UNIT – 1

FUNDAMNETALS OF STRUCTURAL DYNAMICS

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Curriculum

- Static Load v/s Dynamic Load
- Simplified Single Degree of Freedom (SDOF) System
- Mathematical Modelling of SDOF System
- Response of SDOF System to different types of Vibrations like Free, Undamped, Damped and Forced Vibration
- Response of Building to Earthquake Ground Motion
- Multi-degree of Freedom System (MDOF), Work Examples
- Periods and Modes of Vibration for MDOF System

Structural Dynamics

- Study of response of structure under dynamic loading is known as structural dynamics.
- Majority of civil engineering structures are designed with the assumption that all applied loads are static!!!!
- The effect of dynamic load is not considered because the structure is rarely subjected to dynamic loads; more so its consideration in analysis makes the solution more complicated and time consuming.
- This feature of neglecting dynamic loads may sometimes becomes the cause of disaster.
- Hence, Now a days there is grown interest in the process of designing civil engineering structures capable to withstand dynamic loads.

Basic Concepts of Structural Dynamics

Vibration and Oscillation – If motion of structure is oscillating (pendulum) or reciprocating along with deformation of structure, is termed as VIBRATION.

In case, there is no deformation that implies only rigid body motion then it is termed as OSCILLATION.

Free Vibration – Vibration of the system which is initiated by a force which is subsequently withdrawn. Hence this vibration occurs without external force.

Forced Vibration – If the external force is involved during the vibration then it is called as forced vibration.

Damping – All real life structures when subjected to vibration resist it. Due to this the amplitude of the vibration, gradually decreases with respect to time. In case of fee vibration, the motion of the system is damped out eventually. Damping forces depend on a number of factors and it is very difficult to quantify them.

The commonly used representation is viscous damping. Wherein damping force is expressed as $F_d = c\dot{x}$.

Where, $\dot{x}$ = velocity and c = damping constant.

Static Load Vs Dynamic Load

Static Load – A static Load is either constant or applied over long period of time.

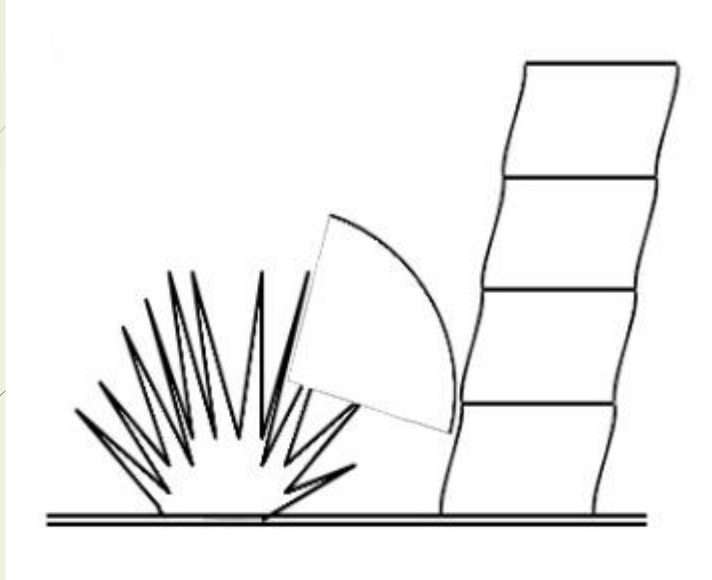
- Resistance due to internal elastic forces of structure.
- All dead loads are static loads.

Dynamic Load – A dynamic load is a load that is variable and applied over short period of time.

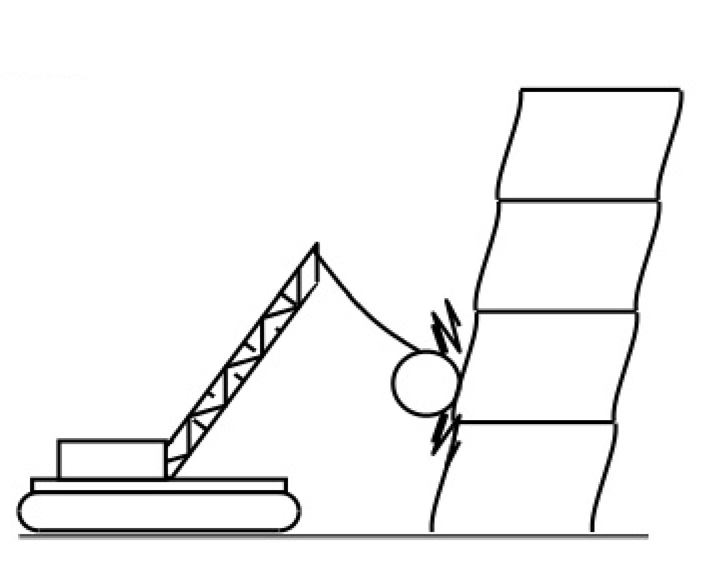
- Accelerations producing inertia forces (Inertia forces from a significant portion of load equilibrated by internal elastic forces of structure)
- Wind load, Moving loads, Machine loads, Impact and Blast loads etc.

- ✓ Structures in general respond very differently to static and dynamic loading.
- ✓ Response due to static loading is displacement only.
- ✓ Response due to dynamic loading is displacement, velocity and acceleration.

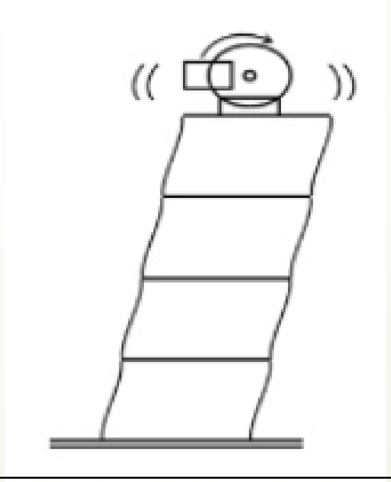
Examples of Dynamic Loading



Blast

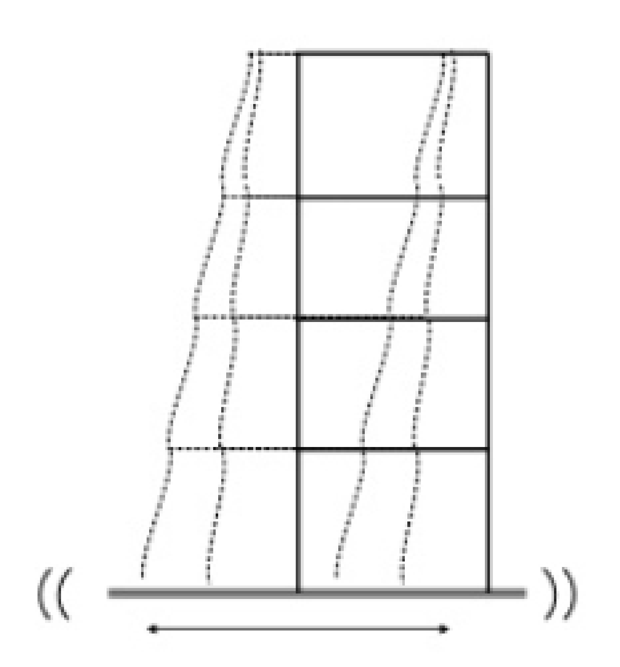


Impact

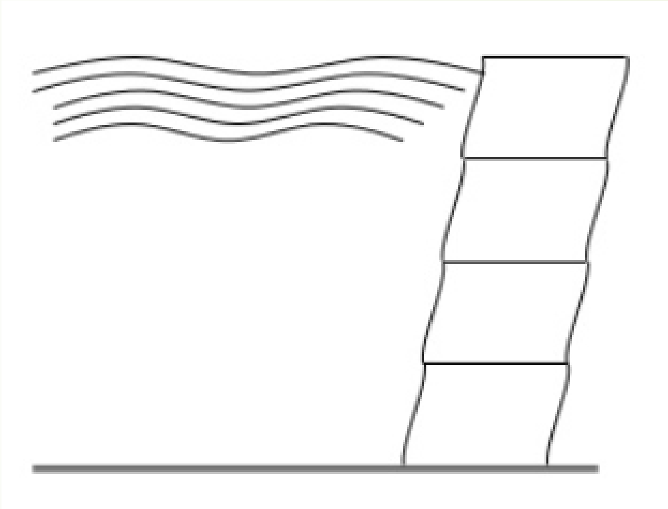


Machine Vibration

Examples of Dynamic Loading

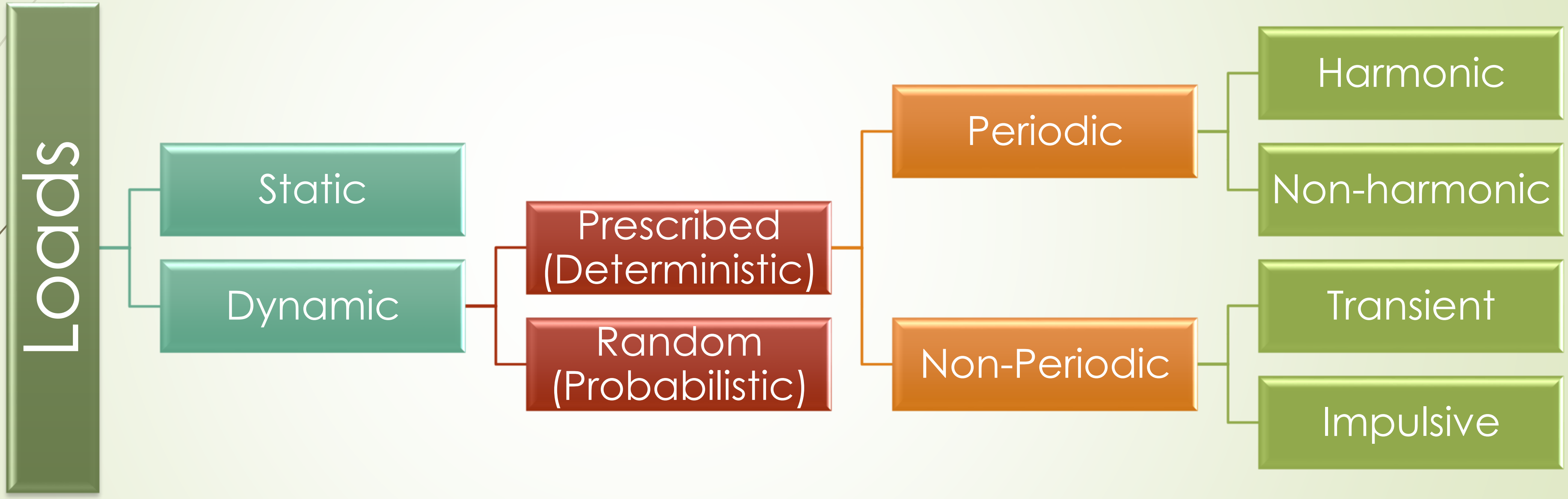


Ground motion
(Earthquake)



Wind

Classification of Loading



Mathematical Modelling

- Study of structural dynamics involves developing an insight into the dynamic behaviour of the structural systems by investigating the behaviour of their models under the influence of dynamic loads.

- The models used in these investigations can be either small / large scale laboratory models for experimental studies or can be mathematical models for analytical studies.

- Real test can not be performed on a structure, so mathematical modelling becomes inherent part of dynamic analysis.

- The link between real physical system and mathematically feasible solution is provided by mathematical model.

- It is the symbolic designation for the substituted idealised system including all assumptions imposed on the physical problem.

Mathematical Modelling

Mass Element, m – Representing mass and inertial characteristic of structure

Spring Element, k – Representing the elastic restoring force and potential energy capacity of structure.

