Born Approximation

Pre-requisite :—

- Let us start with time-indep. Schrödinger eqn.,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

or,

$$(\nabla^2 + k^2)\psi = Q \quad \text{—— (1)}$$

where

$$Q = \frac{2mV}{\hbar^2}\psi$$

$$k^2 = \frac{2mE}{\hbar^2}$$

→ Eqn. (1) is in the form of "Helmholtz eqn.", but the inhomogeneous term of the R.H.S depends on 'ψ'.

Normally, Helmholtz eqn. can be solved using Green's function technique.

$$(\nabla^2+k^2)\,G(r) = \delta^3(r) \quad \text{--- (2)}$$

$G(r)$ → Green's func.

→ $\psi(r) = \int G(r-r_0)\, Q(r_0)\, d^3r_0$

and thus, $G(r) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\cdot\vec{r}}\, g(k)\, d^3k'$ --- (3)

☑ Cauchy's Integral formula

$$\oint \frac{f(z)}{(z-z_0)}\, dz = 2\pi i\, f(z_0) \quad \text{--- (4)}$$

if z_0 Lies within the contour (else it is zero).

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Using these, we find that sol^n to eqn. (2) is

$$G(r) = -\frac{e^{ikr}}{4\pi r} \quad \text{--- (5)}$$

Thus, the general sol^n to Schrödinger eqn. takes the form,

$$\psi(r) = \underbrace{\psi_0(r)}_{\text{free particle solution}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|r-r_0|}}{|r-r_0|} V(r_0)\psi(r_0)\, d^3r_0 \quad \text{--- (6)}$$

1^{st} Born approximation :-

If $|r| >> |r_0|$,

$$|\vec{r}-\vec{r_0}|^2 = r^2 + r_0^2 - 2\vec{r}\cdot\vec{r_0}$$

$$\simeq r^2\left(1 - 2\frac{\vec{r}\cdot\vec{r_0}}{r^2}\right)$$

and $|\vec{r}-\vec{r_0}| \simeq r - \hat{r}\cdot\vec{r_0}$

Now, $\psi_0(r) = \underbrace{Ae^{ikz}}_{\text{incident plane wave}}$

Now, using eqn. (6)

$$\psi(r) = Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\cdot\vec{r_0}} V(r_0)\psi(r_0)\, d^3r_0$$

where one can identify,

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k}\cdot\vec{r_0}} V(r) \times \psi(r_0)\, d^3r_0$$

For localized potential, $V(r_0)$ localized about $r_0 = 0$,

$$\psi(r_0) \simeq \psi_0(r_0) = A e^{ikz_0}$$

$$\therefore \boxed{f(\theta, \varphi) = \frac{-m}{2\pi\hbar^2} \int e^{i(\bar{k} - \bar{k}') \cdot \bar{r}_0} \psi(r_0)\, d^3 r_0}$$

For low energy scattering

$$f(\theta, \varphi) \simeq \frac{-m}{2\pi\hbar^2} \int V(r)\, d^3 r$$