QM-II – Unit-2 (Dr. Tamishree Basak)

OPERATORS, MATRICES AND SPIN.

MATRIX REPRESENTATION OF HARMONIC OSCILLATOR OPERATORS.

The eigenvalue problem of angular momentum

$$\left.\begin{aligned} L^2 Y_{\ell m} &= \hbar^2 \ell(\ell+1) Y_{\ell m} \\ L_z Y_{\ell m} &= \hbar m Y_{\ell m} \end{aligned}\right\} \cdots (1)$$

can be solved by operator methods.

For harmonic oscillator, the state u_n is defined by

$$u_n = \frac{1}{(n!)^{1/2}} (A^+)^n u_0 \quad \cdots \quad (2) \text{ for which}$$

$$H u_n = \hbar\omega\left(n + \frac{1}{2}\right) u_n. \quad \cdots \quad (3)$$

The calculation of action of the raising and lowering operators on u_n can be done with the help of

$$A^+ u_n = \sqrt{n+1}\, u_{n+1} \quad \cdots \quad (4) \text{ and}$$

$$A u_n = \sqrt{n}\, u_{n-1} \quad \cdots \quad (5).$$

Also $\langle u_m | u_n \rangle = \delta_{mn}$ ... (6).

Taking scalar products of (3) to (5) with u_m;

$$\left.\begin{aligned} \langle u_m | H u_n \rangle &\equiv \langle u_m | H | u_n \rangle = \left(n+\frac{1}{2}\right)\hbar\omega\, \delta_{mn}. \\ \langle u_m | A^+ u_n \rangle &\equiv \langle u_m | A^+ | u_n \rangle = \sqrt{n+1}\, \delta_{m,n+1} \\ \langle u_m | A u_n \rangle &\equiv \langle u_m | A | u_n \rangle = \sqrt{n}\, \delta_{m,n-1} \end{aligned}\right\} (7)$$

The more symmetric notation $\langle u_i | O | u_j \rangle \equiv \langle u_i | O u_j \rangle$ is used ... (8)

These quantities may be arranged in arrays called matrices.

The matrix notation is M_{ij} where i = row and j = column. Writing the scalar product $\langle u_m | H | u_n \rangle$ as H_{mn}, we find that

$$H = \hbar\omega \begin{pmatrix} 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 3/2 & 0 & 0 & \cdots \\ 0 & 0 & 5/2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \cdots (9)$$

Similarly for $A^\dagger$ → raising or creating operator

$$A^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} \quad \cdots (10)$$

Illy for A → lowering or annihilating operator.

$$A = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \cdots (11)$$

Calling the array $\langle u_m | F | u_n \rangle$ where F is any operator and u_i are any complete set, a matrix representation of F in the basis provided by u_i.

The product of 2 matrices satisfies

$$(FG)_{ij} = \sum_n (F)_{in} (G)_{nj} \quad \cdots (12)$$

To verify this relation for this 'matrix representations' of the operators F & G, consider the state GU_j and expanding it $GU_j = \sum_n C_n U_n \quad \cdots (13)$

The coefficients C_n are given by

$$C_n = \langle u_n | G | u_j \rangle \quad \cdots (14)$$

Hence, $\langle u_i | F G | u_j \rangle = \langle u_i | F \left(\sum_n C_n u_n \right) \rangle$

$$= \sum_n C_n \langle u_i | F | u_n \rangle$$

$$= \sum_n \langle u_i | F | u_n \rangle \langle u_n | G | u_j \rangle \quad \cdots (15)$$

which is same as equation (12)

Rewriting $\langle u_i | F | u_n \rangle = F_{in} \quad \cdots (16)$

The completeness of the basis vectors $|u_n\rangle$ may be expressed in the form $\sum_n |u_n\rangle \langle u_n| = 1 \quad \cdots (17)$

Further justification for the matrix connection comes from the relation

$$\langle u_m | F | u_n \rangle^* = \langle F u_n | u_m \rangle = \langle u_n | F^\dagger | u_m \rangle \quad \cdots (18)$$

F → matrix $F^\dagger$ → Hermitian conjugate

$$(F^\dagger)_{nm} = F^*_{mn} \quad \cdots (19)$$

The eigen states of the harmonic oscillator Hamiltonian diagonalize the matrix representing H.

MATRIX REPRESENTATION OF ANGULAR MOMENTUM OPERATORS.

Consider the matrix elements of L_z between different angular momentum states $\langle l'm' | L_z | lm \rangle$.

$[L^2, L_z] = 0$. this implies that

$$0 = \langle l'm' | [L^2, L_z] | lm \rangle$$

$$= \langle L^2 l'm' | L_z | lm \rangle - \langle l'm' | L_z | L^2 lm \rangle \quad \cdots (20)$$

$$= \hbar^2 \{ l'(l'+1) - l(l+1) \} \langle l'm' | L_z | lm \rangle$$

We conclude that if $l' \neq l$ then $\langle l'm'|L_z|lm\rangle$ vanishes.

L_z, $L_\pm$ have matrix elements between states that have the same total angular momentum quantum numbers.

$$\langle l'm'|L_z|lm\rangle = \hbar m \delta_{m'm} \quad \cdots (21).$$

The above equation reads the abbreviated notation for a fixed state with states in which only the m-value is variable.

$$\langle lm'|L_\pm|lm\rangle = \hbar[l(l+1) - m(m\pm1)]^{1/2} \delta_{m',m\pm1} \quad \cdots (22)$$

This leads to the matrix representation.

