UNIT – III

DIMENSIONAL & MODEL ANALYSIS

Dimensional Analysis :

It is a mathematical technique used in research work for design & for conducting model tests.

Fixed Dimensions :

Length L

Mass M

Time T

Fundamental dimensions :

Fixed dimensions are called "fundamental dimensions" (or) "fundamental quantity."

secondary (or) Derived Quantities :

secondary (or) Derived quantities are those quantities which posses more than one fundamental dimensions.

S.No	Physical Quantity	Symbol	Dimension
	(a) Fundamental		
1.	Length	L	L
2.	Mass	M	M
3.	Time	T	T
	(b) Geometeric		
1.	Area	A	L^2
2.	Volume	V	L^3
	(c) Kinematic Quantities		
1.	Velocity	v	LT^{-1}
2.	Angular velocity	ω	T^{-1}
3.	Acceleration	a	LT^{-2}
4.	Discharge	Q	L^3T^{-1}
5.	Acceleration Due to Gravity	g	LT^{-2}
6.	Kinematic Viscosity	ν	L^2T^{-1}
	(d) Dynamic Quantities		
1.	Force	F	MLT^{-2}
2.	Weight	W	MLT^{-2}
3.	Density	ρ	ML^{-3}
4.	Specific weight	w	$ML^{-2}T^{-2}$

S.No.	Physical Quantity	Symbol	Dimensions
5.	Dynamic viscosity	μ	$ML^{-1}T^{-1}$
6.	Pressure Intensity	P	$ML^{-1}T^{-2}$
7.	Modulus of Elasticity	K / E	$ML^{-1}T^{-2}$
8.	Surface Tension	σ	MT^{-2}
9.	Shear stress	τ	$ML^{-1}T^{-2}$
10.	Work, Energy	W (or) E	$ML^{2}T^{-2}$
11.	Power	P	$ML^{2}T^{-3}$
12.	Torque	T	$ML^{2}T^{-2}$
13.	Momentum	M	MLT^{-1}

Problems :

1) Determine the dimension of the quanti[ties] given below.

i) Angular velocity ii) Angular Accelerat[ion]

iii) Discharge iv) kinematic viscosity v) Fo[rce]

vi) specific weight vii) Dynamic viscosit[y]

Sol:

i) Angular velocity $= \frac{\text{Angle covered in ra[dians]}}{\text{Time}}$

$= \frac{1}{T}$

Angular velocity $= T^{-1}$

ii) Angular Acceleration = rad / sec^2

$$= \frac{rad.}{T^2} = \frac{1}{T^2}$$

Angular Acceleration $= T^{-2}$

iii) Discharge. = Area x velocity

$$= L^2 \times \frac{L}{T} = \frac{L^3}{T}$$

Discharge $= L^3 T^{-1}$

iv) Kinematic viscosit = $\frac{\mu}{\rho}$

where μ is given by $\tau = \mu \frac{du}{dy}$

$$\mu = \frac{\tau}{\frac{du}{dy}} = \frac{\text{Shear Stress}}{\frac{L}{T} \times \frac{1}{L}}$$

$$= \frac{\text{Force / Area}}{\frac{1}{T}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Area} \times \text{Time}} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}}$$

$$= \frac{ML}{L^2 T^2 \times \frac{1}{T}} = \frac{M}{LT}$$

$$\mu = ML^{-1}T^{-1}$$

$$\rho = \frac{mass}{volume} = \frac{M}{L^3} = ML^{-3}$$

$$\text{Kinematic viscosity} = \frac{\mu}{\rho} = \frac{ML^{-1}T^{-1}}{ML^{-3}} = \frac{T^{-1}}{L^{-2}}$$

Kinematic viscosity $= L^2 T^{-1}$

(v) Force = Mass × Acceleration

$$= M \times \frac{\text{Length}}{(\text{Time})^2} = \frac{ML}{T^2}$$

$$\text{Force} = MLT^{-2}$$

vi) specific weight = $\frac{\text{weight}}{\text{volume}}$

$$= \frac{\text{Force}}{\text{volume}} = \frac{MLT^{-2}}{L^3}$$

$$\text{specific weight} = ML^{-2}T^{-2}$$

vii) Dynamic viscosity, μ is derived

$$\mu = ML^{-1}T^{-1}.$$

P.NO: 569 Dimensional Homogeneity :

Dimensional homogeneity means the dimension of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as "dimensionally homogeneous equation".