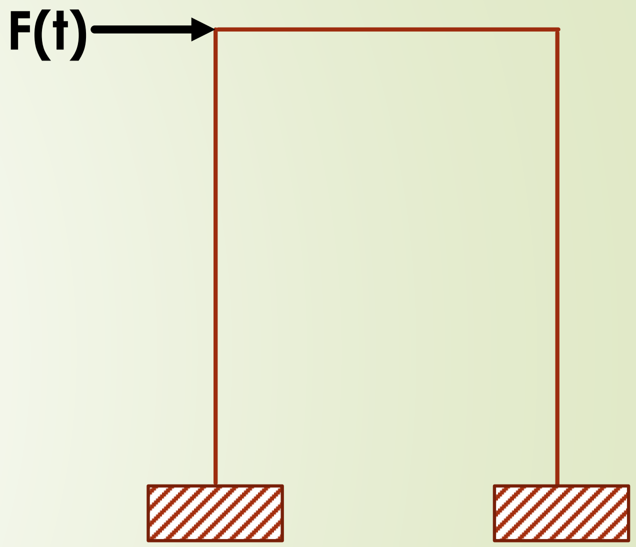
Dashpot, c – Representing frictional characteristics and energy losses of structure

Excitation force, F(t) – Represents the external force acting on structure (F(t) indicates that the force is function of time)

Such pure elements do not exist in our physical world and that mathematical models are only conceptual idealization of real structures.

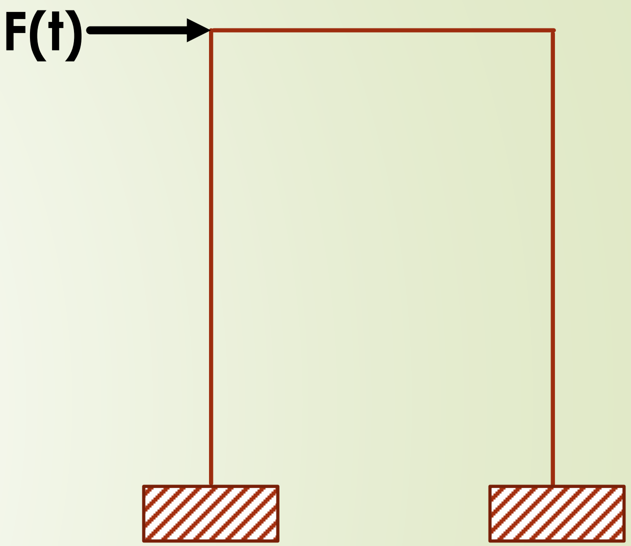
Mathematical Model of a Portal Frame

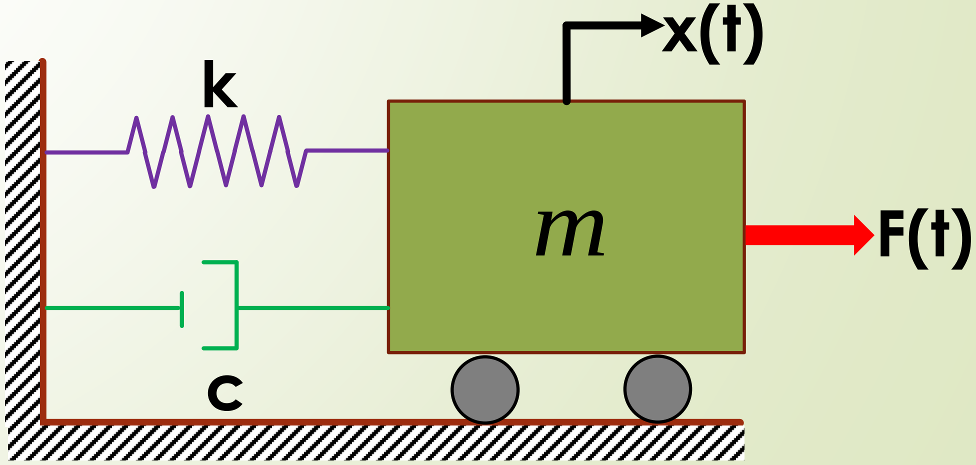
- Consider a development of mathematical model for lateral load analysis of a simple portal frame.
- ✓ The mass of the columns is very small in comparison to that of slab, it is reasonable to assume that the entire mass of portal frame is concentrated at slab level.
- ✓ Axial rigidity of the beam and slab is very large in comparison with the stiffness of columns in lateral load deformations, so it is a good approximation to assume that the beam/slab is infinitely rigid and entire lateral deformation is due to flexural deformations in columns.
- ✓ The change in length of columns due to lateral deformations being small and is not very significant, so it is assumed that the axial stretch in columns is negligible.
- ✓ As the beams are usually cast monolithically with the columns, joints can be assumed to be rigid as the relative rotation between beam and column at the joint will be negligible.



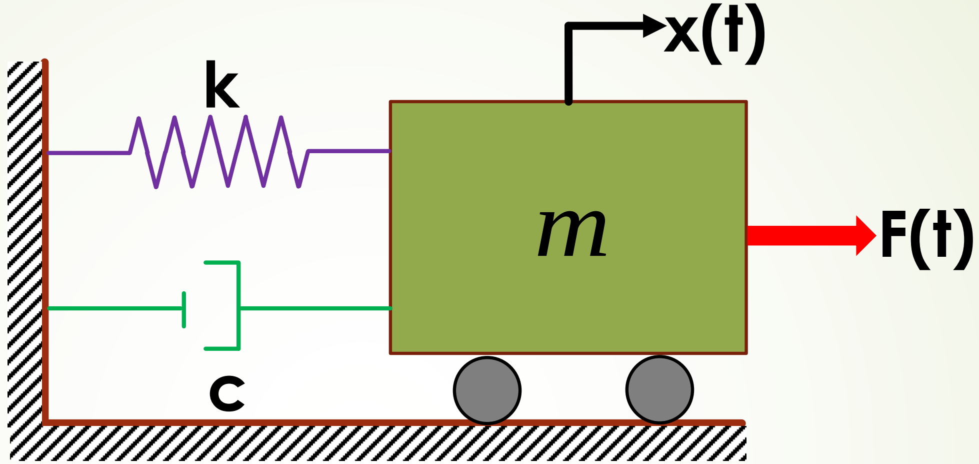
Mathematical Model of a Portal Frame

- **With these simplifying kinematic constraints, the lateral displacement of the rigid beam/slab is the only possible mode of deformation in the system.**
- **Since, the entire mass is concentrated at the slab level, the inertial effects in the model can be completely determined from the knowledge of the motion of slab.**
- **The model resulting from all the above mentioned simplifying assumptions is known as *shear building model*.**
- **The lateral deformation of the portal frame can be represented as the response of SDOF system shown in figure.**





Mathematical Model of a Portal Frame



Mass Element, m – Total mass of the beam and slab of the frame and serves as the storage of kinetic energy

Spring Element, k – Represents the combined stiffness of two columns for lateral deformations and stores the internal strain energy due to column deformations

Dashpot, c – Represents energy dissipation due to various sources

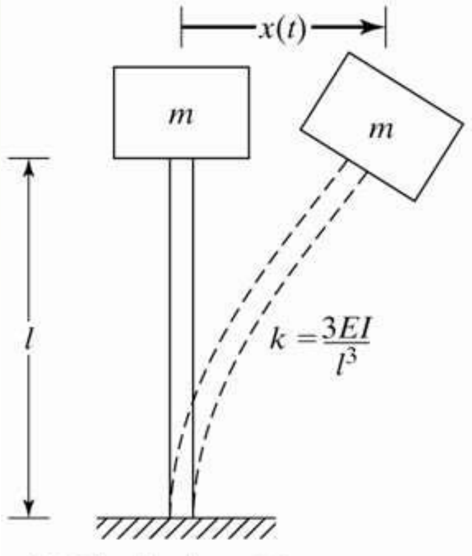
Excitation force, F(t) – Represents the lateral force applied on portal frame

Dynamic Degrees of Freedom

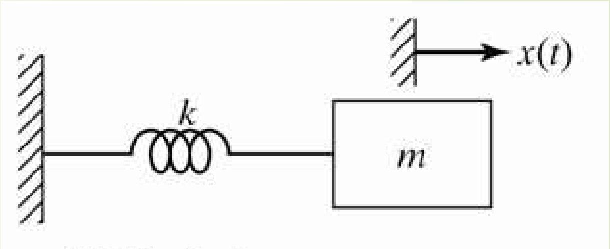
- The number of independent displacement components that must be considered to represent the effects of all significant inertia forces of a structure.
- Depending upon the co-ordinates to describe the motion, we can have following different types of systems

1. Single degree of freedom system (SDoF/SDOF)
2. Multiple degree of freedom (MDoF/MDOF)
3. Continuous system (Distributed System)