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \cdots 23a$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \cdots 23b \qquad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \cdots 23c$$

The commutation relations of the matrices are:

$$[L_+, L_-] = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} - \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= 2\hbar L_z \quad \cdots (24)$$

The general relations between states can also be written in matrix representation.

$$\psi = A\phi \quad \cdots \quad (25)$$

Taking scalar product of this with any member of a complete set u_i,

$$\langle u_i|\psi\rangle = \langle u_i|A\phi\rangle \quad \cdots \quad (26)$$

The insertion of the unit operator in the form in eq" (17) between A & ϕ yields

$$\langle u_i|\psi\rangle = \sum_n \langle u_i|A|u_n\rangle\langle u_n|\phi\rangle \quad \cdots \quad (27)$$

Writing $\langle u_n|\phi\rangle$ as a column vector a_{ij}

$$\langle u_n|\phi\rangle \rightarrow \begin{pmatrix} \langle u_1|\phi\rangle \\ \langle u_2|\phi\rangle \\ \langle u_3|\phi\rangle \\ \vdots \end{pmatrix} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix} \quad \cdots \quad (28)$$

Similarly

$$\langle u_n|\psi\rangle = \begin{matrix} \langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \langle u_3|\psi\rangle \\ \vdots \end{matrix} \equiv \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \quad \cdots \quad (29)$$

Then the matrix representation of (25) is

$$\beta_i = \sum_n A_{in}\alpha_n \quad \cdots \quad (30)$$

Thus matrices represent operators and column vectors represent states. The scalar product $\langle\phi|u_n\rangle = \langle u_n|\phi\rangle^*$ is written conventionally in the form of a row.

$$\langle\phi|u_n\rangle \rightarrow (\alpha_1^*, \alpha_2^*, \alpha_3^* \ldots) \quad \cdots \quad (31)$$

he scalar product $\langle\phi|\psi\rangle$ can be written as

$$\left.\begin{aligned}\langle\phi|\psi\rangle &= \sum_n \langle\phi|u_n\rangle\langle u_n|\psi\rangle \\ &= \sum_n \alpha_n^* \beta_n\end{aligned}\right\} \cdots (32)$$

An eigenvalue equation is a special case of (25).

$$A\phi = a\phi \quad \cdots (33)$$

$$\sum_n A_{in}\alpha_n = a\alpha_i \quad \cdots (34) \rightarrow \text{The matrix form}$$

This is equivalent to

$$\begin{pmatrix} A_{11}-a & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22}-a & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33}-a & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \vdots\end{pmatrix} = 0 \quad \cdots (35)$$

The solution of this equation is non-trivial only if the determinant of the matrix vanishes.

$$\det|A_{in} - a\delta_{in}| = 0 \quad \cdots (36)$$

This would not be so simple for infinite matrices.

THE SPIN OPERATOR AND ITS MATRIX REPRESENTATION

There is always an alternative to represent operators by functions and differentials, since not all operators can be represented in that way. The simplest example is that corresponding to the angular momentum $\ell = \frac{1}{2}$.

The equations $\Theta_{\ell\ell}(\theta) = (\sin\theta)^\ell$ and

$$Y_{\ell,-m} = (-1)^m Y^*_{\ell m}$$ tells us that

$$Y_{\frac{1}{2},\pm\frac{1}{2}} = C_\pm \sqrt{\sin\theta}\; e^{\pm i\phi/2} \quad \cdots (37)$$

$$\text{so } Y_{\ell,\ell-\ell} = c' \frac{e^{-i(\ell-1)}}{(\sin\theta)^\ell}\left(-\frac{d}{d\theta}\right)[(\sin\theta)^\ell(\sin\theta)^\ell] \ldots$$

This equation computes to:

$$L_- Y_{\frac{1}{2},\frac{1}{2}} \propto \frac{\cos\theta}{\sqrt{\sin\theta}} e^{-i\phi/2} \quad \ldots (38)$$

This is not proportional to $Y_{\frac{1}{2},-\frac{1}{2}}$. This indicates problems with extending the established rules to $\ell = \frac{1}{2}$ and thus turn to matrix representation.

We talk about spin $s = \frac{1}{2}$, ℓ = orbital angular momentum associated with $\vec{r} \times \vec{p}$. The spin operators are S_x, S_y & S_z. They are defined by their commutation relations.

$$[S_x, S_y] = i\hbar S_z \quad \ldots (39)$$

Representing them by 2×2 matrices, Equation (21) yields

$$S_z = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \ldots (40) \text{ and}$$

(22) gives $S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ $S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ldots (41)$

This representation is written as

$$S = \frac{1}{2}\hbar\sigma \quad \ldots (42) \text{ where}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ldots (43)$$

These are the Pauli matrices. They satisfy the commutation relations

$$[\sigma_x, \sigma_y] = 2i\sigma_z \quad \ldots (44).$$

Also they satisfy $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1 \quad \ldots (45)$

The Pauli matrices also anticommute

$$\left.\begin{aligned} \sigma_x\sigma_y &= -\sigma_y\sigma_x \\ \sigma_z\sigma_x &= -\sigma_x\sigma_z \\ \sigma_y\sigma_z &= -\sigma_z\sigma_y \end{aligned}\right\} \ldots (46)$$

These relations are peculiar to the spin $\frac{1}{2}$ representations and do not hold for the $\ell = 1$ matrices for example. The eigen states of S_z will be represented by a 2-component column vector, which we call spinor. To find these eigenspinors, we solve

$$S_z \begin{pmatrix} u \\ v \end{pmatrix} = \pm \frac{1}{2} \hbar \begin{pmatrix} u \\ v \end{pmatrix} \quad \cdots (47) \quad \text{i.e.}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u \\ -v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix} \cdots (48)$$

The plus eigenisolution has $v = 0$ and the minus eigensolution has $u = 0$. ...

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \cdots (49)$$ for the eigenspinors corresponding to spin up $[S_z = +(\frac{1}{2})\hbar]$ and spin down $[S_z = -(\frac{1}{2})\hbar]$.

An arbitrary spinor can be expanded in this complete set

$$\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \alpha_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \cdots (50)$$

and the expansion postulate yields the interpretation that $|\alpha_+|^2$ and $|\alpha_-|^2$, when properly normalised so,

$$|\alpha_+|^2 + |\alpha_-|^2 = 1 \quad \cdots (51)$$ yield the probabilities that a measurement of S_z on the state $\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$ yields $+(\frac{1}{2})\hbar$ and $-(\frac{1}{2})\hbar$ respectively.