Let us consider,

$$V = \sqrt{2gH}$$

Dimensions of L.H.S $= V = \frac{L}{T} = LT^{-1}$

Dimensions of R.H.S $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}}$

$$= \frac{L}{T} = LT^{-1}$$

$\frac{T^{-1}}{L^{-2}}$

Dimensions of L.H.S = Dimension of R.H.S $= LT^{-1}$

∴ Equation $V = \sqrt{2gH}$ is dimensianally homogeneous. so it can be used in any system of units.

Methods of Dimensional Analysis :

1) Rayleigh's method

2) Buckingham's π-theorem

1) Rayleigh's method :

This method is used for determining the expression for a variable which depends upon three (or) four variable only. If the number of independent variables becomes more than four, then it is very difficult to find the expression for the dependent variables.

This can also written as

$$X = K x_1^a \cdot x_2^b \cdot x_3^c$$

Where,

K — costant

a, b & c — arbitrary powers.

x — variable [x_1, x_2 & x_3]

2) Buckingham's π - Theorem :

The Rayleigh's method of dimensional analysis becomes more laborious if the variables are more than the number of fundamental dimensions (M, L, T). This difficulty is overcome by using "Buckinoham's π - theorem". If there are n variables [independant & dependent variables] in a physical phenomenon & if these variables contain m fundamental dimension (M, L, T), Then the variables are arranged into (n-m) dimensionless terms. Each term is called "π - term".

$$\pi_1 = \phi\ \pi_2, \pi_3 \ldots \ldots \pi_{n-m}$$

$$\pi_2 = \phi_1\ \pi_1, \pi_3, \ldots \ldots \pi_{n-m}$$

The time period (t) of a pendulum depends upon the length (L) of the pendulum & acceleration due to gravity (g). Derive an expression for the time period.

Sol:

Time period t is a function of

(i) L and (ii) g

$$t = KL^a \cdot g^b$$

where,

K is a constant

Substituting the dimensions on both sides

$$T^1 = KL^a \cdot (LT^{-2})^b$$

Equating the powers of M, L and T on both sides

Power of T, $1 = -2b \quad \therefore b = -\frac{1}{2}$

Power of L, $0 = a + b \quad \therefore a = -b = -(-\frac{1}{2})$

$$a = \frac{1}{2}$$

Substituting the values of a and b in equation,

$$t = k L^{1/2} \cdot g^{-1/2}$$

$$t = k \sqrt{\frac{L}{g}}$$

The value of k is determined from experiments which given as,

$$k = 2\pi$$

$$\boxed{t = 2\pi \sqrt{\frac{L}{g}}}$$

Method of Selecting Repeating variables:

The number of repeating variables are equal to the number of fundamental dimensions of the problem.

1. As far as possible, the dependant variable should not be selected as repeating variable.
2. Variable with
   - i) Geometric property
     - a) Length, l b) d c) Height, H
   - ii) flow property
     - a) velocity, v b) Acceleration
   - iii) fluid property
     - a) μ b) ρ c) w
3. The repeating variables selected should not form a dimensionless group.
4. The same number of fundamental dimensions.
5. No two repeating variable should have the same dimensions.

P.NO : 578

Model :

"Model" is the small scale replica of the actual structure (or) machine.

applications: To Eliminate the defects, Improve performance

Prototype :

The actual structure (or) machine is called "Prototype".

Similitude :

"Similitude" is defined as the similarity between the model & its prototype.

Types :

1) Geometric similarity
2) Kinematic similarity
3) Dynamic similarity

P.NO : 579

1) Geometric similarity :

$$\frac{\forall_p}{\forall_m} = \left(\frac{L_p}{L_m}\right)^3 = \left(\frac{b_p}{b_m}\right)^3 = \left(\frac{D_p}{D_m}\right)^3$$

where,
let,

L_m = Length of model

D_m = Diameter of model

b_m = Breadth of model

A_m = Area of model

$\forall_m$ = Volume of model

$L_p, b_p, D_p, \forall_p$ = Corresponding values of the prototype.

