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Module 1
Quantities

Definition of physical quantities:

The quantities that can be measured are called as physical quantities. Common examples of physical quantities are-

(a) Mass of a body
(b) Position of a particle
(c) Temperature of a gas
(d) Volume of a cube
(e) Magnetic field due to a current carrying wire

Scalar and vector

Some of these physical quantities are scalar and some others are vectors.

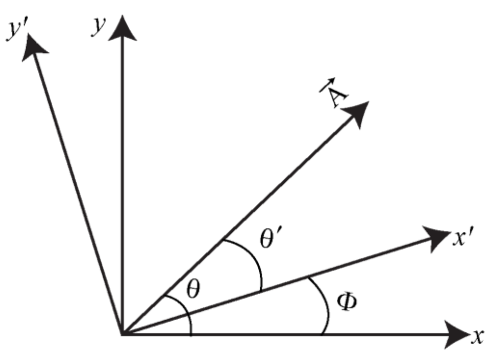
Those physical quantities which have only magnitude and no direction in space are termed as scalar quantities. The examples are (a), (c) and (d).

The other quantities which have both magnitude and some fixed direction in space are called as vectors. (b) and (e) are examples of such quantities. Thus, in a three-dimensional (3D) space, vectors are represented by three real numbers corresponding to each direction of space.

Now the question is: does a set of 3 real numbers would always represent a vector? Does a vector need to have a special property so as to be qualified as a vector? This is true in the light of the example below. Suppose a zoo has n_1 tigers, n_2 elephants and n_3 crocodiles, the set of numbers formed by (n_1, n_2, n_3) does not constitute a vector. Thus a sharper definition of vectors is needed.

Definition of a vector

A vector is a quantity that remains invariant under rotation or translation of the coordinate system. In other words, the components of the vector transform in the same way as the variables of the coordinate system.



Let us start by writing down the components of a vector $\vec{A}$ (in 2D, just for convenience) in x - y coordinate system and also in a rotated $x' - y'$ coordinate system. Let the components of the vector $\vec{A}$ be (A_x , A_y) and (A'_x , A'_y) in the two coordinate systems.

We can write,

A_x = $A\cos\theta$, A_y = $A\sin\theta$ in x - y system where A (=$|\vec{A}|$) is the magnitude (length) of the vector $\vec{A}$.

Likewise, in the $x' - y'$ system,

$A'_x = A\cos\theta = A\cos(\theta - \emptyset) = A\cos\theta\, cos\emptyset + A\sin\theta\sin\emptyset$

Thus $A'_x = A_x\cos\emptyset + A_y\sin\emptyset$ (1)

Similarly for the y component, it can be shown,

$A'_y = -A_x\sin\emptyset + A_y\cos\emptyset$
(2)

It is useful to write equations (1) and (2) in a compact form,

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} cos\,\emptyset & sin\,\emptyset \\ -sin\,\emptyset & cos\,\emptyset \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \tag{3}$$

where the angles θ, θ' and $\emptyset$ are indicated in the Fig 1. Eq.(3) represents the transformation equations for the components of the vectors. It is also known that the components of the position vector $\vec{r}$ transform under rotation as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\,\emptyset & sin\,\emptyset \\ -sin\,\emptyset & cos\,\emptyset \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

With the matrix,

$$\begin{pmatrix} cos\,\emptyset & sin\,\emptyset \\ -sin\,\emptyset & cos\,\emptyset \end{pmatrix}$$

representing a 2D rotation matrix.

The reader may repeat the same exercise for a vector in 3D. For simplicity, you may consider the 3D rotation matrix corresponding to a rotation about z- axis which is,

$$\begin{pmatrix} cos\,\emptyset & sin\,\emptyset & 0 \\ -sin\,\emptyset & cos\,\emptyset & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But before that an introduction to the unit vectors is helpful. Using the unit vectors a constant vector in the x-direction can be written as, $\vec{A} = A\hat{\imath}$ where $\hat{\imath}$ is a unit vector in the direction and has a magnitude 1. Thus $\hat{\imath} = \frac{\vec{A}}{|\vec{A}|}$. Similarly the unit vectors in the other orthogonal directions can be defined.