It is not necessary to keep S_z diagonal. If we look for the eigenstates of the operator $S_x \cos\phi + S_y \sin\phi$, solve

$$(S_x \cos\phi + S_y \sin\phi) \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \hbar \lambda \begin{pmatrix} u \\ v \end{pmatrix} \quad \cdots (52) \quad \text{i.e.,}$$

$$\begin{pmatrix} 0 & \cos\phi - i\sin\phi \\ \cos\phi + i\sin\phi & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$ which implies

$$v e^{-i\phi} = \lambda u, \quad u e^{i\phi} = \lambda v \quad \cdots (53)$$

Taking the products of the left- and right hand sides in the 2 equations we find that.

$$uv(\lambda^2 - 1) = 0 \quad \cdots \quad (54)$$

Hence, $\lambda = \pm 1 \quad \cdots \quad (55)$

The eigenvector corresponding to $\lambda = 1$ satisfies $v = e^{i\phi} u$. and the normalised form

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}.$$

We can multiply a state vector by an arbitrary phase factor which we choose to be $e^{-i\phi/2}$. This yields

$$u_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix} \cdots (56) \text{ and similarly}$$

the eigenstate corresponding to $\lambda = -1$ can be written in the form $u_- = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix} \cdots (57)$.

which is easily seen to be orthogonal to u_+:

$$u_+^* u_- = \frac{1}{2} (e^{i\phi/2}, e^{-i\phi/2}) \begin{pmatrix} e^{-i\phi/2} \\ -e^{i\phi/2} \end{pmatrix} \cdots (58)$$

$$= 0$$

If we change ϕ to $\phi + 2\pi$ the solutions change sign. This is the characteristic of odd half-integer spin wave functions (fermion states) although this does not violate quantum mechanics, since -1 is just a phase factor, it does mean that no classical macroscopic wave packet can be constructed that has odd half-integral angular momentum.

Given an arbitrary state α, the expectation value of S can be calculated.

$$\langle \alpha | S | \alpha \rangle = \sum_i \sum_j \langle \alpha | i \rangle \langle i | S | j \rangle \langle j | \alpha \rangle$$

Or equivalently $(\alpha_+^*, \alpha_-^*) S \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$

thus

$$\langle S_x \rangle = (\alpha_+^*, \alpha_-^*) \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

$$= \frac{1}{2}\hbar (\alpha_+^*, \alpha_-^*) \begin{pmatrix} \alpha_- \\ \alpha_+ \end{pmatrix} = \frac{1}{2}\hbar (\alpha_+^* \alpha_- + \alpha_-^* \alpha_+)$$

$$\langle S_y \rangle = \frac{1}{2}\hbar (\alpha_+^*, \alpha_-^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

$$= \frac{1}{2}\hbar (\alpha_+^*, \alpha_-^*) \begin{pmatrix} -i\alpha_- \\ i\alpha_+ \end{pmatrix} = \frac{-i\hbar}{2} (\alpha_+^* \alpha_- - \alpha_-^* \alpha_+)$$

$$\langle S_z \rangle = \frac{1}{2}\hbar (\alpha_+^*, \alpha_-^*) \begin{pmatrix} \alpha_+ \\ -\alpha_- \end{pmatrix} = \frac{1}{2}\hbar (|\alpha_+|^2 - |\alpha_-|^2)$$

→ All of these are not real, as expected for hermitian operators.

THE INTRINSIC MAGNETIC MOMENT OF SPIN $\frac{1}{2}$ PARTICLES.

When an electron is localised at a crystal lattice site e.g. it is often possible to treat the spin as the only degree of freedom that the e possess. The electron will have an intrinsic magnetic dipole moment by virtue of its spin, and that magnetic moment is

$$M = \frac{-eg}{2mc} S \quad \cdots (60)$$

where g, the gyromagnetic ratio is very close to 2,

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \cdots\right) = 2.0023192 \quad \cdots (61)$$

and m is the electron mass and α is the fine structure constant. For such a localised electron, the Hamiltonian in the presence of an external magnetic field B is just the potential energy

$$H = -\vec{M} \cdot \vec{B} = \frac{egh}{4mc} \vec{\sigma} \cdot \vec{B} \quad \cdots (62)$$

The Schrödinger equation for the state $\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix}$

$$i\hbar \frac{d\psi(t)}{dt} = \frac{egh}{4mc} \vec{\sigma} \cdot \vec{B}\, \psi(t) \quad \cdots (63)$$

If $\vec{B}$ is taken to define the z-axis and if we wri

$$\psi(t) = \begin{pmatrix} \alpha_+(t) \\ \alpha_-(t) \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} \quad \cdots (64)$$ then

the equation becomes

$$\hbar\omega \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \frac{egh B}{4mc} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} \quad \cdots (65)$$

The solution correspond to different frequencies ω. We have, for $\omega = \frac{egB}{4mc}$, $\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and for $\omega = -\left(\frac{egB}{4mc}\right)$,

$\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus if the initial state is

$$\psi(0) = \begin{pmatrix} a \\ b \end{pmatrix} \quad \cdots (66)$$ then the state

at a later time will be

$$\psi(t) = \begin{pmatrix} a\, e^{-i\omega t} \\ b\, e^{i\omega t} \end{pmatrix} \qquad \omega = \frac{geB}{4mc} \quad \cdots (67)$$

Suppose that at $t = 0$ the spin is an eigenstate of S_x with eigenvalue $+\left(\frac{1}{2}\right)\hbar$ that is, it "points in the x-direction." This means that

$$\frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\hbar \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then at a later time

$$\langle S_x \rangle = \frac{1}{2}\hbar \frac{1}{\sqrt{2}} \left(e^{i\omega t}, e^{-i\omega t}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$$

$$= \frac{\hbar}{4} \left(e^{i\omega t}, e^{-i\omega t}\right) \begin{pmatrix} e^{i\omega t} \\ e^{-i\omega t} \end{pmatrix} = \frac{\hbar}{2} \cos 2\omega t \quad \cdots (68)$$

