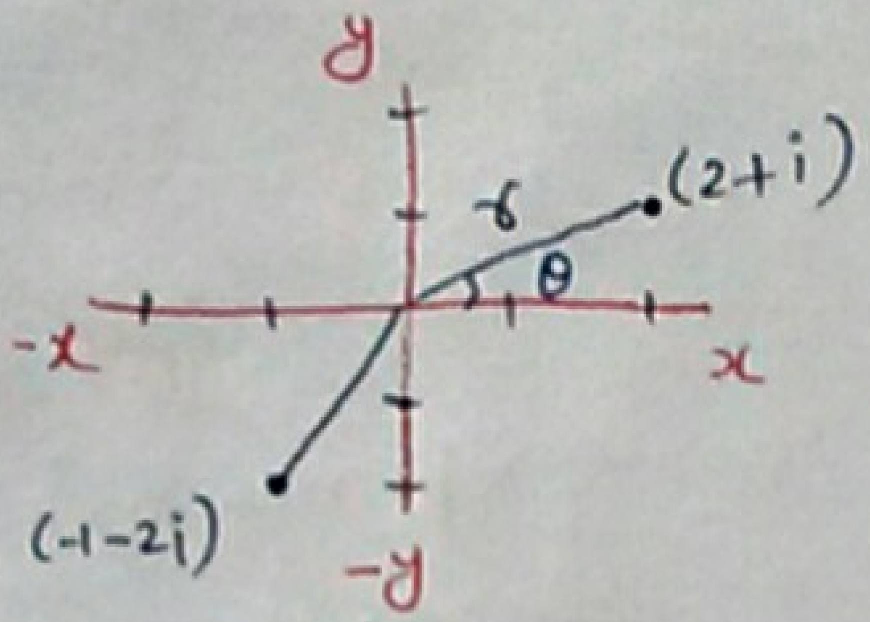
Complex - Numbers

A number of the form $a+ib$ is called complex number where 'a' and 'b' are real Numbers and $i = \sqrt{-1}$

* Argand diagram / graphical representation

$x+iy$ here x is Real Axis and y is imaginary



$\boxed{2+i = z}$

$\boxed{-1-2i = z}$

Conjugate of Any Complex Number 'Z' Can be obtained by just replacing 'i' by '-i' denoted by $\bar{z}$

Ex: $z = x+iy$ then $\bar{z} = x-iy$

Modulus and Argument

if Complex No. is $Z = x+iy$

then its modulus $|z| = \sqrt{x^2+y^2} = r$

$\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$, $\tan\theta = y/x$

here θ is Called Argument of Complex Number

θ is angle of Complex No. with x Axis (see above diagram)

Types of Complex Numbers

1. Cartesian form: $x+iy$

   $x = r\cos\theta$, $y = r\sin\theta$

2. polar form: $r(\cos\theta + i\sin\theta)$

   here $r = \sqrt{x^2+y^2}$

3. exponential form: $re^{i\theta}$

Ex: let $Z = 1+i$ change it into polar form

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here $x=1$, $y=1$ so $r = \sqrt{1^2+1^2} = \sqrt{2}$

and $\theta = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{1}{1} = \pi/4$

put value of r and θ

$z = r(\cos\theta + i\sin\theta)$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

① $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, ② $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots$

③ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

$$\boxed{\cos z + i\sin z = e^{iz}}$$

De Moivre's theorem → $\boxed{(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)}$

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Canbe proved easily by writting in exponential form.

Square root of a Complex Number :-

Let $a+ib$ be a complex number and its square root is $x+iy$

then we can easily find x and y by simple mathematics calculation.

$$\sqrt{a+ib} = x+iy \quad — (1)$$

Squaring both sides of (1)

$$a+ib = (x+iy)^2 = x^2 + i^2y^2 + 2xiy$$

$$a+ib = x^2 - y^2 + 2ixy \qquad (i^2 = -1)$$

equating real and imaginary part

$$x^2 - y^2 = a \quad , \quad 2xy = b \quad — ②$$

also we know that

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

$$x^2+y^2 = \sqrt{a^2+b^2} \quad — ③$$

from ② and ③ we can get x and y put these value in ① which will be desired answer

Hyperbolic function

(1) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ (2) $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