P.NO: 579 2) Kinematic similarity:

$$\frac{a_{p_1}}{a_{m_1}} = \frac{a_{p_2}}{a_{m_2}} = a_r$$

Where,

~~tet~~,

V_{p_1} = velocity of fluid at point 1 in pro

V_{p_2} = " " " point 2 "

a_{p_1} = Acceleration of fluid at point 1 in

a_{p_2} = " " point 2

$V_{m_1}, V_{m_2}, a_{m_1}, a_{m_2}$ = corresponding values at corresponding points of fluid velocity & accel of the model

P.NO: 580 3) Dynamic Similarity:

$$\frac{(F_i)_p}{(F_i)_m} = \frac{(F_v)_p}{(F_v)_m} = \frac{(F_g)_p}{(F_g)_m} = F_r$$

where,

$(F_i)_p$ = Inertia force at a point in Prototyp

$(F_v)_p$ = Viscous force " " " "

$(F_g)_p$ = Gravity force " " " "

$(F_i)_m$ $(F_v)_m$ $(F_g)_m$ = Corresponding values of at the corresponding in model.

580 Types of forces acting in moving fluid:

1) Inertia force, F_i

2) Viscous force, F_v

3) Gravity force, F_g

4) Pressure force, F_p

5) Surface Tension force, F_s

6) Elastic force, F_e

1. Inertia force (F_i):

It is equal to the product of mass & acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration.

2) Viscous force (F_v):

It is equal to the product of shear stress (τ) due to viscosity and surface area of the flow.

3) Gravity force (F_g):

It is equal to the product of mass & acceleration due to gravity of the flowing fluid.

4) Pressure force (F_p):

It is equal to the product of pressure intensity & cross-sectional area of the flowing fluid.

Surface Tension Force (F_s):

It is equal to the product of surface tension & length of surface of the flowing fluid.

Elastic force (F_e):

It is equal to the product of elastic stress & area of the flowing fluid.

Dimensionless Numbers:

Dimensionless numbers are those which are obtained by dividing the inertia force by viscous force (or) gravity force (or) Pressure force (or) Surface tension force (or) Elastic force.

1) Reynold's number
2) Froude's number
3) Euler's number
4) Weber's number
5) Mach's number

1) Reynolds number:

It is defined as the ratio of inertia force of a flowing fluid & the viscous force of the fluid.

$$Re = \frac{V \times d}{\nu} \text{ (or) } \frac{\rho V d}{\mu}$$

2) Froude's number: (Fe)

The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force.

$$Fe = \frac{V}{\sqrt{Lg}} \quad (or) \quad Fe = \sqrt{\frac{Fi}{Fg}}$$

3) Euler's number: (Eu)

It is defined as the square root of the ratio of inertia force to the pressure force of flowing fluid.

$$Eu = \sqrt{\frac{Fi}{Fp}}$$

Calculation,

$$Eu = \frac{V}{\sqrt{P/\rho}}$$

4) Weber's number: (We)

It is defined as the square root of the ratio ~~of~~ to the surface tension force of the flowing fluid.

$$We = \sqrt{\frac{Fi}{Fs}}$$

calculation,

$$We = \frac{V}{\sqrt{\sigma/\rho L}}$$

5) Match's number : (M)

It is defined as the square root of the ratio of the inertia force to the elastic force of a flowing fluid

$$M = \sqrt{\frac{F_i}{F_e}}$$

Calculation,

$$M = \frac{V}{C}$$

P.NO: 583

Model Laws (or) Similarity Laws :

The dynamic similarity between the model & the prototype, the ratio of the corresponding forces acting at the corresponding points in the model and prototype should be equal.

1. Reynold's model law - 583
2. Froude model law - 587
3. Euler model law - 595
4. Weber model law - 596
5. Mach mode law - 596

ITE, INDUS UNIVERSITY
DEPARTMENT OF CIVIL ENGINEERING
FLUID MECHANICS-II (CV0415)

1) Reynold's model law :

Reynold's model law is the law in which models are based on Reynold's number. Models based on Reynold's number includes :

i) Pipe flow

ii) Resistance experinced by sub-marines, airplanes, fully immersed bodies etc.

$$[Re]_m = [Re]_p \text{ (or) } \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

where,

V_m = velocity of fluid in model

ρ_m = Density of fluid in model

L_m = Length (or) linear dimension of the model

μ_m = viscosity (or) fluid in model

V_p, ρ_p, L_p and μ_p = Corresponding values of velocity, density, linear & viscosity of fluid in prototype.

2) Froude model Law :

Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model & prototype, the Froude number for both of them should be equal.

$$[Fe]_{model} = [Fe]_{Prototype}$$

[or]

$$\frac{V_m}{\sqrt{g_m L_m}} = \frac{V_P}{\sqrt{g_P L_P}}$$

where,

g_m = Acceleration due to gravity at a pl[ace where] model is tested.