Similarly, $\langle S_y \rangle = \frac{1}{2}\hbar \frac{1}{\sqrt{2}} \left(e^{i\omega t}, e^{-i\omega t}\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$

$$= \frac{\hbar}{4} \left(-i e^{2i\omega t} + i e^{-2i\omega t}\right) \quad (69)$$

$$= \frac{\hbar}{2} \sin 2\omega t$$

thus the spin processes about the direction of B with frequency

$$2\omega = \frac{egB}{2mc} \simeq \frac{eB}{mc} \equiv \omega_c \quad \cdots \quad (70)$$

this is the cyclotron frequency. This precision occurs is spin initially points at an arbitrary angle θ with respect to the z-axis. For a magnetic field of the order of 10^4 gauss (1 T),

$$\omega_c = \frac{(4.8 \times 10^{-10}\ \text{esu})(10^4\ \text{gauss})}{(0.9 \times 10^{-27}\ \text{g})(3 \times 10^{10}\ \text{cm/sec})}$$

$$\approx 1.8 \times 10^{11}\ \text{rad/sec}$$

THE ADDITION OF TWO SPINS

Suppose we have two electrons, whose spins are described by the operators $\mathbf{S}_1$ and $\mathbf{S}_2$. Each of these sets of operators satisfies the standard angular momentum commutation relations

$$[S_{1x}, S_{1y}] = i\hbar S_{1z}$$

and so on,

$$[S_{2x}, S_{2y}] = i\hbar S_{2z} \tag{15-1}$$

and so on, but the two sets of operators commute with each other, since the degrees of freedom associated with different particles are independent, that is,

$$[\mathbf{S}_1, \mathbf{S}_2] = 0 \tag{15-2}$$

Let us now define the total spin $\mathbf{S}$ by

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \tag{15-3}$$

The commutation relations obeyed by the components of $\mathbf{S}$ are

$$\begin{aligned} [S_x, S_y] &= [S_{1x} + S_{2x}, S_{1y} + S_{2y}] \\ &= [S_{1x}, S_{1y}] + [S_{2x}, S_{2y}] \\ &= i\hbar(S_{1z} + S_{2z}) = i\hbar S_z \end{aligned} \tag{15-4}$$

and so on. We are therefore justified in calling S the total *spin*. We may now determine the eigenvalues and eigenfunctions of $\mathbf{S}^2$ and S_z.

The two-spin system actually has four states. If we denote the spinor of the first electron by $\chi_{\pm}^{(1)}$, so that

$$\begin{aligned} \mathbf{S}_1^2\chi_{\pm}^{(1)} &= \tfrac{1}{2}(\tfrac{1}{2}+1)\,\hbar^2\chi_{\pm}^{(1)} \\ S_{1z}\chi_{\pm}^{(1)} &= \pm\tfrac{1}{2}\hbar\chi_{\pm}^{(1)} \end{aligned} \tag{15-5}$$

and similarly for the spinor $\chi_{\pm}^{(2)}$ of the second electron, then the four states are

$$\chi_{+}^{(1)}\chi_{+}^{(2)},\ \chi_{+}^{(1)}\chi_{-}^{(2)},\ \chi_{-}^{(1)}\chi_{+}^{(2)},\ \chi_{-}^{(1)}\chi_{-}^{(2)} \tag{15-6}$$

The eigenvalues of S_z for the four states are

$$\begin{aligned} S_z\chi_{\pm}^{(1)}\chi_{\pm}^{(2)} &= (S_{1z}+S_{2z})\chi_{\pm}^{(1)}\chi_{\pm}^{(2)} \\ &= (S_{1z}\chi_{\pm}^{(1)})\,\chi_{\pm}^{(2)} + \chi_{\pm}^{(1)}(S_{2z}\chi_{\pm}^{(2)}) \end{aligned}$$

that is,

$$\begin{aligned} S_z\chi_{+}^{(1)}\chi_{+}^{(2)} &= \hbar\chi_{+}^{(1)}\chi_{+}^{(2)} \\ S_z\chi_{+}^{(1)}\chi_{-}^{(2)} &= S_z\chi_{-}^{(1)}\chi_{+}^{(2)} = 0 \\ S_z\chi_{-}^{(1)}\chi_{-}^{(2)} &= -\hbar\chi_{-}^{(1)}\chi_{-}^{(2)} \end{aligned} \tag{15-7}$$

There are two states with m-value 0. One might expect that one linear combination of them will form an $S = 1$ state, to form a triplet with the $m = 1$ and $m = -1$ states, and the orthogonal combination will form a singlet $S = 0$ state. To check this expectation, let us construct the lowering operator

$$S_- = S_{1-} + S_{2-} \tag{15-8}$$

and apply this to the $m = 1$ state. This should give us the $m = 0$ state that belongs to the $S = 1$ triplet, aside from a coefficient in front. Indeed, using the fact that

$$S_-^{(i)}\chi_+^{(i)} = \hbar\chi_-^{(i)} \tag{15-9}$$

which can be established by noting that

$$\tfrac{1}{2}\hbar\left[\begin{pmatrix}0&1\\1&0\end{pmatrix} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right]\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\1\end{pmatrix} \tag{15-10}$$

we get

$$\begin{aligned} S_-\chi_+^{(1)}\chi_+^{(2)} &= (S_{1-}\chi_+^{(1)})\,\chi_+^{(2)} + \chi_+^{(1)}S_{2-}\chi_+^{(2)} \\ &= \hbar\chi_-^{(1)}\chi_+^{(2)} + \hbar\chi_+^{(1)}\chi_-^{(2)} \\ &= \sqrt{2}\hbar\,\frac{\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}}{\sqrt{2}} \end{aligned} \tag{15-11}$$

The linear combination has been normalized, and the compensating factor in front, $\sqrt{2}\hbar$, agrees with what one would expect from (11-36) and (11-48) with $l = m = 1$. If we now apply S_- to this linear combination, and note that

$$S_-^{(i)}\chi_-^{(i)} = 0 \tag{15-12}$$

we get

$$\begin{aligned} S_- \frac{\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}}{\sqrt{2}} &= \frac{\hbar}{\sqrt{2}}(\chi_-^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_-^{(2)}) \\ &= \sqrt{2}\hbar\chi_-^{(1)}\chi_-^{(2)} \end{aligned} \tag{15-13}$$

as we should, for an angular momentum state $S = 1$. The remaining state, constructed to be orthogonal to (15-11) and properly normalized, is

$$\frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}) \tag{15-14}$$

and because it has no partners, we conjecture that it is an $S = 0$ state. In order to check this, we compute $\mathbf{S}^2$ for the two states

$$X_\pm = \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} \pm \chi_-^{(1)}\chi_+^{(2)}) \tag{15-15}$$

We have

$$\begin{aligned} \mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \\ &= \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+} \end{aligned} \tag{15-16}$$