The unit vectors in the three orthogonal directions are perpendicular to each other. These set of unit vectors are determined locally using the surfaces of constant coordinates at the point of intersection. We can illustrate the concept via a specific example. The function $f = z$ denotes a set of planes in the z-direction for different values of z. The unit vector $\hat{z}$ is perpendicular to all these planes and points in the direction of the planes with increasing of the planes with increasing values of z.

During the course of lectures, we shall denote the unit vectors along the Cartesian coordinate axes as $\hat{x}, \hat{y}, \hat{z}$ or $\hat{\imath}, \hat{\jmath}, \hat{k}$.

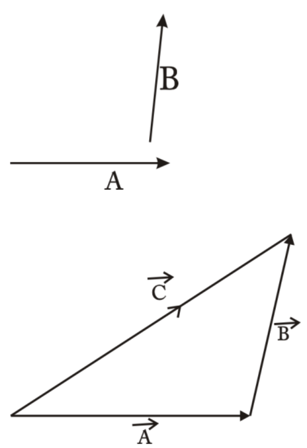
Vector operations

The different vector operations that we are going to discuss are---

(a) Addition
(b) Subtraction
(c) Multiplication

Let us discuss them one by one

(a) Addition of vectors



If $\vec{A}$ and $\vec{B}$ are two vectors as shown, then $\vec{C}$ (=$\vec{A}$+$\vec{B}$) is a resultant vector that obeys triangle law of vector addition. So what is triangle law of vector addition? It states that-

If two vectors are represented by two sides of a triangle taken in order (in the above figure, they are in anticlockwise order), then their sum (or equivalently the resultant) is represented by the third side of the triangle taken in opposite (clockwise) order.

Some useful properties of vector addition are-

(i) Commutative law : $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
(ii) Associative law : $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$
(iii) Additive inverse: $\vec{A} + (-\vec{A}) = 0$

Example:

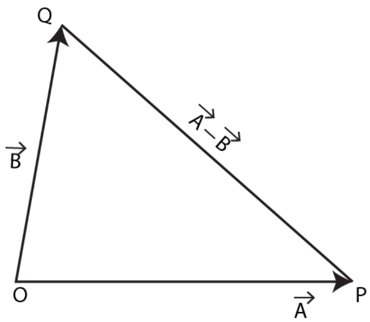
$$\left.\begin{aligned} \text{If } \vec{A} &= A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \\ \text{and } \vec{B} &= B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \end{aligned}\right\}\ \hat{i},\hat{j},\hat{k} \text{ are unit vectors along } x,y,z \text{ direction}$$

$\vec{A} + \vec{B} = (A_x + B_x)\ \hat{i} + (A_y + B_y)\ \hat{j} + (A_z + B_z)\ \hat{k}$

Addition of two vectors implies addition of their components along different directions.

Subtraction of vectors

The subtraction of two vectors $\vec{A}$ - $\vec{B}$ is defined as the addition of -$\vec{B}$ to $\vec{A}$. Thus to subtract $\vec{B}$ from $\vec{A}$, reverse the direction of $\vec{B}$ and add to $\vec{A}$.



Let $\overrightarrow{OP} = \vec{A}$ and $\overrightarrow{OQ} = \vec{B}$

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

$= \overrightarrow{OP} + \overrightarrow{QO} = \overrightarrow{QO} + \overrightarrow{OP}$

$= \overrightarrow{QP}$

Vector Multiplication

The product of two vectors can either be a scalar quantity or a vector quantity. Thus we can define a scalar product and a vector product.

Scalar product

The scalar product of two vectors is defined as the product of their individual magnitudes multiplied by the cosine of the angles between them. This is written as,

$\vec{A} . \vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$

If the two vectors are parallel or in the same direction, then

$\vec{A} . \vec{B} = |\vec{A}||\vec{B}|$ as $\theta = 0$

If the vectors are perpendicular, then

$\vec{A}\,.\,\vec{B} = 0$ as θ (angle between $\vec{A}$ and $\vec{B}$) is 90^0 .

<u>Vector product</u>

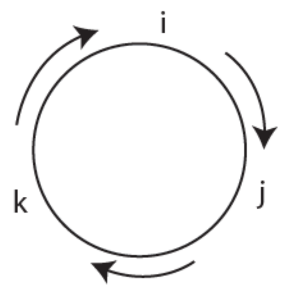
The magnitude of the vector products of two vectors is,

$|\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$.