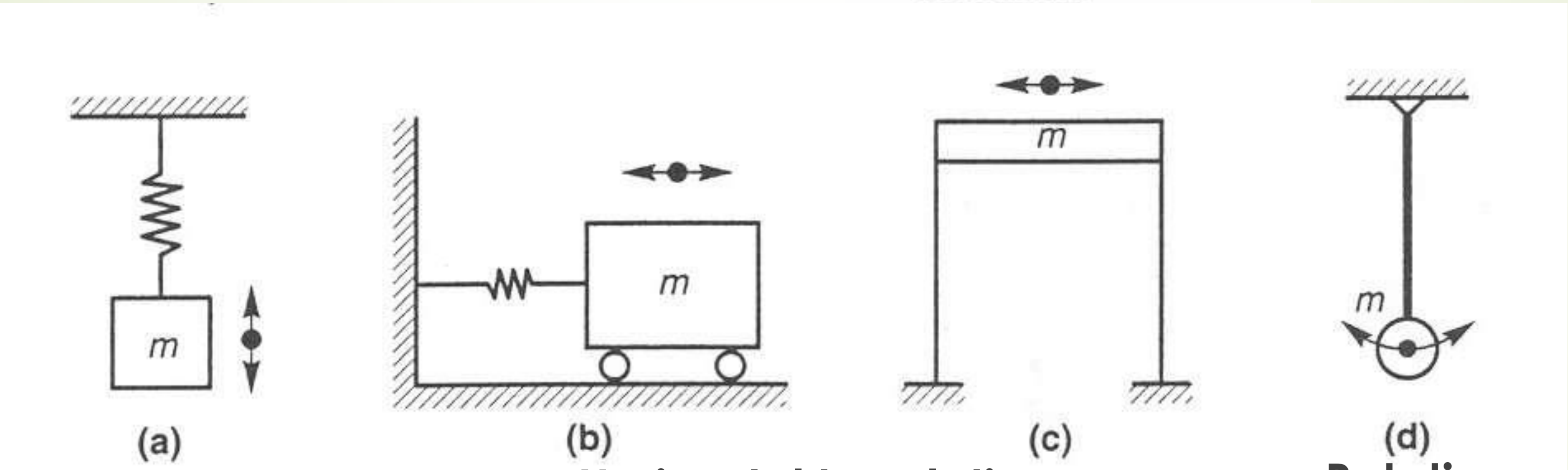
Single Degree of Freedom System



(a) Idealization of the tall structure



(b) Equivalent spring-mass system

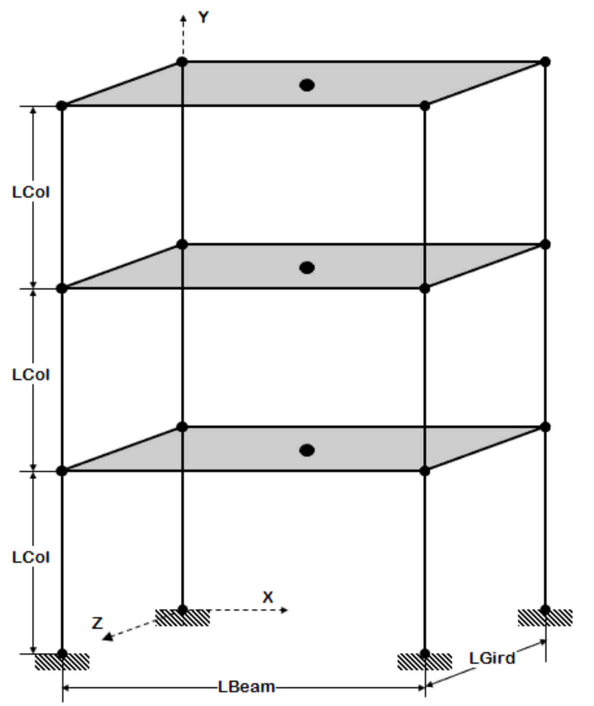


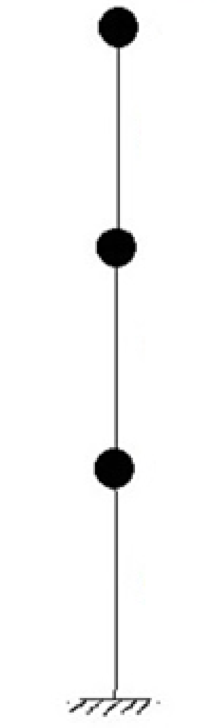
V

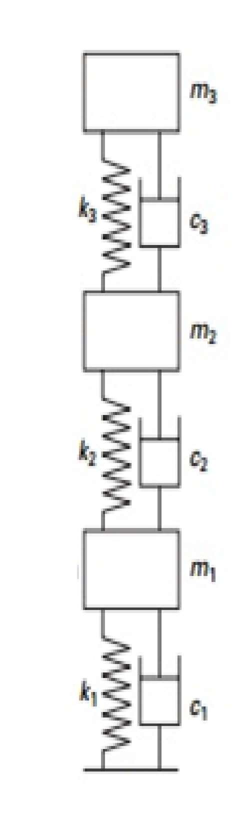
Horizontal translation

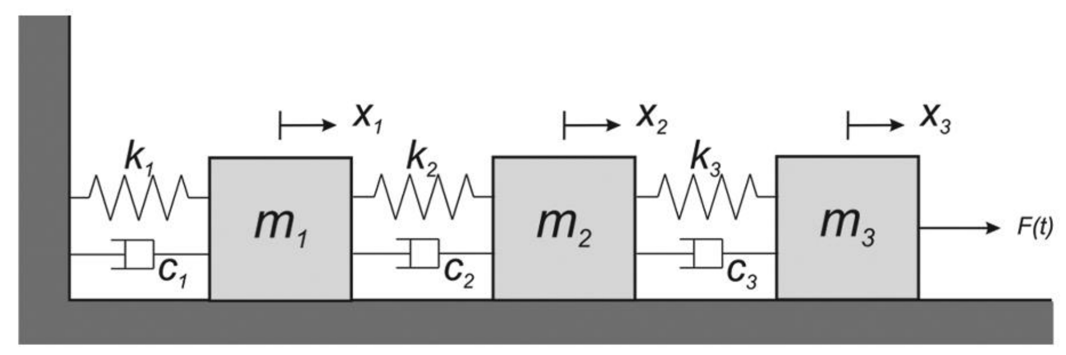
Rotation

Multiple Degree of Freedom System



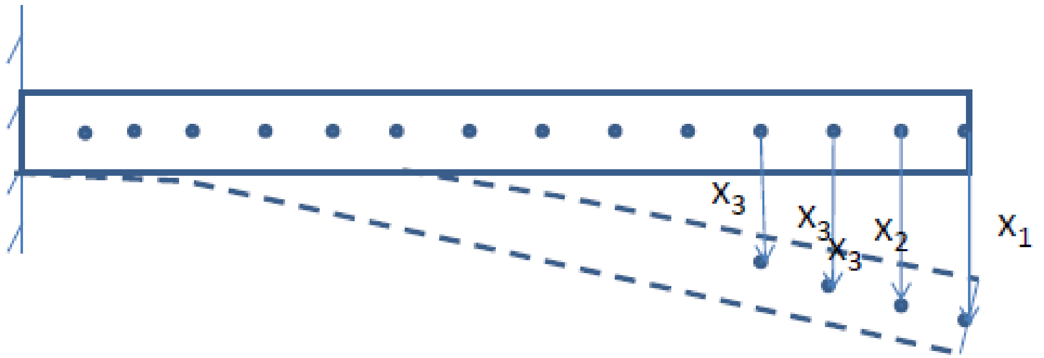






Continuous System

- If the mass of a system may be considered to be distributed over its entire length as shown in figure, in which the mass is considered to have infinite degrees of freedom, it is referred to as a continuous system.
- It is also known as distributed system.



Cantilver beam with infinite number of degrees of freedom

Approaches to Develop Equation of Motion for SDOF System

- **Differential equation describing the motion is known as equation of motion.**

1. Newtown's second law of motion $m\ddot{x} = -\mathrm{kx}$ $\ddot{x} + {\omega_n}^2 x = 0$
2. D'Alembert's Principle Inertia force, $F_i = m\ddot{x}$ $-m\ddot{x} - kx = 0$
3. Principle of Virtual work
4. Hamilton's principle
5. Lagrange's equation

- ✓ 1, 2 are based on vector principles of vector mechanics.
- ✓ 4, 5 are based on variational principles.
- ✓ 3rd one is extension of equilibrium methods.

D'Alembert's Principle

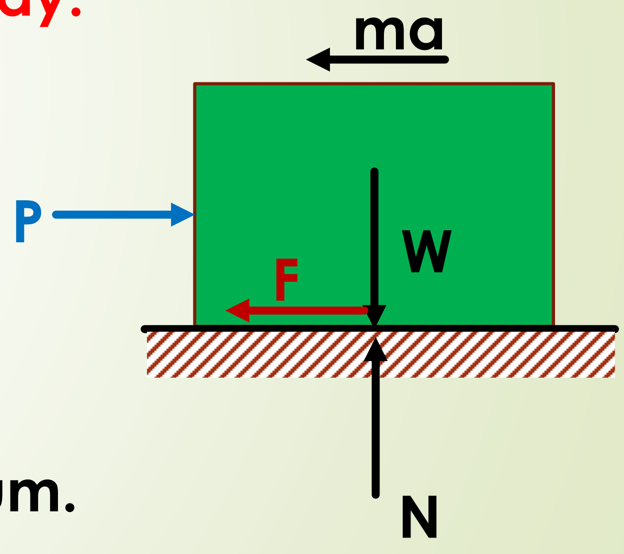
- It States that

"The body will be in dynamic equilibrium under the action of external force and inertia force of the body."

$$\Sigma F = m a$$

$$\Sigma F - m a = 0$$

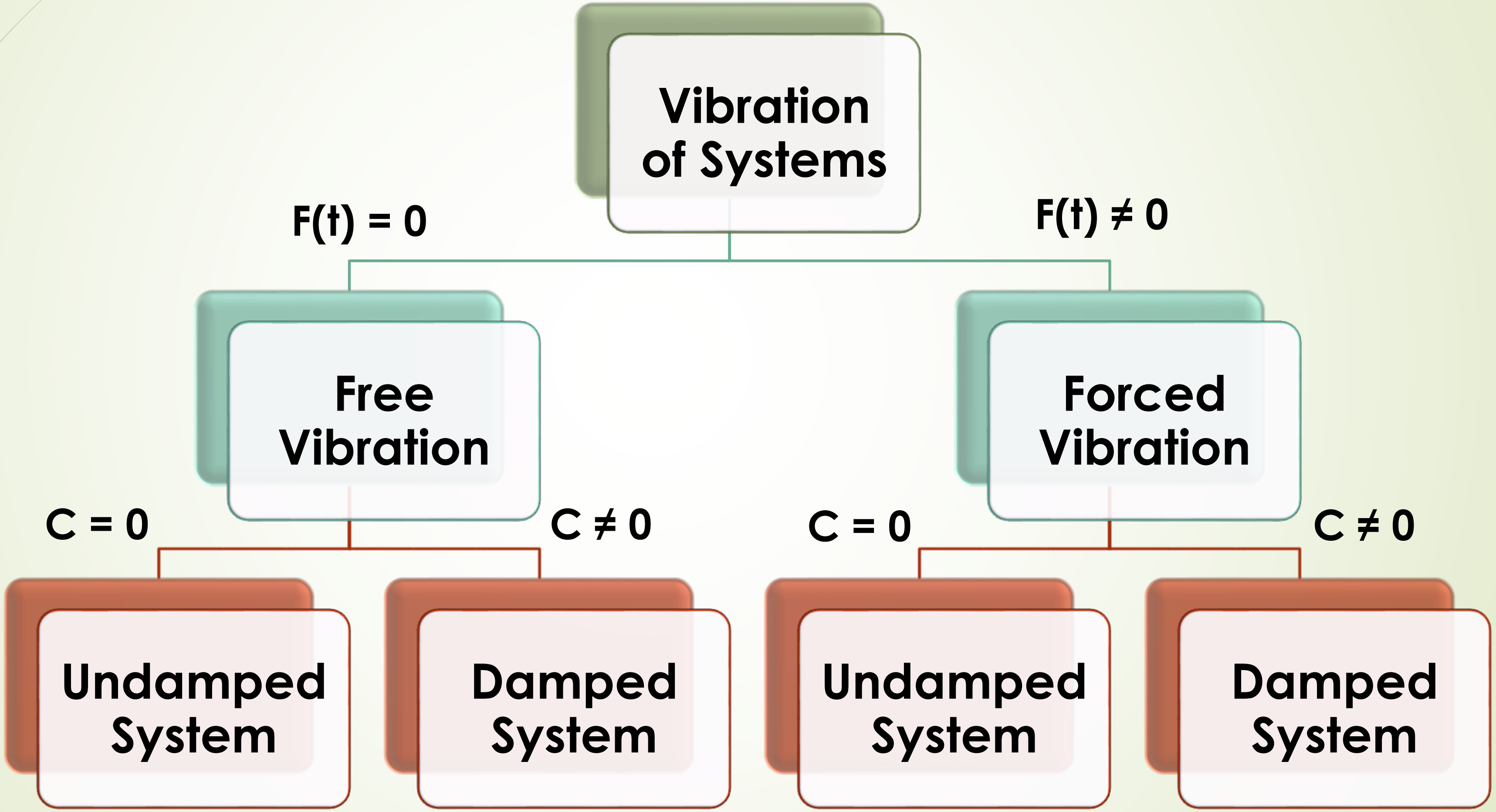
$$\Sigma F + (- m a) = 0$$



EXAMPLE

It is also known as equation of dynamic equilibrium.