First of all,

$$\begin{aligned} \mathbf{S}_1^2 X_\pm &= \frac{1}{\sqrt{2}}(\chi_-^{(2)}\mathbf{S}_1^2\chi_+^{(1)} \pm \chi_+^{(2)}\mathbf{S}_1^2\chi_-^{(1)}) \\ &= \tfrac{3}{4}\hbar^2 X_\pm \end{aligned} \tag{15-17}$$

and similarly

$$\mathbf{S}_2^2 X_\pm = \tfrac{3}{4}\hbar^2 X_\pm \tag{15-18}$$

Next, we calculate

$$2S_{1z}S_{2z}X_\pm = 2(\tfrac{1}{2}\hbar)(-\tfrac{1}{2}\hbar)\, X_\pm = -\tfrac{1}{2}\hbar^2 X_\pm \tag{15-19}$$

Finally

$$\begin{aligned} (S_{1+}S_{2-} + S_{1-}S_{2+})\, X_\pm = \frac{1}{\sqrt{2}}&(S_{1+}\chi_+^{(1)}S_{2-}\chi_-^{(2)} + S_{1-}\chi_+^{(1)}S_{2+}\chi_-^{(2)} \\ &\pm S_{1+}\chi_-^{(1)}S_{2-}\chi_+^{(2)} \pm S_{1-}\chi_-^{(1)}S_{2+}\chi_+^{(2)}) \end{aligned}$$

which, with the help of (15-9) and (15-12) yields

$$(S_{1+}S_{2-} + S_{1-}S_{2+})\, X_{\pm} = \pm\hbar^2 X_{\pm} \tag{15-20}$$

Thus

$$\begin{aligned} \mathbf{S}^2 X_{\pm} &= \hbar^2(\tfrac{3}{4} + \tfrac{3}{4} - \tfrac{1}{2} \pm 1)\, X_{\pm} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \hbar^2 X_{\pm} \\ &= \hbar^2 S(S+1)\, X_{\pm} \end{aligned} \tag{15-21}$$

with $S = 1$ and 0 corresponding to the $\pm$ states.

What we have shown is that the totality of the four states of two spin 1/2 particles may be recombined into a triplet and into a singlet total spin state. It is important to note that the two descriptions are entirely equivalent. In one case we have as our complete set of commuting observables $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z}, and S_{2z}. In the other case we have as our complete set of commuting observables $\mathbf{S}^2$, S_z, $\mathbf{S}_1^2$, $\mathbf{S}_2^2$. By the expansion theorem, any function can be expanded in terms of a complete set of eigenstates. *What we have demonstrated here is the expansion of the eigenstates of the second set of observables in terms of the complete set of states of the first set of observables.* This is quite analogous to the expression of the eigenstates of the hydrogen atom in terms of the eigenstates of the momentum operator, in which the coefficients (the analogs of the $1/\sqrt{2}$'s here) are the momentum-space wave functions. It is a simple exercise to invert the process and to find the products of the $\chi^{(1)}\chi^{(2)}$ in terms of triplet and singlet combinations.

In physical problems it frequently happens that to first approximation two sets of completely commuting observables are equally useful in the construction of eigenstates. In next approximation, when additional terms in the Hamiltonian are taken into account, only one of these sets remains useful. A simple example occurs in low energy nuclear physics.

In early studies of the potential $V(r)$ that describes the interaction between neutrons and protons at low energies, it became clear that the strength of the interaction depended on whether the two interacting particles were in a total spin $S = 1$ state or a spin $S = 0$ state. For example, a deuteron had $S = 1$, while the corresponding $S = 0$ state of a neutron and a proton did not bind. This can be described in terms of a spin-dependent potential. Suppose we have

$$V(r) = V_1(r) + \frac{1}{\hbar^2}\, \mathbf{S}_1 \cdot \mathbf{S}_2 V_2(\mathbf{r}) \tag{15-22}$$

we can easily see that S_{1z} and S_{2z} do not commute with the second term, so that the eigenstates of H containing this potential cannot just be simple products of eigenstates of S_{1z} and S_{2z}. If we observe, however, that

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \tfrac{1}{2}(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2) \tag{15-23}$$

so that this term can be replaced by the eigenvalue, when acting on an eigenfunction of $\mathbf{S}^2$, $\mathbf{S}_1^2$, and $\mathbf{S}_2^2$, then

$$\begin{aligned} V(r) &= V_1(r) + \frac{1}{2} V_2(r) \left[S(S+1) - \frac{3}{2} \right] \\ &= V_1(r) + \frac{1}{4} \begin{pmatrix} 1 \\ -3 \end{pmatrix} V_2(r) \begin{cases} S = 1 \\ S = 0 \end{cases} \end{aligned} \tag{15-24}$$

Such a spin-dependent potential is actually observed in the neutron–proton system. The bound state is an $S = 1$ state—this is the deuteron—but there is also an unbound $S = 0$ state. This implies that $V_1 - (3/4)V_2$ is a less attractive potential than $V_1 + (1/4)V_2$. This is only possible if $V_2(r) \neq 0$.

The spin singlet wave function (15-14) implies that, if in a measurement electron (2) is found in an "up" state, then electron (1) *must be* in a spin "down" state. Although the electrons are identical, we may consider a singlet state in which the electrons are moving to the right and to the left with equal and opposite momenta, so that the two-electron system is still at rest in the center of mass. Thus electron (2) could be the one moving to the right, and electron (1) the one moving to the left, and saying that the right one is in an "up" state has a well-defined meaning.

We may ask a more interesting question. Suppose a measurement of S_x is made on electron (2) and it is found that the eigenvalue is $\hbar/2$, that is, electron (2) is in an "up" state along the x-axis. What will a measurement of S_x on electron (1) produce? Since the two electrons are separated by a large distance, one might think that either $+\hbar/2$ of $-\hbar/2$ might be found, perhaps with equal probability, on the grounds that the information about the result of a "projection" of electron (2) into a particular eigenstate of S_x cannot propagate with infinite speed to affect the measurement on electron (1). This is indeed what one would be led to expect if one were to accept certain criteria of what a *complete* physical theory should have, as articulated in a paper by A. Einstein, N. Rosen, and B. Podolsky.[1] On the other hand, quantum mechanics maintains that the two-spin system is described by a single wave function in which the spins are correlated. A measurement of part of the system, the S_x of one electron, combined with the knowledge that the system is in a spin singlet state, is really a measurement of the whole wavefunction. Thus if S_x of electron (2) yields $\hbar/2$, then a measurement of S_x of electron (1) must yield $-\hbar/2$. To see this formally, note that the eigenstates $\chi_\pm$ can be decomposed into eigenstates of S_x, which we denote by $\xi_\pm$. We saw [see (14-56 and (14-57) with $\phi = 0$] that