How to find the direction of the resultant vector $\vec{C}$ (= $\vec{A} \times \vec{B}$) ? Keep your right hand palm along the first vector (here, $\vec{A}$), curl your fingers toward the second vector ($\vec{B}$), the thumb points towards the resultant ($\vec{C}$).

Unlike scalar products, vector products are non commutative and non-associative.(see properties (i) and (ii)).

Cross products of two vectors can easily be computed by following a simple cyclic multiplication rule for the unit vectors, such as



Cyclic multiplications are defined as,

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

Cross products taken in the anticlockwise sense will be accompanied by a negative sign. For example,

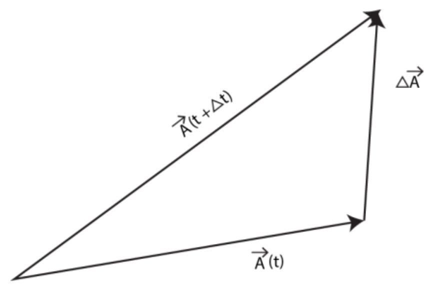
$$\hat{i} \times \hat{k} = -\hat{j}$$

For three vectors, whether they are in the same plane can be tested by using a scalar triple product, viz. $\vec{A}\,.\,(\vec{B} \times \vec{C})$ Which represents the volume enclose by the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Thus a zero volume indicates that the vectors are coplanar.

Derivative of a vector

Any vector in general, depends on space and time, that is $\vec{A}$ ($\vec{r}$, t). Derivative of a vector with respect to the space coordinates are going to be more prominent features of our discussion. However, in several physical situations, it becomes important to compute time derivative of a vector.

For the following discussion, only consider the dependence on time, that is, $\vec{A}$ (t). The change in $\vec{A}$ during the interval from t to $t + \Delta t$ is



$$\Delta\vec{A} = \vec{A}\,(t + \Delta t) - \vec{A}\,(t)$$

The differentiation with time can be carried out using,

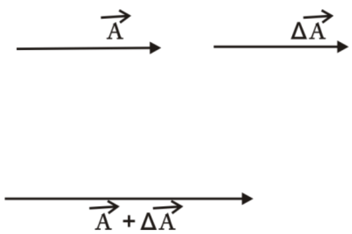
$$\frac{d\vec{A}(t)}{dx} = \underset{\Delta t \to 0}{lt} \; \frac{\vec{A}\,(t + \Delta t) - \vec{A}(t)}{\Delta t}$$

$\frac{d\vec{A}(t)}{dx}$ is a new vector (say $\vec{B}$) which can have any magnitude and direction, depending on the behavior of $\vec{A}$.

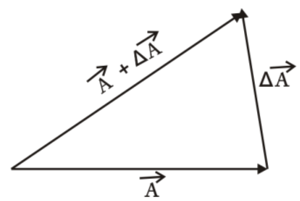
There is a very important difference between the time derivative of a vector and a scalar. Derivative of a scalar has only magnitude, while the derivative of a vector has both magnitude and direction.

Consider two cases to appreciate the difference.

Case I



case II

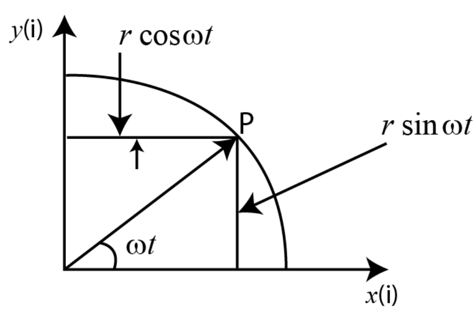


(Time dependences are not written for brevity)

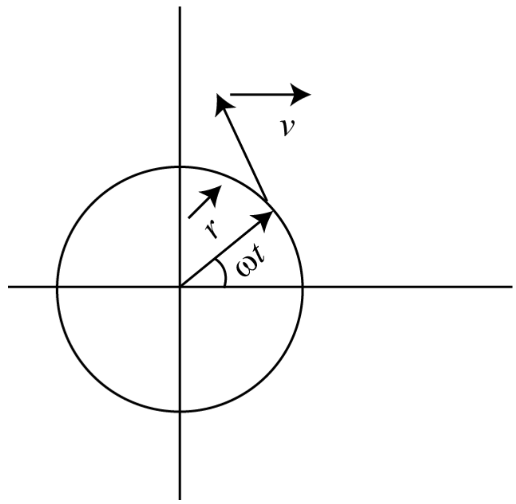
In case I, $\Delta\vec{A}$ is coplanar with $\vec{A}$ and in the other case, $\Delta\vec{A}$ is perpendicular to $\vec{A}$. Thus the direction is unaltered, while the magnitude changes for the vector $\vec{A}$. In case II, the direction is altered, leaving the magnitude unchanged. In a general sense, the time derivative of a vector will change both magnitude and direction.