Classification of Vibration of Systems

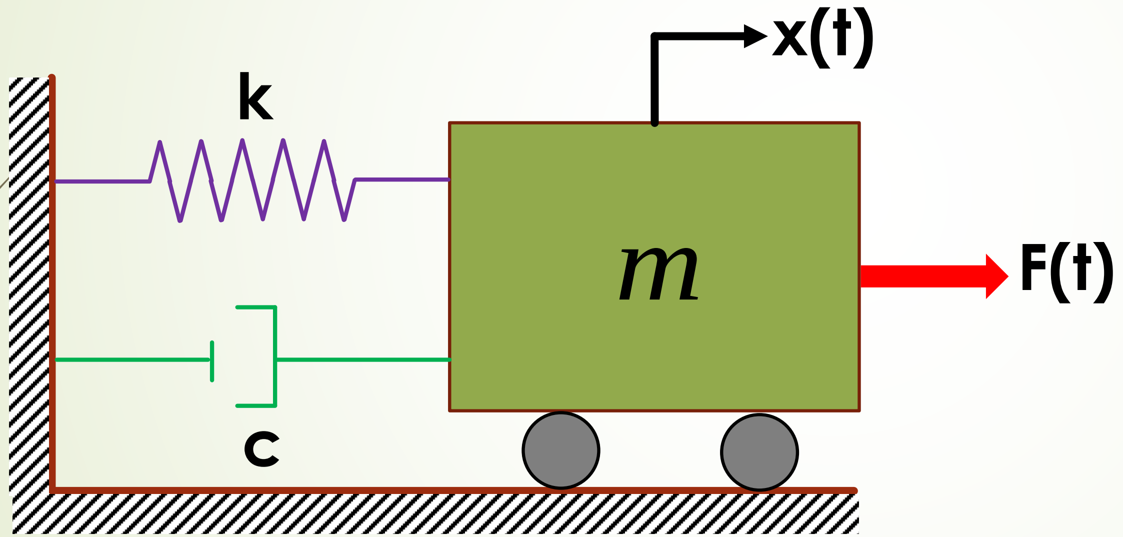


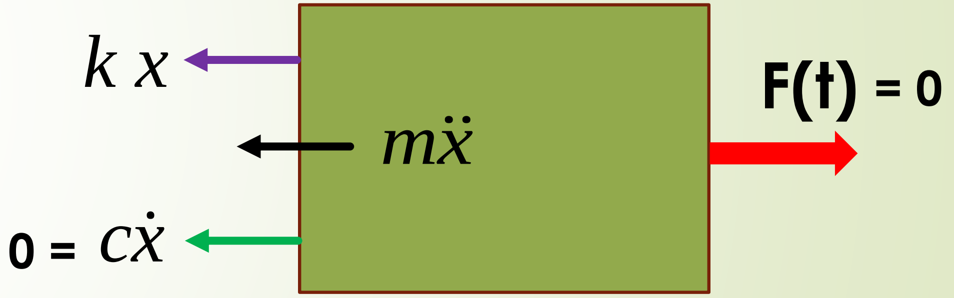
Free Undamped Vibration of SDOF System

Free Undamped Vibration of SDOF System

$m\ddot{x} =$ Inertia force, $c\dot{x} =$ Damping force and

$kx =$ Restoring force





Free Body Diagram

Free Undamped Vibration of SDOF System

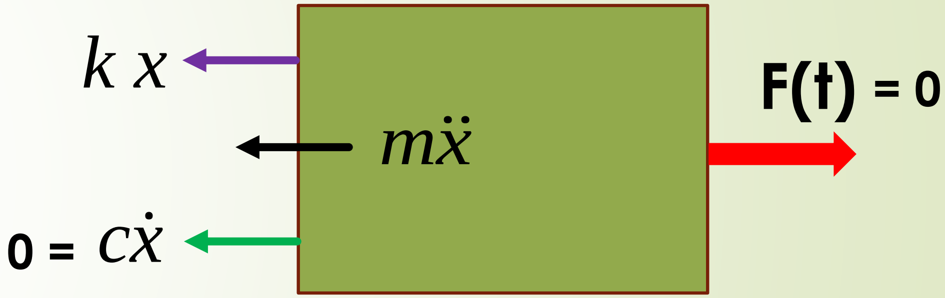
Applying D'Alembert's principle,

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ --------- (i)}$$

For, free undamped vibrating system

$$F(t) = 0 \text{ and } c = 0$$

$$\Rightarrow m\ddot{x} + kx = 0 \text{ --------- (A)}$$



Free Body Diagram

- **Equation (A) is higher order homogeneous differential equation.**
- **Assuming x = e^{st} as a general solution,**
- **D = $\dot{x}$ = s. e^{st} and D^2 = $\ddot{x}$ = s^2. e^{st}**

$\dot{x}$

Free Undamped Vibration of SDOF System

- Substituting above values in equation (A) we get,
- $m(s^2 . e^{st}) + k (e^{st}) = 0 \Rightarrow e^{st} (ms^2 + k) = 0$
- for above equation there are two possible solutions,
- $e^{st} = 0$ or $(ms^2 + k) = 0$
- Out of these two solutions $e^{st} = 0$ cannot be a solution because x = 0 is always a solution for homogeneous equation (Trivial solution).
- So, considering $(ms^2 + k) = 0$ as a solution (Non-trivial solution).

$$\Rightarrow s = \sqrt{-\frac{k}{m}}$$

$$\Rightarrow s = \pm i\omega_n (\text{where, } \omega_n = \text{natural angular frequency})$$

$\dot{x}$

Free Undamped Vibration of SDOF System

$$\Rightarrow s = \sqrt{-\frac{k}{m}}$$

$$\Rightarrow s = \pm i\omega_n \text{(where, } \omega_n = \text{natural angular frequency)}$$

- **Above statement shows two complex roots are possible.**

$$x(t) = c_1 e^{+i\omega_n t} + c_2 e^{-i\omega_n t} \text{ --------- (ii)}$$

$$\text{we know that, } e^{i\theta} = \cos\theta + i\sin\theta \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta$$

- **Substituting above values in equation (ii) we get**

$$x(t) = c_1(\cos\omega_n t + i\sin\omega_n t) + c_2(\cos\omega_n t - i\sin\omega_n t)$$

$\dot{x}$

Free Undamped Vibration of SDOF System

$$x(t) = c_1(\cos\omega_n t + i\sin\omega_n t) + c_2(\cos\omega_n t - i\sin\omega_n t)$$

$$= (c_1 + c_2)\cos\omega_n t + (c_1 - c_2)\ i\sin\omega_n t$$

$$= A\cos\omega_n t + B\sin\omega_n t \quad ------- \text{(iii)}$$

Differentiating equation (iii)

$$\dot{x} = \text{-}A\ \omega_n \sin\omega_n t + B\ \omega_n \cos\omega_n t \quad ------ \text{(iv)}$$

The boundary conditions are,

At $t = 0,\ x = x_0$

substituting in equation (iii)

$$\Rightarrow \text{A} = x_0$$

$\dot{x}$

Free Undamped Vibration of SDOF System

The boundary conditions are,

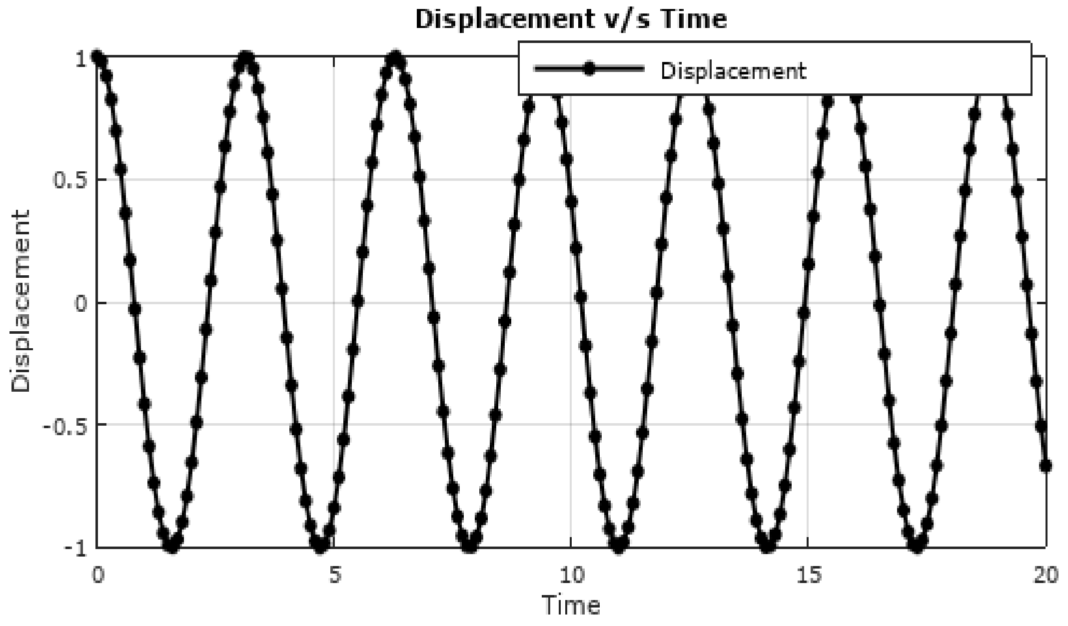
At $t = 0, \ \dot{x} = \dot{x}_0$

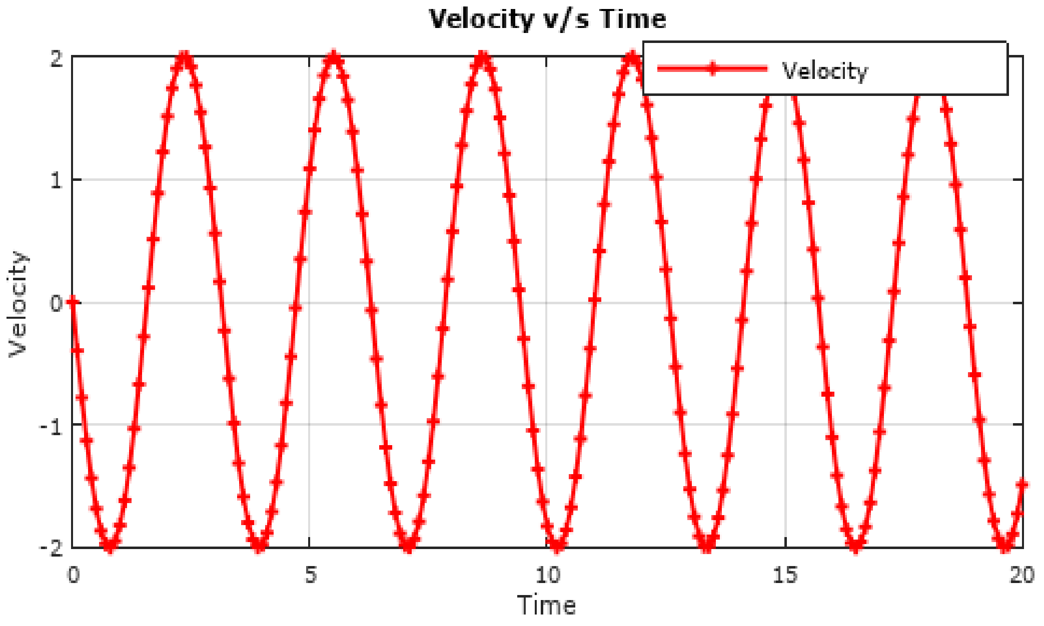
substituting in equation (iv)

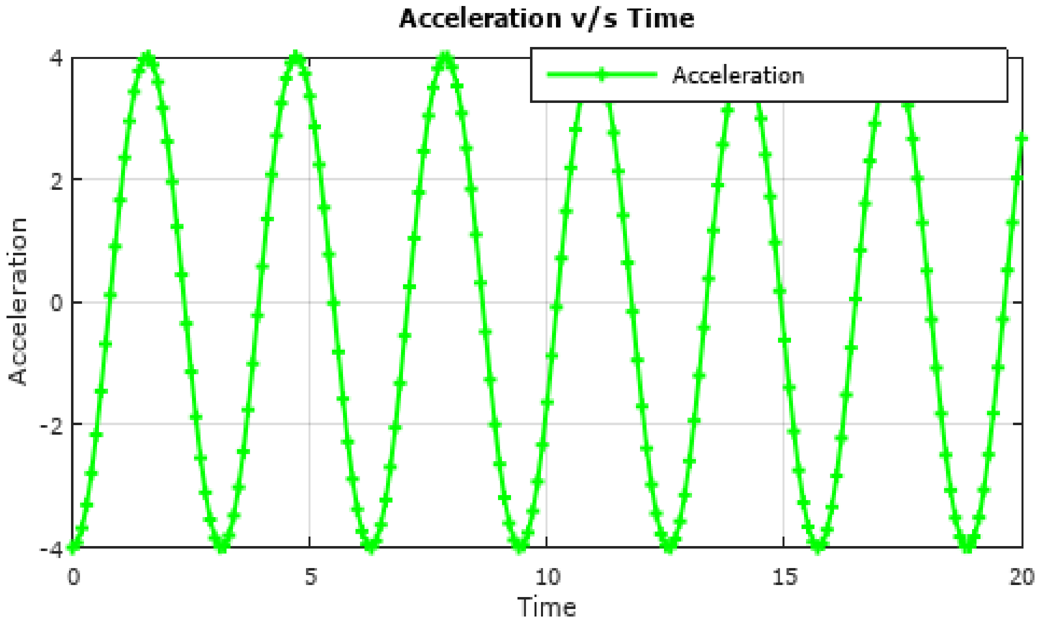
$$\Rightarrow B = \frac{\dot{x}_0}{\omega_n}$$

