalue characterized by l.

$$L_z L_+ Y_{lm} = (L_+ L_z + \hbar L_+) Y_{lm}$$
$$= m\hbar L_+ Y_{lm} + \hbar L_+ Y_{lm}$$
$$= \hbar(m+1) L_+ Y_{lm} \quad \cdots (34)$$

So that $L_+ Y_{lm}$ is also an eigenfunction of L_z but with m-value increased by unity.

Ill^{ly}

$$L_z L_- Y_{lm} = \hbar(m-1) L_- Y_{lm} \quad \cdots (35)$$

So that $L_- Y_{lm}$ is an eigen function of L_z with m-value lowered by unity. ∴ $L_\pm$ are called raising and lowering opertors respectively.

$$L_\pm Y_{lm} = C_\pm(l, m) Y_{lm\pm 1} \quad \cdots (36)$$

It follows from the hermicity of L_x & L_y that

$$L_\pm^\dagger = (L_x \pm iL_y)^\dagger = L_x \mp iL_y = L_\mp \quad \cdots (37)$$

Hence a consequence of $\langle L_\pm Y_{lm} | L_\pm Y_{lm} \rangle \geq 0 \quad \cdots (38)$

is that $\langle Y_{lm} | L_\mp L_\pm Y_{lm} \rangle \geq 0 \quad \cdots (39)$

∴ (27) & (28) imply that

$$\langle Y_{lm} | (\vec{L}^2 - L_z^2 \pm \hbar L_z) Y_{lm} \rangle \geq 0 \quad \cdots (40)$$

i.e.
$$\left.\begin{aligned} l(l+1) &\geq m^2 + m \\ l(l+1) &\geq m^2 - m \end{aligned}\right\} \quad \cdots (41)$$

Since $l(l+1) \geq 0$, we can take $l \geq 0$ without loss of generality.

(ii) show that $-l \leq m \leq l$ ... (42)

If there is minimum value of $m = (m_-)$, then for the corresponding eigenstate $L_- Y_{lm_-} = 0$ ... (43)

Calculating m_- using eq^n (27) and applying it to Y_{lm_-},

$$l(l+1)\hbar^2 = m_-^2 \hbar^2 - m_- \hbar^2 \quad \cdots (44)$$

III^ly if there is a maximum value of $(m = m_+)$ then

$$L_+ Y_{lm_+} = 0 \quad \cdots (45)$$

$$l(l+1)\hbar^2 = m_+^2 \hbar^2 + m_+ \hbar^2 \quad \cdots (46)$$

Hence

$$m_- = l \qquad m_+ = +l \quad \cdots (47)$$

Since the maximum value is to be reached from the minimum value by unit steps, there are $(2l+1)$ states, i.e (a) $2l+1$ is an integer.

(b) m can take on the values $m = -l, -l+1, -l+2, \ldots, l-1, l$.

Calculating the coefficients $C_\pm(l,m)$ as in eq^n (36),

$$|C_\pm(l,m)|^2 \langle Y_{l,m\pm1} | Y_{l,m\pm1} \rangle = \langle L_\pm Y_{lm} | L_\pm Y_{lm} \rangle.$$

$$= \hbar^2 [l(l+1) - m(m\pm1)]$$

With a convenient choice of phase,

$$C_+(l,m) = \hbar\sqrt{l(l+1) - m(m+1)} = \hbar\sqrt{(l-m)(l+m+1)}$$

$$C_-(l,m) = \hbar\sqrt{l(l+1) - m(m-1)} = \hbar\sqrt{(l+m)(l-m+1)}$$

$$\cdots (48)$$

THE SPHERICAL HARMONICS:

For obtaining the expressions for the eigenfunctions in terms of spherical angles θ and ϕ, we use the explicit form of the operators L_z and $L_\pm$

$$Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\exp im\phi \quad \cdots (1)$$

The solution for $L_+ Y_{lm_f} = 0$ can be found as

$$\Theta_{ll}(\theta) = (\sin\theta)^l \quad \cdots (2)$$

An arbitrary state is obtained by the lowering procedure

$$Y_{lm}(\theta,\phi) = C(L_-)^{l-m}(\sin\theta)^l \exp il\phi \quad \cdots (3)$$

Consider

$$L_- Y_{ll}(\theta,\phi) = \hbar \exp -i\phi\left(\frac{-\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)(\sin\theta)^l \exp il\phi$$

$$= \hbar e^{i(l-1)\phi}\left(\frac{-\partial}{\partial\theta} - l\cot\theta\right)(\sin\theta)^l.$$

For an arbitrary function $f(\theta)$

$$\left(\frac{d}{d\theta} + l\cot\theta\right)f(\theta) = \frac{1}{(\sin\theta)^l}\frac{d}{d\theta}\left[(\sin\theta)^l f(\theta)\right] \cdots (4)$$

It can be obtained that

$$Y_{l,l-1} = \frac{c'\exp i(l-1)\phi}{(\sin\theta)^l}\left(\frac{-d}{d\theta}\right)\left[(\sin\theta)^l(\sin\theta)^l\right]. \cdots (5)$$

The general form is

$$Y_{\ell m} = \frac{C \exp im\phi}{(\sin\theta)^m} \left(\frac{d}{du}\right)^{\ell-m} [(1-u^2)^{\ell}] \quad \ldots (6)$$

where $u = \cos\theta$ $\quad \frac{d}{du} = \frac{-1}{\sin\theta \left(\frac{d}{d\theta}\right)}$

→ The eigenfunctions are to be normalised.
The range for this spherical functions is $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi$ and integral over the surface of the sphere is

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \quad \ldots (7)$$

we must impose

$$\langle Y_{\ell m} | Y_{\ell m} \rangle = 1 = \int_0^{2\pi} d\phi \int_{-1}^{1} du \, |C|^2 \left[\frac{1}{(1-u^2)^{m/2}} \left(\frac{d}{du}\right)^{\ell-m} (1-u^2)^{\ell}\right]^2.$$

∴ The normalised $Y_{\ell m}(\theta,\phi)$ with the phases can be written as

$$Y_{\ell m}(\theta,\phi) = (-1)^m \sqrt{\left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}\right]} P_\ell^m(\cos\theta) e^{im\phi} \quad \ldots (8)$$

for $m \geq 0$,

$$Y_{\ell,-m} = (-1)^m Y^*_{\ell m} \quad \ldots (9)$$

The associated Legendre Polynomials are given by $(m \geq 0)$

$$P_\ell^m(u) = (-1)^{\ell+m} \frac{(\ell+m)!}{(\ell-m)!} \frac{(1-u^2)^{-m/2}}{2^\ell \ell!} \left(\frac{d}{du}\right)^{\ell-m} (1-u^2)^{\ell} \quad \ldots (10)$$

with the value for negative m obtained from.

$$P_l^{-m}(u) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(u) \quad \ldots (11)$$

$m = l$ or $-l$ we get

$$Y_{ll}(\theta, \phi) = K (\sin\theta)^l \exp il\phi$$

K = constant.

The probability distribution in polar angle w.r.t z-axis is given by.

$$|Y_{ll}|^2 = K^2 (\sin\theta)^{2l} \quad \ldots (12)$$

For $m = l$, L_z has its largest possible value so that $L^2 = L_z^2$. In classical limit $l >>> 1$

$$\frac{\vec{L}^2 - L_z^2}{\vec{L}^2} = \frac{1}{l} \to 0.$$

i.e it is possible to line up the angular momentum in a particular direction.

List of few of the eigenstates are as follows:

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi} \sin\theta$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}}\, e^{2i\phi} \sin^2\theta$$