Velocity and acceleration in a uniform circular motion

Consider a particle executing uniform circular motion. The position vector of the particle at a time instant t is given by $\vec{r}$ = $\hat{i}\, r\cos\omega t \;+\; \hat{j}\, r\sin\omega t$.



The velocity and acceleration of the body are obtained by successively differentiating the position vector with respect to time.

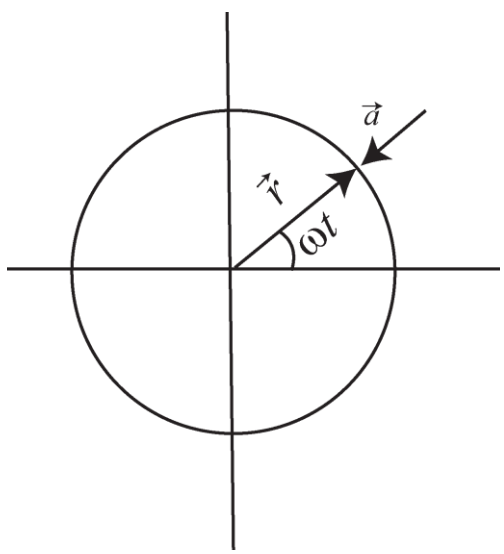


$$\vec{v} = r\omega\,(-\hat{i}\,\sin \omega t + \hat{j}\,\cos \omega t)$$

To get a feel of the direction, take $\vec{r}\,.\,\vec{v} = 0$ Thus the velocity and position vectors are perpendicular to each other. The acceleration vector can be obtained by computing the time derivative of the velocity vector.

$$\vec{a} = -\,\omega^2 r\,(\hat{i}\,\cos\omega t + \hat{j}\,\sin\omega t)$$

$$= -\omega^2 \vec{r}$$



Thus $\vec{a}$ and $\vec{r}$ are oppositely directed. Since acceleration is proportional to force, the force is called as centripetal force.

Example problems

1. Prove that $(\vec{A}\text{x}\vec{B})\,.\,(\vec{A}\text{x}\vec{B}) = A^2\ B^2 - (\vec{A}.\vec{B})^2$

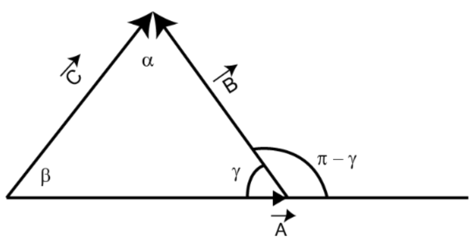
   Solution

   $(\vec{A} \times \vec{B})^2 = A^2B^2sin^2(\vec{A},\ \vec{B})$

   Where sin $(\vec{A}, \vec{B})$ is the angle between vectors $\vec{A}$ and $\vec{B}$

   Thus $(\vec{A} \times \vec{B})^2 = A^2B^2$ [1- $cos^2(\vec{A}, \vec{B})$]
   $= A^2B^2 - A^2B^2\ cos^2\ (\vec{A}, \vec{B})$
   $=A^2B^2 - (\vec{A}.\vec{B})^2$ (proved)

2. Derive the law of sines, i.e. prove
   $\frac{A}{sin\ \alpha} = \frac{B}{sin\ \beta} = \frac{C}{sin\ \gamma}$ where the angles α, β and γ are defined as follows



   Solution:

   $\vec{B} = \vec{C} - \vec{A}$
   Thus $\vec{B}$ x $(\vec{C} - \vec{A}) = 0$ (cross products of two equal vectors is zero)