substituting values of A and B in $x(t) = A\cos\omega_n t + B\sin\omega_n t$

$$x = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$







Free Vibration of SDOF

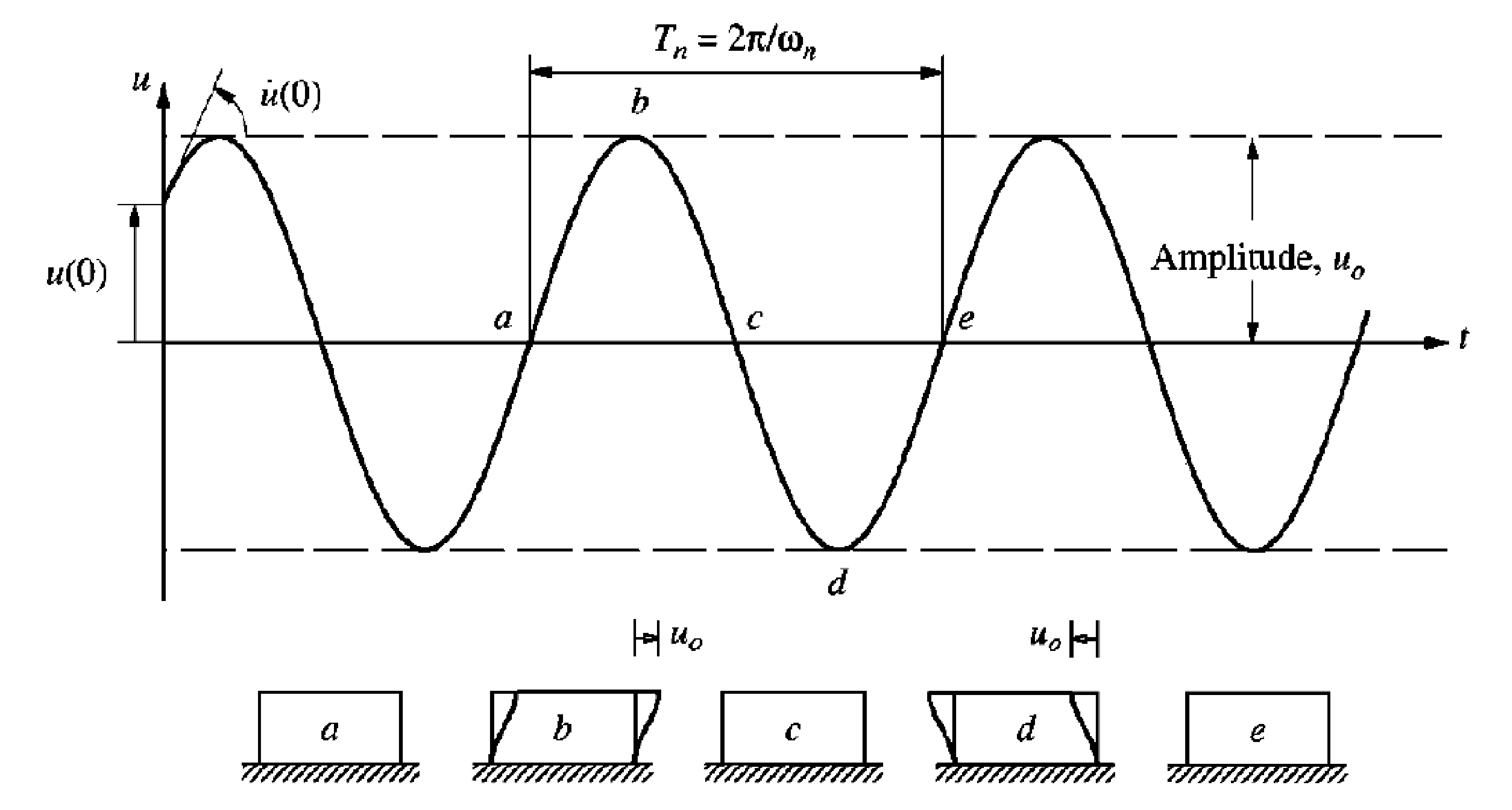
$$y_o = 1\ m$$

$$\dot{y}_0 = 0\ \text{m/s}$$

$$m = 2\ \text{kg}$$

$$k = 8\ \text{N/m}$$

Free Undamped Vibration of SDOF System



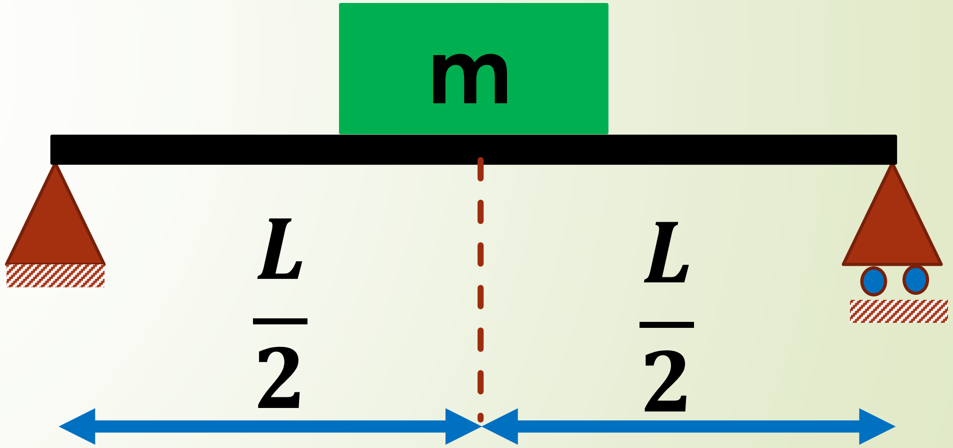
$$\text{Amplitude} = \sqrt{(x_0)^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2}$$

EX-1 A mass 'm' is attached to the mid point of a simply supported beam of length L. The mass of the beam is small as compared to mass 'm'. Determine the spring constant and the frequency of the vibration of the beam in vertical direction. The beam has uniform flexural rigidity EI.

$$\delta = \frac{PL^3}{48EI} \qquad \Rightarrow k = \frac{48EI}{L^3}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$f = 1.102\sqrt{\frac{EI}{mL^3}}$$



EX-2 Consider a rigid frame shown in figure, having infinitely rigid girder which is distributed horizontally (37 kg/m) by initial condition of $x_0 = 0,\ \dot{x}_0 = 3\ m/s$ at $t = 0$.

Find:

(a) Natural period and frequency

(b) Displacement and velocity at any time 't'.

(c) Forces in columns AB and CD at t = 2sec.

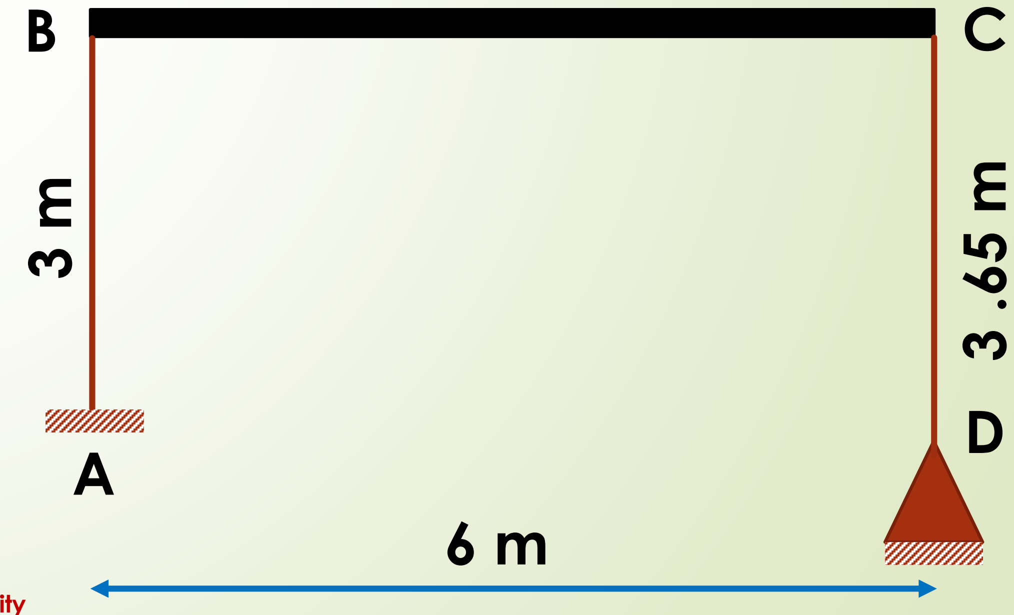
Take I = 6938 cm^4 and E = 20684 kN/cm^2.

$$m = 37*6 = 222\ kg$$

$$k = \left[\frac{12EI}{L^3}\right]_{AB} + \left[\frac{3EI}{L^3}\right]_{CD}$$

$$k = 7263086\ N/m$$

$$EI = 14.35*10^6\ Nm^2$$



$\dot{x}$

CONTINUED….

$$\omega = 180.87\ rad/\sec$$

$$f = 28.78\ Hz$$

$$T = 0.0347\ \sec$$

using given intial conditions in equations of motion of free undamped vibration, we get

$$x = \frac{3}{180.87}\sin 180.87\,t \quad \text{and}$$

$$\dot{x} = \frac{3}{180.87}\cos 180.87\,t\ (180.87)$$

CONTINUED….

Forces in column

Column AB

A = S D

$D = -7.31*10^{-3}$m

$S = 6.37 * 10^{6}$ N/m

A = -46.56 kN

Column CD

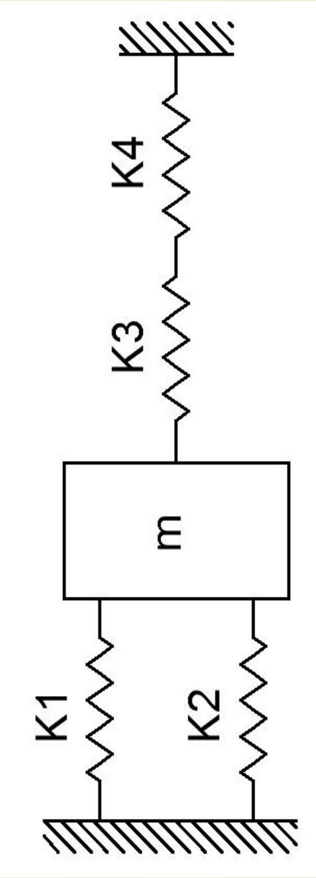
A = S D

$D = -7.31*10^{-3}$m

S = 885308 N/m

A = -6.47 kN

EX-3 For a system shown in figure determine the displacement and velocity after 1 second, if the initial displacement and velocity are 2.5 cm and 5 cm/sec respectively for the mass. Also calculate amplitude of vibration. Take, EI = 3 x 10^9 Ncm^2 and W = 15000 N. take, $k_1 = k_4 = 150$ N/m and $k_2 = k_3 = 100$ N/m.



$\dot{x}$

CONTINUED….