$$\xi_\pm = \frac{1}{\sqrt{2}}(\chi_+ \pm \chi_-)$$

or equivalently

$$\chi_\pm = \frac{1}{\sqrt{2}}(\xi_+ \pm \xi_-)$$

Let us substitute this into

$$\psi = \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)})$$

We get

$$\begin{aligned}\psi &= (1/\sqrt{2})^{(3)}[(\xi_+^{(1)} + \xi_-^{(1)})(\xi_+^{(2)} - \xi_-^{(2)}) - (\xi_+^{(1)} - \xi_-^{(1)})(\xi_+^{(2)} + \xi_-^{(2)})] \\ &= \frac{1}{\sqrt{2}}(\xi_+^{(1)}\xi_-^{(2)} - \xi_-^{(1)}\xi_+^{(2)})\end{aligned} \tag{15-25}$$

[1]This is nicely discussed in D. Bohm, *Quantum Theory* and from a more modern point of view, connected with the researches of J. S. Bell, in J. J. Sakurai *Modern Quantum Mechanics*. See also the Afterword in David J. Griffiths, *Introduction to Quantum Mechanics*, Prentice Hall, Englewood Cliffs, N.J. 1995

which clearly indicates that if electron (2) is in the "up" state, then electron (1) must be in the "down" state.

THE ADDITION OF SPIN 1/2 AND ORBITAL ANGULAR MOMENTUM

Much more important for future applications is the combination of a spin with an orbital angular momentum. Since **L** depends on spatial coordinates and **S** does not, they commute

$$[\mathbf{L}, \mathbf{S}] = 0 \tag{15-26}$$

It is therefore evident that the components of the total angular momentum **J**, defined by

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{15-27}$$

will satisfy the angular momentum commutation relations.

In asking for linear combinations of the Y_{lm} and the $\chi_\pm$ that are eigenstates of

$$J_z = L_z + S_z \tag{15-28}$$

and

$$\begin{aligned} \mathbf{J}^2 &= \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}\cdot\mathbf{S} \\ &= \mathbf{L}^2 + \mathbf{S}^2 + 2L_zS_z + L_+S_- + L_-S_+ \end{aligned} \tag{15-29}$$

we are again looking for the expansion coefficients of one complete set of eigenfunctions in terms of another set of eigenfunctions.

Let us consider the linear combination

$$\psi_{j,m+1/2} = \alpha Y_{lm}\chi_+ + \beta Y_{l,m+1}\chi_- \tag{15-30}$$

It is, by construction, an eigenfunction of J_z with eigenvalue $(m + \frac{1}{2})\hbar$. We now determine α and β such that it is also an eigenfunction of J^2. We shall make use of the fact that

$$\begin{aligned} L_+Y_{lm} &= [l(l+1) - m(m+1)]^{1/2}\,\hbar Y_{l,m+1} \\ &= [(l+m+1)(l-m)]^{1/2}\,\hbar Y_{l,m+1} \\ L_-Y_{lm} &= [(l-m+1)(l+m)]^{1/2}\,\hbar Y_{l,m-1} \\ S_+\chi_+ &= S_-\chi_- = 0 \qquad S_\pm\chi_\mp = \hbar\chi_\pm \end{aligned} \tag{15-31}$$

Then

$$\begin{aligned} \mathbf{J}^2\psi_{j,m+1/2} = \alpha\hbar^2 \{&l(l+1)\,Y_{lm}\chi_+ + \tfrac{3}{4}Y_{lm}\chi_+ + 2m(\tfrac{1}{2})\,Y_{lm}\chi_+ \\ &+ [(l-m)(l+m+1)]^{1/2}\,Y_{l,m+1}\chi_-\} + \beta\hbar^2\{l(l+1)\,Y_{l,m+1}\chi_- \\ &+ \tfrac{3}{4}Y_{l,m+1}\chi_- + 2(m+1)(-\tfrac{1}{2})\,Y_{l,m+1}\chi_- \\ &+ [(l-m)(l+m+1)]^{1/2}\,Y_{lm}\chi_+\} \end{aligned} \tag{15-32}$$

This will be of the form

$$\hbar^2 j(j+1)\,\psi_{j,m+1/2} = \hbar^2 j(j+1)(\alpha Y_{lm}\chi_+ + \beta Y_{l,m+1}\chi_-) \tag{15-33}$$

provided that

$$\begin{aligned} \alpha[l(l+1) + \tfrac{3}{4} + m] + \beta[(l-m)(l+m+1)]^{1/2} &= j(j+1)\,\alpha \\ \beta[l(l+1) + \tfrac{3}{4} - m - 1] + \alpha[(l-m)(l+m+1)]^{1/2} &= j(j+1)\,\beta \end{aligned} \tag{15-34}$$

This requires that

$$\begin{aligned} (l-m)(l+m+1) = [\,&j(j+1) - l(l+1) - \tfrac{3}{4} - m] \\ &\times [\,j(j+1) - l(l+1) - \tfrac{3}{4} + m + 1] \end{aligned}$$

which evidently has two solutions,

$$j(j+1) - l(l+1) - \tfrac{3}{4} = \begin{cases} -l-1 \\ l \end{cases} \tag{15-35}$$

that is,

$$j = \begin{cases} l - \frac{1}{2} \\ l + \frac{1}{2} \end{cases} \tag{15-36}$$

For $j = l + 1/2$, we get, after a little algebra

$$\alpha = \sqrt{\frac{l+m+1}{2l+1}} \qquad \beta = \sqrt{\frac{l-m}{2l+1}} \tag{15-37}$$

(Actually we just get the ratio; these are already normalized forms.) Thus

$$\psi_{l+1/2,m+1/2} = \sqrt{\frac{l+m+1}{2l+1}}\,Y_{lm}\chi_+ + \sqrt{\frac{l-m}{2l+1}}\,Y_{l,m+1}\chi_- \tag{15-38}$$

We can guess that the $j = l - 1/2$ solution must have the form

$$\psi_{l-1/2,m+1/2} = \sqrt{\frac{l-m}{2l+1}}\,Y_{lm}\chi_+ - \sqrt{\frac{l+m+1}{2l+1}}\,Y_{l,m+1}\chi_- \tag{15-39}$$

in order to be orthogonal to the $j = l + 1/2$ solution.