(3) $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

Relation betwn Circular and Hyperbolic fn

(1) $\sin ix = i \sinh x$ (2) $\sinh ix = i \sin x$

(3) $\cos ix = \cosh x$ (4) $\cosh ix = \cos x$

(5) $\tan ix = i \tanh x$ (6) $\tanh ix = i \tan x$

Formulae of Hyperbolic fn

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\operatorname{sech}^2 x = 1 - \tanh^2 x$

(3) $\operatorname{cosech}^2 x = \coth^2 x - 1$

(other formulas are same as trigonometric Functions)

Sinh and Cosh series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

Logarithmic Series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots$$

Gregory's Series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \tan^{-1} x$$

$$\frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \tanh^{-1} x$$

Gamma Function

$$\boxed{\int_0^\infty x^m e^{-\alpha x^n}\,dx = \frac{\frac{1}{n}\Gamma\frac{m+1}{n}}{\alpha^{(m+1)/n}}} \quad \text{— (B)}$$

$\boxed{\Gamma_{n+1} = n\Gamma_n}$ — (A) $\boxed{\Gamma_1 = 1}$ — (C) $\boxed{\Gamma_{1/2} = \sqrt{\pi}}$ (D) $\boxed{\Gamma_{n+1} = n!}$

Γ ← this is a notation of Gamma $\boxed{\Gamma_n = \text{gamma } n}$ (read)

Ex:

① $\Gamma_5 = ?$ $\Rightarrow \Gamma_5 = \Gamma_{4+1} = 4! = 4\times3\times2\times1$
↑ used D
$= 24$

② $\Gamma_{\frac{5}{2}} = ?$ $\Gamma_{\frac{5}{2}} = \Gamma_{\frac{3}{2}+1} = \frac{3}{2}\Gamma_{\frac{3}{2}} = \frac{3}{2}\Gamma_{\frac{1}{2}+1}$
used A

$\frac{3}{4}\sqrt{\pi} = \frac{3}{2}\times\frac{1}{2}\Gamma_{\frac{1}{2}}$
↑ used (C)

$$\boxed{\Gamma_{\frac{5}{2}} = \frac{3}{4}\sqrt{\pi}}$$

③ $\Gamma_{-\frac{1}{2}} = ?$ $\Gamma_{-\frac{1}{2}+1} = -\frac{1}{2}\Gamma_{-\frac{1}{2}}$ ← (used A)

$\Gamma_{\frac{1}{2}} = -\frac{1}{2}\Gamma_{-1/2}$

used C →
$\sqrt{\pi} = -\frac{1}{2}\Gamma_{-1/2}$

$$\boxed{-2\sqrt{\pi} = \Gamma_{-1/2}}$$

Formula

$$\int_0^\infty x^m e^{-\alpha x^n}\,dx = \frac{\frac{1}{n}\Gamma\frac{m+1}{n}}{\alpha^{(m+1)n}}$$

Very imp. formula in integration
m, n and α are Constants

Beta function (β)

$$\beta(l, m) = \beta(m, l)$$

$$\beta(l, m) = \int_0^1 x^{m-1}(1-x)^{l-1}\,dx = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

both Imp. formula in Integration

Ex: Using Beta and Gamm f^n Evalute

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx = ?$$

Sol^n putting $x^3 = \sin^2\theta \Rightarrow dx = \frac{2}{3}\sin^{-1/3}\theta\cos\theta\, d\theta$

put the values

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx = \int_0^{\pi/2} \left(\frac{\sin^2\theta}{1-\sin^2\theta}\right)^{1/2} \frac{2}{3}\sin^{-1/3}\theta\cos\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \sin^{2/3}\theta\, d\theta$$

use $\beta(l,m)$ formula of Integral

here $p = 2/3$ $q = 0$

$$\int_0^1 \left[\frac{x^3}{(1-x^3)}\right]^{1/2} dx = \frac{\frac{2}{3}\,\Gamma\frac{\frac{2}{3}+1}{2}\,\Gamma\frac{0+1}{2}}{\Gamma\frac{\frac{2}{3}+0+2}{2}} = \frac{2\,\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{3\,\Gamma\frac{4}{3}}$$

$$= \frac{2}{3}\sqrt{\pi}\,\frac{\Gamma 5/6}{\Gamma 4/3}$$ Ans

$\Gamma 5/6$ and $\Gamma 4/3$ can be solved by $\Gamma n+1 = n\Gamma n$ For further Ans