Equivalent Stiffness of system, k = 310 N/m

W = 15000 N

m = 1529.05 kg (Assuming, g = 9.81 m/s^2)

$\omega_n = 0.4502\ rad/\sec$

Amplitude = 113.84 mm

Displacement at t = 1 sec, $x_1 = 70.83$ mm

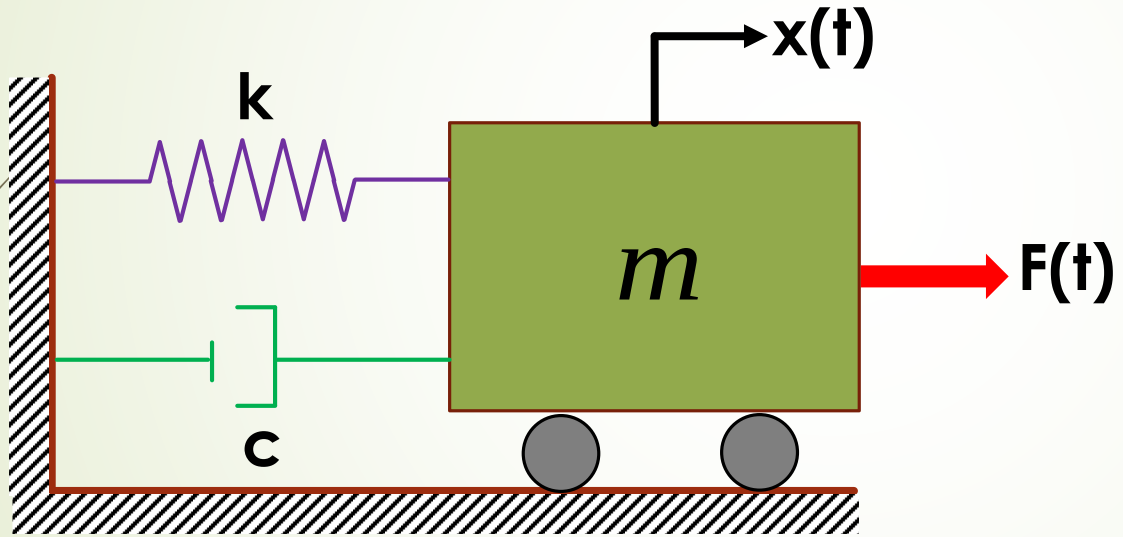
velocity at t = 1 sec, $\dot{x}_1 = 40.13$ mm/sec

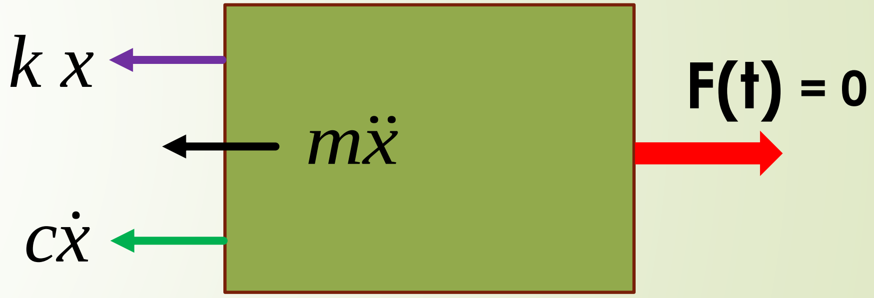
Free Damped Vibration of SDOF System

Free Damped Vibration of SDOF System

$m\ddot{x} =$ Inertia force, $c\dot{x} =$ Damping force and

$kx =$ Restoring force





Free Body Diagram

Free Damped Vibration of SDOF System

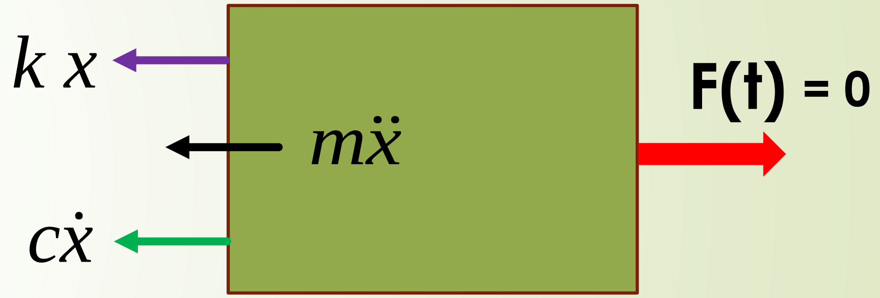
Applying D'Alembert's principle,

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ --------- (i)}$$

For, free damped vibrating system

$$F(t) = 0$$

$$\Rightarrow m\dot{x} + c\dot{x} + kx = 0 \text{ --------- (A)}$$



Free Body Diagram

- **Equation (A) is higher order homogeneous differential equation.**
- **Assuming x = e^{st} as a general solution,**
- **D = $\dot{x}$ = s. e^{st} and D^2 = $\ddot{x}$ = s^2. e^{st}**

$\dot{x}$

Free Damped Vibration of SDOF System

- Substituting above values in equation (A) we get,
- $m(s^2 . e^{st}) + c(s. e^{st}) + k (e^{st}) = 0 \Rightarrow e^{st} (ms^2 + cs + k) = 0$
- for above equation there are two possible solutions,
- $e^{st} = 0$ or $(ms^2 + cs + k) = 0$
- Out of these two solutions $e^{st} = 0$ cannot be a solution because x = 0 is always a solution for homogeneous equation (Trivial solution).
- So, considering $(ms^2 + cs + k) = 0$ as a solution (Non-trivial solution).
- Dividing the above equation by m and then using basic relation we get,

$$s^2 + \frac{c}{m}s + \frac{\mathrm{k}}{\mathrm{m}} = 0 \Rightarrow s^2 + \frac{c}{m}s + \omega_{\mathrm{n}}^2 = 0$$

$\dot{x}$

Free Damped Vibration of SDOF System

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \Rightarrow s^2 + \frac{c}{m}s + \omega_n^2 = 0$$

- **Assuming (c/2m) = n,**

$$s^2 + 2sn + \omega_n^2 = 0$$

- **Above equation is quadratic equation in terms of s.**
- **Solution of quadratic equation can be given by,**

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $a = 1$, $b = 2n$ and $c = \omega_n^2$ or ω^2

$$s = -n \pm \sqrt{n^2 - \omega^2}$$

$\dot{x}$

Free Damped Vibration of SDOF System

$$s = -n \pm \sqrt{n^2 - \omega^2}$$

n > ω

Over Damped System

Two real roots possible

n = ω

Critically Damped System

Two same roots are possible

n < ω

Under Damped System

Two complex roots are possible

$\dot{x}$

Free Damped Vibration of SDOF System

Case – (I): When n > ω (Over Damped System)

- Let s_1 and s_2 be the two real roots.

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

Case – (II): When n = ω (Critically Damped System)

- Let $s_1 = s_2 = s$ be the two roots.

$$x = (c_1 + c_2 t)e^{st}$$

$$n = \omega \; and \; n = \frac{c}{2m} \Rightarrow \frac{c}{2m} = \sqrt{\frac{k}{m}}$$

$$c = 2\sqrt{mk} \Rightarrow Cc = 2m\omega_n$$

Free Damped Vibration of SDOF System

Case – (III): When n < ω (Under Damped System)

- **Let S_1 and S_2 are two complex roots.**

$$s_1 = -n + i\sqrt{\omega^2 - n^2} \text{ and}$$

$$s_2 = -n - i\sqrt{\omega^2 - n^2}$$

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

$$x = c_1 e^{(-n + i\sqrt{\omega^2 - n^2})t} + c_2 e^{(-n - i\sqrt{\omega^2 - n^2})t}$$

$$x = c_1 e^{-nt} . e^{(i\sqrt{\omega^2 - n^2})t} + c_2 e^{-nt} . e^{(-i\sqrt{\omega^2 - n^2})t}$$

$\dot{x}$

Free Damped Vibration of SDOF System

$$x = c_1 e^{-nt}\left[\cos\left(\sqrt{\omega^2 - n^2}\right)t + i\sin\left(\sqrt{\omega^2 - n^2}\right)t\right]$$

$$+ c_2 e^{-nt}\left[\cos\left(\sqrt{\omega^2 - n^2}\right)t - i\sin\left(\sqrt{\omega^2 - n^2}\right)t\right]$$

$$\Rightarrow x = (c_1 + c_2)e^{-nt}.\cos\left(\sqrt{\omega^2 - n^2}\right)t$$

$$+ i(c_1 - c_2)e^{-nt}.\sin\left(\sqrt{\omega^2 - n^2}\right)t$$

$$Let, (c_1 + c_2) = A \text{ and } \ i(c_1 - c_2) = B$$

$\dot{x}$

Free Damped Vibration of SDOF System

$$\Rightarrow x = Ae^{-nt}.\cos\left(\sqrt{\omega^2 - n^2}\right)t + Be^{-nt}.\sin\left(\sqrt{\omega^2 - n^2}\right)t$$

Differentiating with respect to t,

$$\dot{x} = e^{-nt}\begin{bmatrix} -A\left(\sqrt{\omega^2 - n^2}\right)\sin\left(\sqrt{\omega^2 - n^2}\right)t \\ +B\left(\sqrt{\omega^2 - n^2}\right)\cos\left(\sqrt{\omega^2 - n^2}\right)t \end{bmatrix}$$

$$+\left[A\cos\left(\sqrt{\omega^2 - n^2}\right)t + B\sin\left(\sqrt{\omega^2 - n^2}\right)t\right]e^{-nt}(-n)$$

$\dot{x}$

Free Damped Vibration of SDOF System

The boundary conditions are,

(a) At $t = 0, \; x = x_0$

substituting above Boundary Condition in equation of displacement $\Rightarrow A = x_0$

(b) At $t = 0, \; \dot{x} = \dot{x}_0$

substituting above Boundary Condition in equation of velocity $\Rightarrow B = \dfrac{\dot{x}_0 + n x_0}{\sqrt{\omega^2 - n^2}}$

$\dot{x}$

Free Damped Vibration of SDOF System

substituting above values of A and B in equation of displacement

$$x = e^{-nt}\left[x_0 \cos\left(\sqrt{\omega^2 - n^2}\right)t + \frac{\dot{x}_0 + n x_0}{\sqrt{\omega^2 - n^2}} \sin\left(\sqrt{\omega^2 - n^2}\right)t\right]$$

Let, $\xi = \dfrac{n}{\omega} \Rightarrow n = \xi\omega$

Term $\sqrt{\omega^2 - n^2}$ represents damped natural circular frequency.

$\dot{x}$

Free Damped Vibration of SDOF System

$$\omega_d = \sqrt{\omega^2 - n^2} = \omega\sqrt{1-\xi^2}$$

Where, ξ is called as damping ratio and is defined by,

$$\xi = \frac{c}{c_c}$$ (c_c = critical damping co-efficient in Ns/m)

Equation of displacement can be reduced in form of

$$x = e^{-\xi\omega t}\left[x_0 \cos\omega_d t + \frac{\xi\omega x_0 + \dot{x}_0}{\omega_d}\sin\omega_d t\right]$$

$\dot{x}$

Free Damped Vibration of SDOF System

Time period for damped vibration

$$T_d = \frac{2\pi}{\omega_d}$$

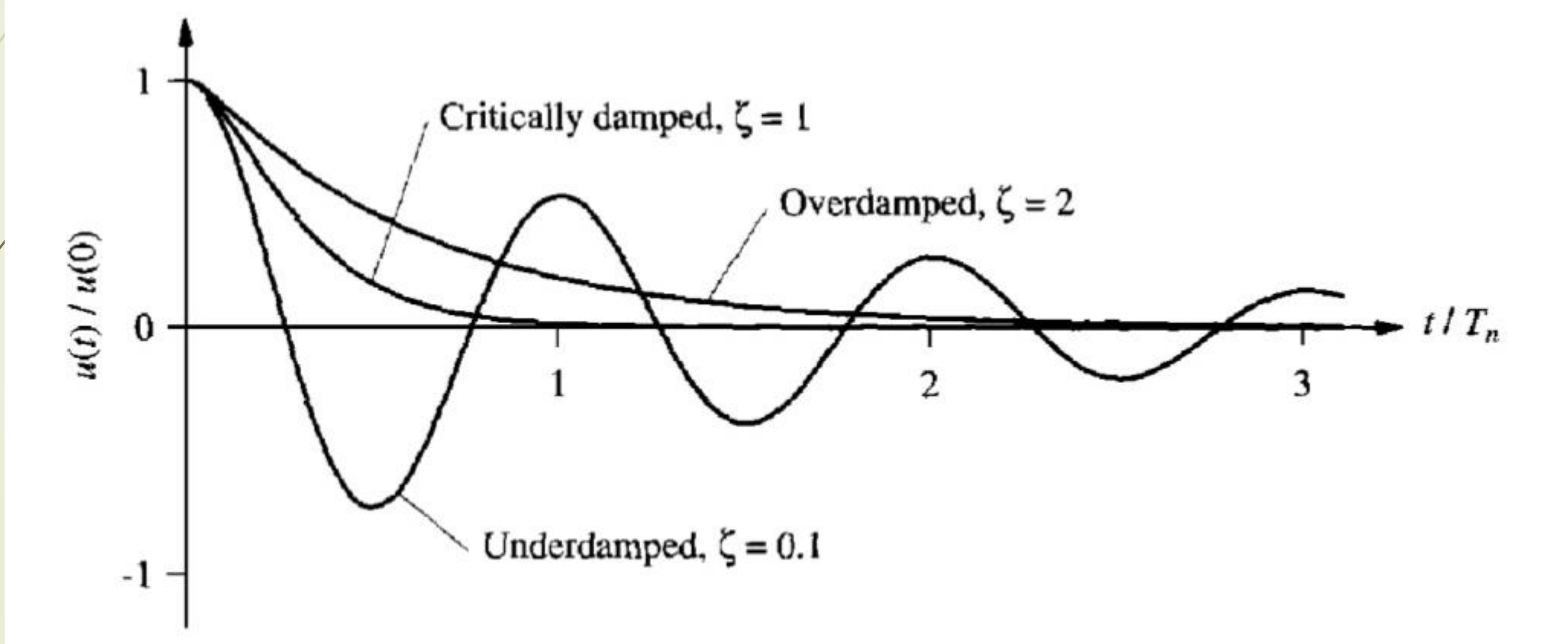
But, $\omega_d = \omega\sqrt{1-\xi^2}$

$$T_d = \frac{2\pi}{\omega\sqrt{1-\xi^2}}$$

Amplitude????