GENERAL RULES FOR ADDITION OF ANGULAR MOMENTA, AND IMPLICATIONS FOR IDENTICAL PARTICLES

These two examples illustrate the general features that are involved in the addition of angular momenta: If we have the eigenstates $Y^{(1)}_{l_1 m_1}$ of $\mathbf{L}_1^2$ and L_{1z}, and the eigen-

states $Y^{(2)}_{l_2 m_2}$ of $\mathbf{L}_2^2$ and L_{2z}, then we can form $(2l_1 + 1)(2l_2 + 1)$ product wave functions

$$Y^{(1)}_{l_1 m_1} Y^{(2)}_{l_2 m_2} \begin{Bmatrix} -l_1 \le m_1 \le l_1 \\ -l_2 \le m_2 \le l_2 \end{Bmatrix} \tag{15-40}$$

These can be classified by the eigenvalue of

$$J_z = L_{1z} + L_{2z}. \tag{15-41}$$

which is $m_1 + m_2$, and which ranges from a maximum value of $l_1 + l_2$ down to $-l_1 - l_2$. As in the simple cases discussed earlier, different linear combinations of functions with the same m value will belong to different values of j. In the following table we list the possible combinations for the special example of $l_1 = 4$, $l_2 = 2$. We shall use the simple abbreviation (m_1, m_2) for $Y^{(1)}_{l_1 m_1} Y^{(2)}_{l_2 m_2}$.

m-value	*m_1, m_2 combinations*	*number*
6	(4, 2)	1
5	(4, 1) (3, 2)	2
4	(4, 0) (3, 1) (2, 2)	3
3	(4, −1) (3, 0) (2, 1) (1, 2)	4
2	(4, −2) (3, −1) (2, 0) (1, 1) (0, 2)	5
1	(3, −2) (2, −1) (1, 0) (0, 1) (−1, 2)	5
0	(2, −2) (1, −1) (0, 0) (−1, 1) (−2, 2)	5
−1	(1, −2) (0, −1) (−1, 0) (−2, 1) (−3, 2)	5
−2	(0, −2) (−1, −1) (−2, 0) (−3, 1) (−4, 2)	5
−3	(−1, −2) (−2, −1) (−3, 0) (−4, 1)	4
−4	(−2, −2) (−3, −1) (−4, 0)	3
−5	(−3, −2) (−4, −1)	2
−6	(−4, −2)	1

There are a total of 45 combinations, consistent with $(2l_1 + 1)(2l_2 + 1)$.

The highest state has total angular momentum $l_1 + l_2$ as can easily be checked by applying J^2 to $Y^{(1)}_{l_1 l_1} Y^{(2)}_{l_2 l_2}$:

$$\begin{aligned} \mathbf{J}^2 Y^{(1)}_{l_1 l_1} Y^{(2)}_{l_2 l_2} &= (\mathbf{L}_1^2 + \mathbf{L}_2^2 + 2L_{1z}L_{2z} + L_{1+}L_{2-} + L_{1-}L_{2+})\, Y^{(1)}_{l_1 l_1} Y^{(2)}_{l_2 l_2} \\ &= \hbar^2[l_1(l_1 + 1) + l_2(l_2 + 1) + 2l_1 l_2]\, Y^{(1)}_{l_1 l_1} Y^{(2)}_{l_2 l_2} \\ &= \hbar^2(l_1 + l_2)(l_1 + l_2 + 1)\, Y^{(1)}_{l_1 l_1} Y^{(2)}_{l_2 l_2} \end{aligned} \tag{15-42}$$

This is $j = 6$ in the example discussed in the table. Successive applications of

$$J_- = L_{1-} + L_{2-} \tag{15-43}$$

will pick out one linear combination from each row in the table. These will form the 13 states that belong to $j = 6$. When this is done, there remains a single state with $m = 5$, two with $m = 4, \ldots$, one with $m = -5$. It is extremely plausible, and

can, in fact, be checked, that the $m = 5$ state belongs to $j = 5$. Again successive applications of J_- pick out another linear combination from each row in the table, forming 11 states that belong to $j = 5$. Repetition of this procedure shows that we get, after this, sets that belong to $j = 4$, $j = 3$, and finally $j = 2$. The multiplicities add up to 45:

$$13 + 11 + 9 + 7 + 5 = 45$$

We shall not work out the details of this decomposition, as it is beyond the scope of this book. We merely state the results.

(a) The products $Y^{(1)}_{l_1 m_1} Y^{(2)}_{l_2 m_2}$ can be decomposed into eigenstates of $\mathbf{J}^2$, with eigenvaues $j(j+1)\hbar^2$, where j can take on the values

$$j = l_1 + l_2, l_1 + l_2 - 1, \ldots, |l_1 - l_2| \tag{15-44}$$

We can verify that the multiplicities check in (15-44): if we sum the number of states, we get ($l_1 \geq l_2$)

$$\begin{aligned} &[2(l_1 + l_2) + 1] + [2(l_1 + l_2 - 1) + 1] + \cdots + [2(l_1 - l_2) + 1] \\ &\quad = \sum_{n=0}^{2l_2} [2(l_1 - l_2 + n) + 1] \\ &\quad = (2l_2 + 1)(2l_1 + 1) \end{aligned} \tag{15-45}$$

(b) It is possible to generalize (15-38) and (15-39) to give the Clebsch-Gordan series

$$\psi_{jm} = \sum_{m1} C(jm;\, l_1 m_1 l_2 m_2)\; Y^{(1)}_{l_1 m_1} Y^{(2)}_{l_2 m_2} \tag{15-46}$$

The coefficients $C(jm;l_1m_1l_2m_2)$ are called Clebsch-Gordan coefficients, and they have been tabulated for many values of the arguments. We calculated the coefficients for $l_2 = 1/2$, and summarize (15-37) and (15-38) in the table below. Note that $m = m_1 + m_2$, so that the m in (15-37) and (15-38) is really m_1 below.