   So $\vec{B}\text{x}\vec{C} = \vec{B}\text{x}\vec{A}$

   BC $sin\ \alpha$ = AB $sin\ (\pi\text{-}\gamma)$ = AB $sin\ \gamma$
   Thus
   $\frac{A}{sin\ \alpha} = \frac{C}{sin\ \gamma}$ similarly $\frac{A}{sin\ \alpha} = \frac{B}{sin\ \beta}$ (proved)

Rules of vector differentiation

1) $\frac{d}{dt}\ (\vec{A} + \vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$

2) $\frac{d}{dt}(\varphi\ \vec{A}) = \frac{d\varphi}{dt}\vec{A} + \varphi\ \frac{d\vec{A}}{dt}$

(φ: a scalar, $\vec{A}$ and $\vec{B}$ are vectors)

3) $\frac{d}{dt}(\vec{A}\ .\ \vec{B}) = \frac{d\vec{A}}{dt}\ .\ \vec{B} + \vec{A}\ .\ \frac{d\vec{B}}{dt}$

4) $\frac{d}{dt}(\vec{A}\ \mathrm{x}\ \vec{B}) = \frac{d\vec{A}}{dt}\ \mathrm{x}\ \vec{B} + \vec{A}\ \mathrm{x}\ \frac{d\vec{B}}{dt}$

5) $\frac{d}{dt}(\vec{A}\ .\ \vec{A}) = 2\vec{A}\ .\ \frac{d\vec{A}}{dt}$

Differential Calculus

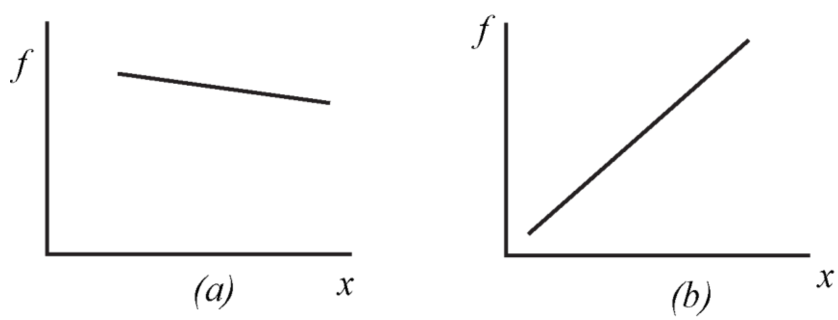
Vector derivatives

An important ingredient of studying electromagnetics is learning vector derivatives. The operations that are most frequently required are gradient, divergence and curl which yield vector and scalar derivatives of scalar and vector fields.

Suppose we talk about a function which is of one variable, say x, that is $f(x)$. Let us talk about the derivative $\frac{df}{dx}$ which will provide an estimate of how rapidly the function $f(x)$ varies as x is changed by a small amount. This is written as,

$$df = (\frac{df}{dx})dx$$

Thus if one changes x by a small dx, then f changes by df and $\frac{df}{dx}$ is the proportionality constant for a change. For example, $\frac{df}{dx}$ is small in fig (a) as f varies slowly with x. Whereas, below in Fig (b), $\frac{df}{dx}$ is large. Thus $\frac{df}{dx}$ provides a measure of the slope of $f$vs x.



However, things are not as simple for a function that depends on three variables, i.e. $f(x, y, z)$. To measure how $f(x, y, z)$ vary as x, y and z are changed by a small amount dx, dy and dz . If you choose one direction to compute this change, the result could be very different than if one chooses another direction to compute this change. Thus there are infinitely large numbers of results possible corresponding to each direction.

However things may not be as bad as it looks, if we ask a specific question, that is, in which direction, the function vary the fastest and how rapidly does it vary in this direction? The answer is provided by the 'gradient' operation.

For vectors, the differentiation procedure involves a scalar derivative, called as 'divergence' and a vector derivative, called as 'curl'. In the next section, we shall see the physical significance of gradient, divergence and curl.

Linear transformation and matrices

Linear transformations on physical quantities are usually described by matrices. Consider $\vec{u}$ be a column vector that denotes a physical quantity, and T be a transformation. Thus $T\vec{u} = A\vec{u}$ where A is a $m \times n$ matrix that denotes the linear transformation. Many a time, the main job is to find the transformation matrix. We shall illustrate it by a specific example. In the example, we discuss orthogonal transformation.