3) Euler's model law:

$$[Eu]_{model} = [Eu]_{Prototype}$$

(or)

$$\frac{V_m}{\sqrt{P_m}} = \frac{V_P}{\sqrt{P_P}}$$

4) Weber model law:

$$[We]_{model} = [We]_{Prototype}$$

(or)

$$\frac{V_m}{\sqrt{\sigma_m / \rho_m \cdot L_m}} = \frac{V_P}{\sqrt{\sigma_P / \rho_P \cdot L_P}}$$

5) Mach model Law:

$$[M]_{model} = [M]_{Prototype}$$

(or)

$$\frac{V_m}{\sqrt{K_m / \rho_m}} = \frac{V_P}{\sqrt{K_P / \rho_P}}$$

-568

Problems :

1) a) State Buckingham's π-theorem.

b) The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor & the discharge Q. Express η in terms of dimensionless parameters.

a plac

sol :

$$\eta = f(\rho, \mu, \omega, D, Q)$$

(or)

$$f(\eta, \rho, \mu, \omega, D, Q) = 0 \longrightarrow (1) \text{ eqn}$$

The total no. of variables $n = 6$

η = Dimensionless

$\rho = ML^{-3}$

$\mu = ML^{-1}T^{-1}$

$\omega = T^{-1}$

$D = L$

$Q = L^3 T^{-1}$

$\therefore m = 3 \qquad f(\pi_1, \pi_2, \pi_3) = 0 \rightarrow (2)$ eqn

Number of π-terms $= n - m = 6 - 3 = 3$

$$\pi_1 = D^{a_1} \cdot \omega^{b_1} \cdot \rho^{c_1} \cdot \eta$$

$$\pi_2 = D^{a_2} \cdot \omega^{b_2} \cdot \rho^{c_2} \cdot \mu$$

$$\pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$$

First π-term :

subs. dimensions on both sides of π_1

$$M^0 L^0 T^0 = L^{a_1} . (T^{-1})^{b_1} . (ML^{-3})^{c_1} . M^1 L^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_1 + 0$, $\therefore c_1 = 0$

Power of L $0 = a_1 + 0$, $\therefore a_1 = 0$

Power of T $0 = -b_1 + 0$ $\therefore b_1 = 0$

subtituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = D^0 \omega^0 \rho^0 . \eta = m$$

second π-term : $\pi_2 = D^{a_2} . \omega^{b_2} . \rho^{c_2} . \mu$

substituting the dimensions on both sides

$$M^0 L^0 T^0 = L^{a_2} . (T^{-1})^{b_2} . (ML^{-3})^{c_2} . M L^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_2 + 1$ $\therefore c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1$ $\therefore a_2 = 3c_2 + 1$

$= -3 + 1$

$a_2 = -2$

Power of T, $0 = -b_2 - 1$ $\therefore b_2 = -1$

sub. the values a_2, b_2 and c_2 in π_2

$$\pi_2 = D^{-2} . \omega^{-1} . \rho^{-1} . \mu = \frac{\mu}{D^2 \omega \rho}$$

Third π-term :

$$\pi_3 = D^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$$

substituting the dimensions on both sides,

$$M^0 L^0 T^0 = L^{a_3} \cdot (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot L^3 T^{-1}$$

Equating the powers of M, L and T on both sides,

Power of M, $0 = c_3$ $\therefore c_3 = 0$

Power of L, $0 = a_3 - 3c_3 + 3$ $\therefore a_3 = 3c_3 - 3$
$= -3$

Power of T, $0 = -b_3 - 1$ $\therefore b_3 = -1$

sub. the values of a_3, b_3 and c_3 in π_3

$$\pi_3 = D^{-3} \omega^{-1} \cdot \rho^0 \cdot Q = \frac{Q}{D^2 \omega}$$

substituting the values of π_1 and π_2 in

(2) - eqn

$$f_1\left(\eta, \frac{\mu}{D^2 \omega \rho}, \frac{Q}{D^2 \omega}\right) = 0$$

[or]

$$\eta = \phi\left[\frac{\mu}{D^2 \omega \rho}, \frac{Q}{D^2 \omega}\right]$$

2) In 1 in 40 model of a spillway, the velo[city]
P.No : 590
and discharge are 2 m/s and 2.5 m^3/s.
Find the corresponding velocity & discharge in the prototype.