$C(jm;l_1m_1, 1/2, m_2)$

	$m_2 = 1/2$	$m_2 = -1/2$
$j = l_1 + 1/2$	$\sqrt{\dfrac{l_1 + m + 1/2}{2l_1 + 1}}$	$\sqrt{\dfrac{l_1 - m + 1/2}{2l_1 + 1}}$
$j = l_1 - 1/2$	$-\sqrt{\dfrac{l_1 - m + 1/2}{2l_1 + 1}}$	$\sqrt{\dfrac{l_1 + m + 1/2}{2l_1 + 1}}$

Another useful table is

$$C(jm; l_1 m_1, 1, m_2)$$

	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j = l_1 + 1$	$\sqrt{\frac{(l_1 + m)(l_1 + m + 1)}{(2l_1 + 1)(2l_1 + 2)}}$	$\sqrt{\frac{(l_1 - m + 1)(l_1 + m + 1)}{(2l_1 + 1)(l_1 + 1)}}$	$\sqrt{\frac{(l_1 - m)(l_1 - m + 1)}{(2l_1 + 1)(2l_1 + 2)}}$
$j = l_1$	$-\sqrt{\frac{(l + m)(l_1 - m + 1)}{2l_1(l_1 + 1)}}$	$\frac{m}{\sqrt{l_1(l_1 + 1)}}$	$\sqrt{\frac{(l_1 - m)(l_1 + m + 1)}{2l_1(2l_1 + 1)}}$
$j = l_1 - 1$	$\sqrt{\frac{(l_1 - m)(l_1 - m + 1)}{2l_1(2l_1 + 1)}}$	$-\sqrt{\frac{(l_1 - m)(l_1 + m)}{l_1(2l_1 + 1)}}$	$\sqrt{\frac{(l_1 + m)(l_1 + m + 1)}{2l_1(2l_1 + 1)}}$

A final comment is in order. We noted, when discussing identical particles, that a system of two electrons (or more generally, two fermions) must be in a state that is antisymmetric under the interchange of the two particles. This interchange involves not only the exchange of the spatial coordinates, but also of the spin labels. For a system of two identical spin 1/2 particles, the $S = 1$ triplet of states

$$\begin{gathered} \chi_+^{(1)}\chi_+^{(2)} \\ \frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}) \\ \chi_-^{(1)}\chi_-^{(2)} \end{gathered} \tag{15-47}$$

is symmetric under spin label interchange, while the $S = 0$ (singlet)

$$\frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}) \tag{15-48}$$

is antisymmetric. Thus for a triplet state, the spatial wave function must be antisymmetric, and for a singlet state, it must be symmetric. The spatial wave function of a two-particle system in their center-of-mass system is of the general form

$$u(\mathbf{r}) = R_{nlm}(r)\, Y_{lm}(\theta, \phi) \tag{15-49}$$

An interchange of the coordinates of the two particles is equivalent to the change

$$\begin{gathered} r \to r \\ \theta \to \pi - \theta \\ \phi \to \phi + \pi \end{gathered} \tag{15-50}$$

Thus the radial function remains unchanged. However, under this transformation

$$\begin{aligned} Y_{lm}(\theta\, \phi) &\to Y_{lm}(\pi - \theta, \phi + \pi) \\ &= (-1)^l\, Y_{lm}(\theta, \phi) \end{aligned} \tag{15-51}$$

Thus triplet states must have odd orbital angular momentum l, and singlet states must have even orbital angular momentum. We shall see an application of this when we discuss the states of helium.

SOME COMMENTS ON PARITY

Note that the same argument can be applied to checking the properties of the Y_{lm} under inversion. The transformation $x \to -x$, $y \to -y$, and $z \to -z$ is identical to (15-50). Thus we see that a particle that is in an orbital angular momentum state will have its wave function changed by $(-1)^l$. Even l orbital states are also even-parity states, and odd l orbital states are odd-parity states. It should be noted, however, that the particles themselves can have an intrinsic parity. We can define the intrinsic parity of the electron and the proton and neutron to be even. In that case, the parity of a hydrogen $l = 1$ state, for example, is odd, while the parity of the ground state is even.

In relativistic quantum mechanics one can show that the intrinsic parity of an antiparticle of a fermion is opposite to that of the fermion. Thus the e^+ has negative intrinsic parity, and hence the ground state of positronium for which $l = 0$, has *negative* parity.

An interesting application of these remarks occurs in elementary particle physics. One of the first unstable elementary particles to be discovered was the π meson predicted by Yukawa. This particle, which plays an important role in nuclear forces, comes in three charge states π^+, π^0, π^-. It was found to have spin 0, and the question arose whether the wave function of a pion—as this meson came to be called—was even or odd under reflection, assuming that the known particles, the proton and the neutron, had positive intrinsic parity. The following experiment was suggested.

Consider the capture of a π^- by a deuteron. A slow pion in liquid deuterium loses energy by a variety of mechanisms, until it finally ends up in the lowest Bohr orbit about the (pn) nucleus, and is then captured through the action of the nuclear forces. In the nuclear reaction

$$\pi^- + d \to n + n$$

the angular momentum is 1; the pion has zero spin, the orbital angular momentum is zero in the lowest Bohr state, so that the only contribution is the angular momentum of the deuteron, which is 1. The two neutrons must therefore be in an angular momentum 1 state. If the total spin of the two neutrons is 0, then the orbital angular momentum must be 1. If the total spin of the two-neutron state is 1, then orbital angular momentum 0, 1, and 2 is possible, since adding two angular momenta of one unit each can yield 0, 1, and 2, and adding one unit to two units of angular momentum can yield 3, 2, and 1. However, a singlet state of two identical fermions must have even angular momentum, and is thus excluded. A triplet state must have odd orbital angular momentum, and this is possible if the orbital angular momentum is 1. Such a state, however, has odd parity by (15-51), and hence the pion must have odd parity. In terms of the spectroscopic notation, which we shall use, where a state is labeled according to

$$^{2S+1}L_j \qquad (15\text{-}52)$$

the two neutron states, from the total class of states $^1S_0$, $^1P_1$, $^1D_2$, $^1F_3, \ldots, ^3S_1$, $^3P_2$, $^3P_1$, $^3P_0$, $^3D_3$, $^3D_2$, $^3D_1$, $^3F_4$, $^3F_3$, $^3F_2, \ldots$, are restricted to $^1S_0$, $^1D_2, \ldots, ^3P_{2,1,0}$, $^3F_{4,3,2}$, ... by the Fermi-Dirac statistics argument, and of these there is only one state, the $^3P_1$ state, that has angular momentum 1.