Orthogonal transformations are transformations of the form $y = Ax$ where A is an orthogonal matrix. For the transformation corresponding to a rotation by an angle θ in a plane is expressed as

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} cos\theta - sin\theta \\ sin\theta \; cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= R\,x$$

is an orthogonal transformation. Verify that,

$$R^T = R^{-1}.$$

The inner product of two vectors is same as the dot product that obeys a few relations as given below (with $\vec{u}, \vec{v}$ and $\vec{w}$ as vectors and α is a scalar)

$$\langle \vec{u} + \vec{v} | \vec{w} \rangle = \langle \vec{u} | \vec{w} \rangle + \langle \vec{v} | \vec{w} \rangle$$
$$\langle \alpha\vec{u} | \boldsymbol{\vec{v}} \rangle = \alpha \langle \vec{u} | \vec{v} \rangle$$
$$\langle \vec{u} | \vec{v} \rangle = \langle \vec{v} | \vec{u} \rangle \geq 0$$

a) The importance of orthogonal transformation is that the inner product remains invariant under an orthogonal transformation.

$$< a|b > = a^T b.$$

The proof is as follows.

$\vec{u} = A\vec{a};\quad \vec{v} = A\vec{b}$ where $\vec{A}$ is orthogonal (A is a matrix, while u, v are column vectors)

$$< u|v > = u^T v = (Aa)^T Ab = a^T A^T Ab.$$
$$= < a|b >$$

b) A real square matrix is orthogonal if and only if it's column vectors $\vec{a}_1 \ldots \ldots \ldots \vec{a}_n$(and also it's row vectors) form an orthonormal system, that is,

$$< a_i | a_j > = \delta_{ij} \text{ where } \delta_{ij} = 1 \text{ when } i = j \text{ and } 0 \text{ otherwise.}$$

c) The determinant of an orthogonal matrix has values +1 or -1. The proof is as follows

$det\,(A\,B) = (det\,A)\,(det\;B)$

and $det\;A^T = det\,A$.

$\det I = 1 \Rightarrow \det(AA^{-1}) = \det(AA^T) = (detA)(detA^{-1})$

$\Rightarrow (detA)^2 = 1$

Thus $\det A = \pm 1$.

d) The eigenvalues of an orthogonal matrix are either real or complex conjugate in pairs. Carrying along the same lines we define some more matrices here.

Complex, Hermitian, skew Hermitian, Unitary matrices

Hermitian : $A^{-T} = A$; $a_{ji} = a_{ij}$ (bar indicates complex conjugate)
Skew-Hermitian: $A^{-T} = -A, \bar{a}_{ji} = -a_{ij}$
Unitary: $A^{-T} = A^{-1}$

Again for Hermitian matrices the diagonal elements are purely real. Similarly for a skew Hermitian matrices, the diagonals are purely imaginary or zero.

1. The eigenvalues of a Hermitian matrix are real.
2. The eigenvalues of a skew Hermitian are purely imaginary and zero.
3. The eigenvalues of a unitary matrix have absolute value 1.

Similarity of Matrices, Basis of Eigenvectors

A $n \times n$ matrix A' is similar to another $n \times n$ matrix, $A' = P^{-1}AP$ for a non-singular $n \times n$ matrix P (Singular matrices are those for which the inverse does not exist, thus non-singular ones are invertible). The transformation which gives A' from A is called a similarity transformation. Similarity transformation are important as they preserve eigenvalues. If A' is similar to A, then both of them have same eigenvalues.

1. Furthermore if x is an eigenvector of A, then $y = P^{-1}x$ is an eigenvector of A', corresponding to the same eigenvalue.

Proof of the above

$Ax = \lambda x,$ left multiply by λP^{-1},

$P^{-1}Ax = P^{-1}\lambda x$

$\Rightarrow p^{-1}AIx = \lambda p^{-1}x$

$\Rightarrow P^{-1}APP^{-1}x = \lambda P^{-1}x$

$\Rightarrow A'P^{-1}x = \lambda P^{-1}x$

$\Rightarrow A'y = \lambda y$ (proved)

2. If $\lambda_1, \lambda_2 \dots \dots \dots \lambda_n$ are distinct eigenvalues of an $n \times n$ matrix, then corresponding eigenvectors $x_1, x_2 \dots \dots \dots x_n$ form a linearly independent set.