Free Damped Vibration of SDOF System

Response of Free Damped SDOF Vibration System Based on different values of damping ratio



free vibration of undamped, critically damped and overdamped systems

Free Damped Vibration of SDOF System

Critically Damped Systems

- Damping co-efficient c_{cr} is called the critical damping co-efficient. (Smallest value of c that inhibits oscillation completely)
- It represents the dividing line between oscillatory and non-oscillatory motion.
- The automobile shock absorber, scale measuring deadweight are an example of a critically damped devices.

Rear Shock Absorber

Free Damped Vibration of SDOF System

Underdamped Systems

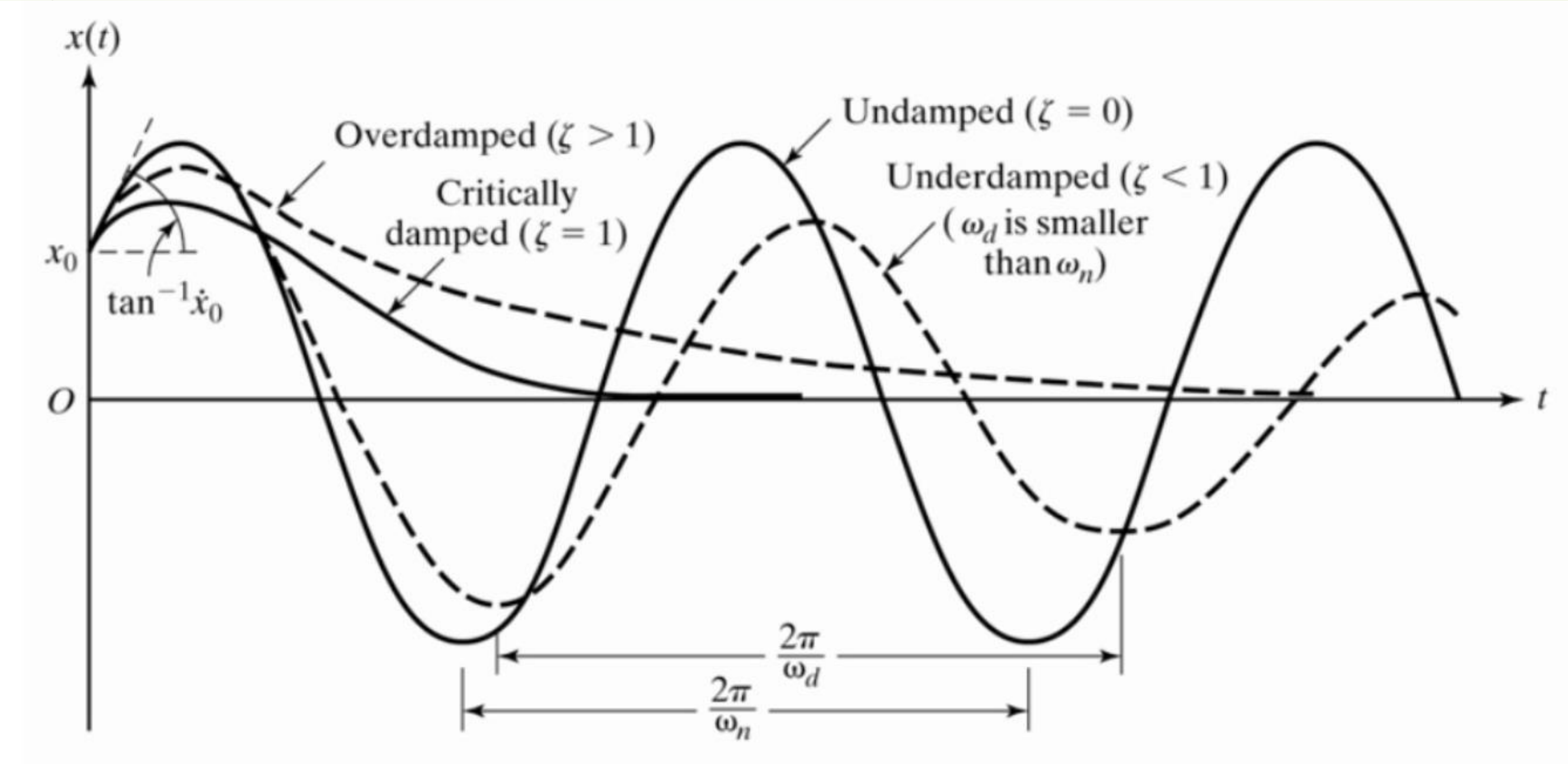
- Most of the structures of interest will fall in category of Underdamped systems.
- Buildings, bridges, dams, nuclear power plants, offshore structures, etc. will fall in underdamped systems as typically their damping ratio is less than 0.10.

Overdamped Systems

- In overdamped case the system does not oscillate and returns to equilibrium position as in case of critically damped systems but at a slower rate.
- Common automatic door closer is normally over damped system.

Free Damped Vibration of SDOF System

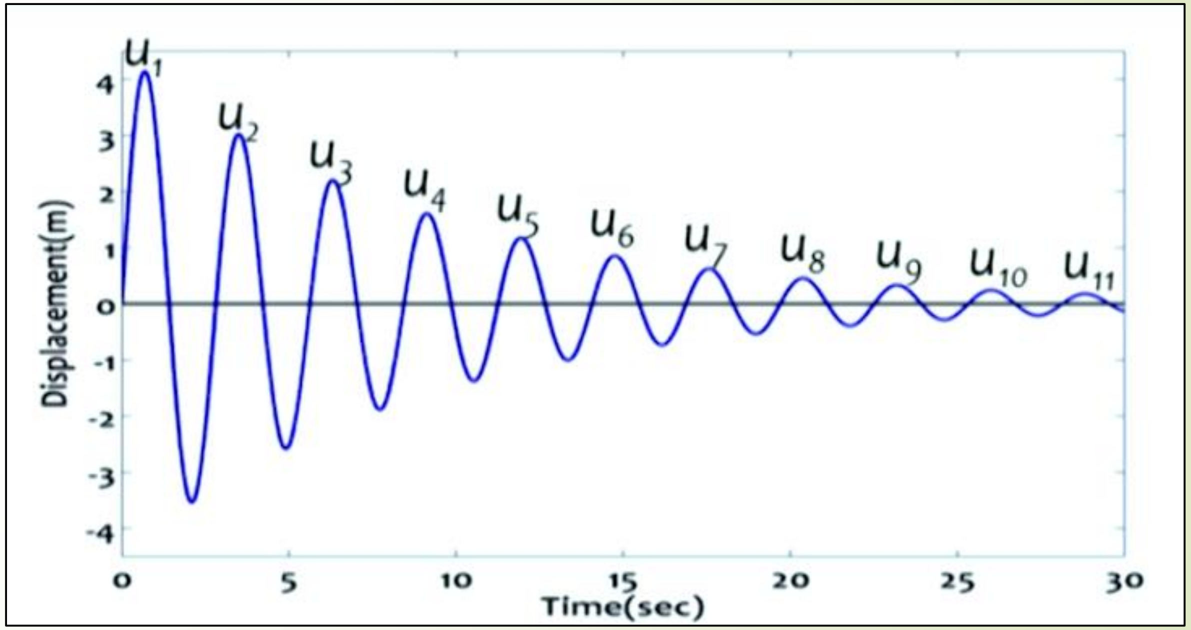
Response of Free Damped SDOF Vibration System for typical ranges of damping ratio



Free Damped Vibration of SDOF System

- The response is governed by Two terms in equation.
- Bracket portion gives oscillatory motion while exponential term is decaying the response.
- Taking ratio of two consecutive amplitudes,

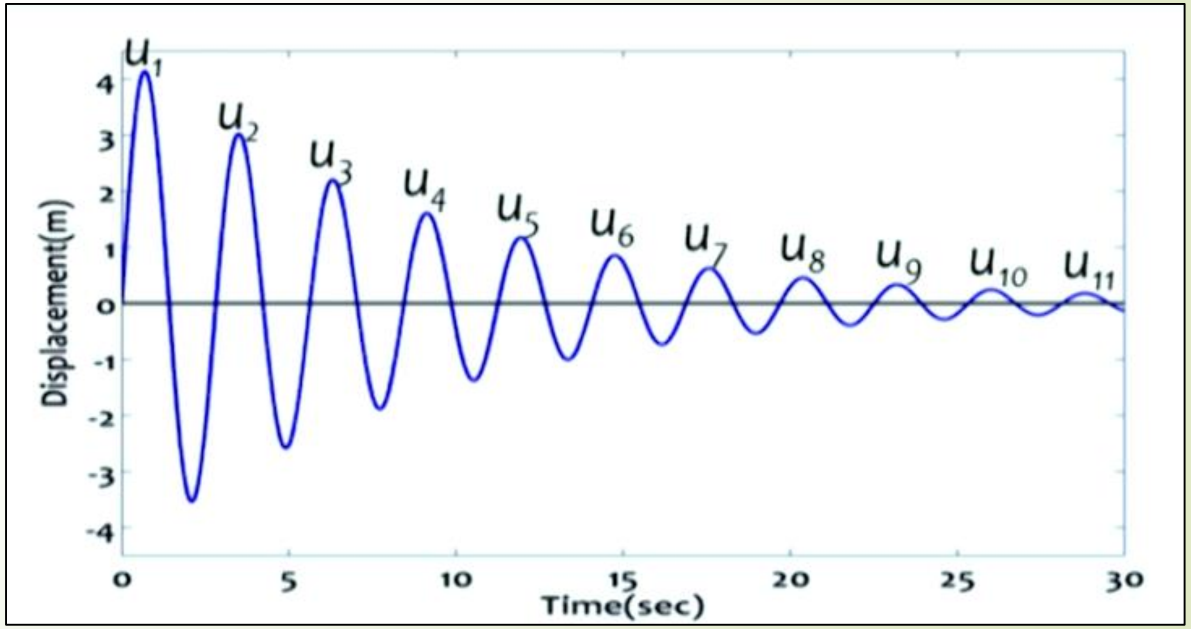
$$\frac{u_2}{u_1} = \frac{u_{t_2}}{u_{t_1}} = \frac{u_{(t_1+T_D)}}{u_{t_1}}$$



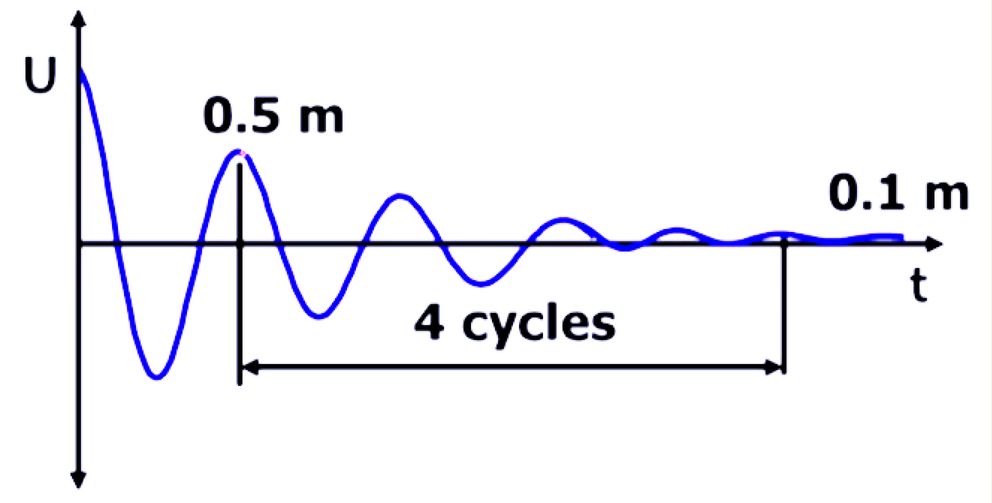
Free Damped Vibration of SDOF System

Logarithmic Decrement

$$\delta = \frac{1}{n} ln \frac{u_1}{u_2} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \cong 2\pi\xi$$



EX-4 A water tank is set to vibrate freely. Amplitude of vibration reduces from 0.5m to 0.1m in 4 cycles in 8 seconds. Find the damped natural period and damping ratio of the system.