Data:

Scale ratio of length, $L_r = 40$

velocity of model, $V_m = 2$ m/s

Discharge of model, $Q_m = 2.5\ m^3/s$

Find:

i) velocity of Prototype, $V_p = ?$

ii) Discharge of Prototype, $Q_p = ?$

sol:

i) velocity of Prototype, V_p

$$\text{velocity ratio}, \frac{V_p}{V_m} = \sqrt{L_r}$$

$$V_p = V_m \times \sqrt{L_r}$$

$$= 2 \times \sqrt{40}$$

$$V_p = 12.64\ m/s$$

ii) Discharge of Prototype, Q_p

$$\text{Discharge ratio}, \frac{Q_p}{Q_m} = L_r^{2.5}$$

$$Q_p = (L_r)^{2.5} \times Q_m$$

$$= (40)^{2.5} \times 2.5$$

$$Q_p = 25298.22\ m^3/s$$

P.NO : 604

classification of models :

1) Undistorted models
2) Distorted models

1) Undistorted models :

"Undistorted models" are those models which are geometrically similar to their prototypes (or) in other words if the scale ratio for the linear dimensions of the model & its prototype is same, the model is called "Undistorted model".

2) Distorted models :

A model is said to be distorted if it is not geometrically similar to its prototype. For a distorted model different scale ratios for the linear dimensions are adopted.

Advantages of Distorted models :

1) The vertical dimensions of the model can be measured accurately.

2) The cost of the model can be reduced.

3) Turbulant flow in the model can be maintained.

Scale Ratios for Distorted Models:

1. Scale ratio for velocity:

Let,

V_P = velocity in prototype

V_m = velocity in model

Then,

$$\frac{V_P}{V_m} = \frac{\sqrt{2gh_P}}{\sqrt{2gh_m}} = \sqrt{\frac{h_P}{h_m}} = \sqrt{(L_r)_V}$$

$$\therefore \frac{h_P}{P_m} = (L_r)_V$$

2. Scale ratio for area of flow.

Let,

A_P = Area of flow in prototype = $B_P \times h_P$

A_m = Area of flow in model = $B_m \times h_m$

Then,

$$\frac{A_P}{A_m} = \frac{B_P \times h_P}{B_m \times h_m} = \frac{B_P}{B_m} \times \frac{h_P}{h_m} = (L_r)_H \times (L_r)_V$$

3. Scale ratio for discharge:

let,

Q_P = Discharge through prototype = $A_P \times V_P$

Q_m = Discharge " model = $A_m \times V_m$

Then,

$$\frac{Q_P}{Q_m} = \frac{A_P \times V_P}{A_m \times V_m} = (L_r)_H \times (L_r)_V \times \sqrt{(L_r)_V} = (L_r)_H \times [(L_r)_V]^{3/2}$$

P.No: 608

The discharge through a weir is $1.5\ m^3/s$. Find the discharge through the model of the weir if the horizontal dimension of the model $= \frac{1}{50}$ the horizontal dimension of the prototype & vertical dimension of the model $= \frac{1}{10}$ the vertical dimension of the prototype.

Sol:

Discharge through weir (prototype) $= Q_p = 1.5\ m^3/s$

$$\left.\begin{array}{r}\text{Horizontal dimension of} \\ \text{model}\end{array}\right\} = \frac{1}{50} \times \begin{array}{l}\text{Horizontal} \\ \text{dimension of} \\ \text{prototype}\end{array}$$

$$\therefore \frac{\text{Horizontal dimension of prototype}}{\text{Horizontal dimension of model}} = 50 \ [\text{or}]$$

$$(L_r)_H = 50$$

$$\text{vertical dimension of model} = \frac{1}{10} \times \begin{array}{l}\text{vertical dimension} \\ \text{of prototype}\end{array}$$

$$\therefore \frac{\text{vertical dimension of Prototype}}{\text{vertical dimension of model}} = 10$$

$$(L_r)_V = 10$$

Using equation,

$$\frac{Q_P}{Q_m} = (L_r)_H \times \left[(L_r)_V\right]^{3/2}$$

$$= 50 \times 10^{3/2}$$

$$\frac{Q_P}{Q_m} = 1581.14$$

$$Q_m = \frac{Q_P}{1581.14} = \frac{1.50}{1581.4}$$

$$Q_m = 0.00094\ m^3/s$$

$$\Rightarrow \boxed{Q_m = 0.948\ lit/s}$$

$B_p \times h_p$

$\times h_m$

$\times (L_r)_V$

$A_p \times V_p$

$\times V_m$

$\times \left[(L_r)_V\right]^{3/2}$