EX-5 A vibrating system consists of a mass of 454 kg and spring of stiffness 3506N/m is viscously damped so that the two consecutive amplitudes are 1.0 m and 0.85 m. Determine:

(a) Natural frequency of undamped system
(b) Logarithmic decrement
(c) Damping ratio
(d) Damping co-efficient
(e) Damped natural frequency and time period

EX-6 In the laboratory, a model of simply supported beam is displaced at the middle from stable condition and allowed to vibrate. It is found that it vibrates at a natural time period of 0.1 sec and the amplitude of motion decreased from 4 mm to 3.5 mm after 5 cycles. The same experiment was repeated by adding 5 kg at the midspan and it was found to vibrate with time period of 1.1 sec.

Calculate:

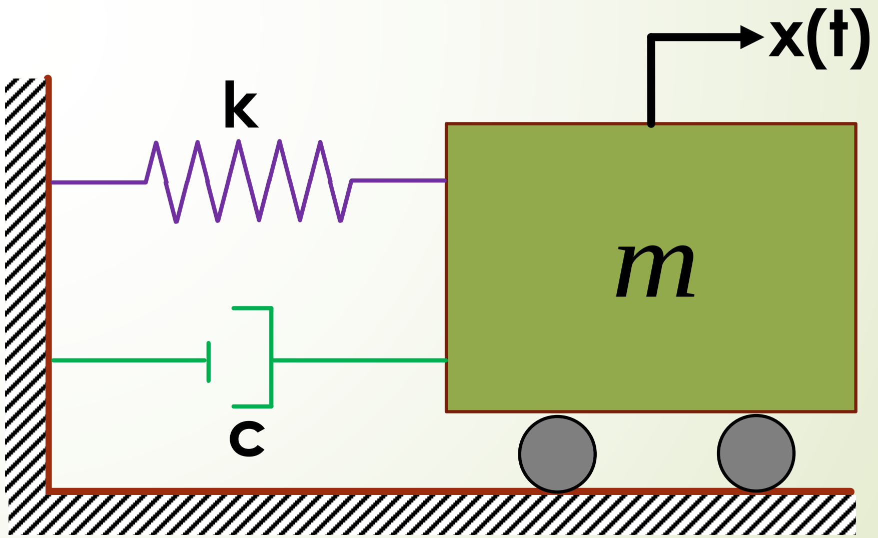
(a) Equivalent mass of the system

(b) Equivalent stiffness of the system

(c) Damping ratio of beam

EX-7 Determine the equation for free vibration response of a SDOF system as shown in figure, at time t = 0.2 sec for the following data:

Natural circular frequency = 12 rad/sec
Damping factor = 0.15
Initial velocity = 10 cm/sec
Initial displacement = 5 cm



Solution:

Since, $\xi = 0.15 < 1 \Rightarrow$ The system is under-damped.

Equation of response for free under-damped SDOF System is given by,

$$x = e^{-nt}\left[x_0 \cos\left(\sqrt{\omega^2 - n^2}\right)t + \frac{\dot{x}_0 + n x_0}{\sqrt{\omega^2 - n^2}} \sin\left(\sqrt{\omega^2 - n^2}\right)t \right]$$

Which can also be written as,

$$x = e^{-\xi\omega t}\left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \xi\omega x_0}{\omega_d} \sin\omega_d t \right]$$

Solution:

$$\omega_d = \omega\sqrt{1-\xi^2} = 11.86 \text{ rad/sec}$$

At, t = 0, $x = x_0 = 0.05$ m

Substituting above values and condition in equation of response,

A = 0.05

At t = 0, $\dot{x} = 0.1$ m/sec

Similarly Substituting above values and condition in equation of motion for free under-damped SDOF system

B = 0.016

Solution:

Using these values of A, B in equation of response we can get, generalised equation of response as follows:

$$x = e^{-1.8t}\left[0.05\cos 11.8642t + 0.016\sin 11.8642t\right]$$

Putting, t = 0.2 sec

$$x_{(t=0.2sec)} = -17.31 \text{ mm}$$

Similarly, Putting t = 0.2 sec in generalised equation of motion

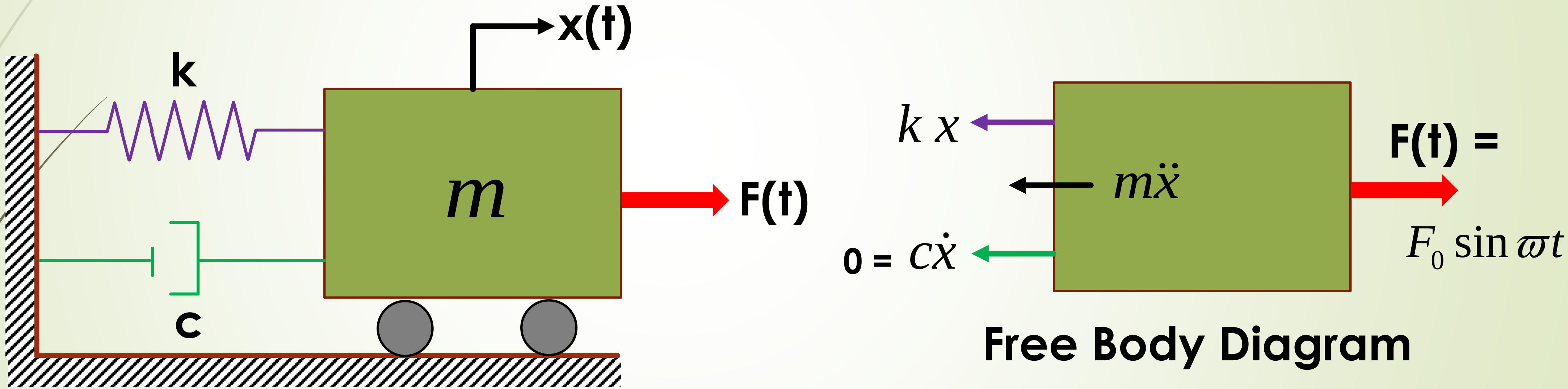
$$\dot{x}_{(t=0.2sec)} = -0.352 \text{ m/s}$$

Forced Undamped Vibration of SDOF System

Forced Undamped Vibration of SDOF System

$m\ddot{x} =$ Inertia force, $c\dot{x} =$ Damping force and

$kx =$ Restoring force



Free Body Diagram

Forced Undamped Vibration of SDOF System

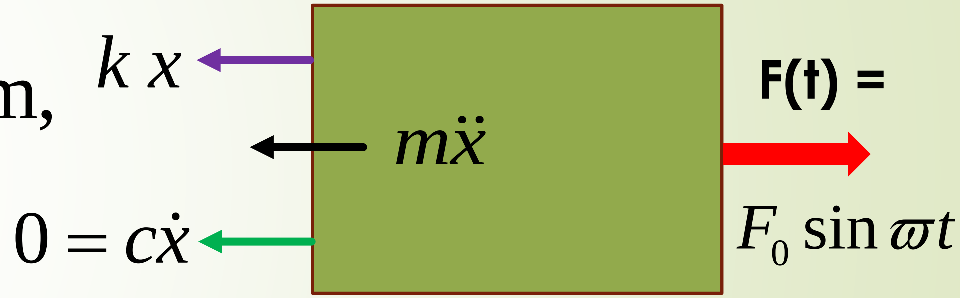
Applying D'Alembert's principle,

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ --------- (i)}$$

For forced undamped vibrating system,

$$c = 0$$

$$\Rightarrow m\ddot{x} + kx = F_0 \sin \varpi t \text{ --------- (A)}$$



Free Body Diagram

- **Equation (A) is higher order non-homogeneous differential equation.**
- **Solution of such equation consists of two parts namely: Complimentary Function and Particular Integral.**

$\dot{x}$

Forced Undamped Vibration of SDOF System

$$x(t) = x_c(t) + x_p(t)$$

Where, $x_c(t) =$ Complimentary solution, satisfying homogeneous differential equation and

$x_p(t) =$ particular solution, satisfying non-homogeneous part of equation

To determine $x_c(t)$:

$m\ddot{x} + kx = 0$ -------(Equation represnts free -undamped vibration)

Solution of this equation is, $x_c(t) = \mathrm{A}\cos\omega t + \mathrm{B}\sin\omega t$

$\dot{x}$

Forced Undamped Vibration of SDOF System

To determine $x_p(t)$:

Particular solution may be taken as,

$$x_p(t) = \mathrm{Y} \sin \varpi \mathrm{t}$$

Where, Y = peak value of particular solution

$$\dot{x}_p(t) = \mathrm{Y} \ \varpi \ \cos \varpi \mathrm{t} \text{ and}$$

$$\ddot{x}_p(t) = -\mathrm{Y} \ \varpi^2 \ \sin \varpi \mathrm{t}$$

Forced Undamped Vibration of SDOF System

Substituting these values in equation (A)

$$m(-Y\varpi^2 \sin\varpi t) + k(Y\sin\varpi t) = F_0 \sin\varpi t$$

$$Y(-m\varpi^2 + k) = F_0$$

$$Y = \frac{F_0}{k - m\varpi^2} = \frac{F_0}{k\left[1 - \frac{m}{k}\varpi^2\right]} = \frac{F_0}{k\left[1 - \frac{\varpi^2}{\omega^2}\right]}$$

$$= \frac{F_0}{k\left[1 - \left(\frac{\varpi}{\omega}\right)^2\right]}$$

Term $\frac{\varpi}{\omega} = r$ is known as frequency ratio

$\dot{x}$

Forced Undamped Vibration of SDOF System

$$= \frac{F_0}{k\left[1-r^2\right]}$$

Where, r = $\dfrac{\varpi}{\omega} = \dfrac{\text{Frequency of applied force}}{\text{Natural frequency of vibration}}$

So, exact solution can be given by,

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{F_0}{k\left[1-r^2\right]}\sin\varpi t$$

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t + \frac{F_0}{k\left[1-r^2\right]}\cos\varpi t\,(\varpi)$$

$\dot{x}$

Forced Undamped Vibration of SDOF System

Applying boundary condition in equation of displacment,

At $t = 0, x = x_0$

$x_0 = A$

Applying boundary condition in equation of velocity,

At $t = 0, \dot{x} = \dot{x}_0$

$$\dot{x}_0 = B\omega(1) + \frac{F_0}{k\left[1 - r^2\right]} (\cos\varpi t)(\varpi)$$

$$B = \frac{\dot{x}_0}{\omega} - \frac{r\,F_0}{k\left[1 - r^2\right]}$$

$\dot{x}$

Forced Undamped Vibration of SDOF System

$$x(t) = x_0 \cos \omega t + \left[\frac{\dot{x}_0}{\omega} - \frac{\text{r F}_0}{\text{k}\left[1 - r^2\right]} \right] \sin \omega t + \frac{\text{F}_0}{\text{k}\left[1 - r^2\right]} \sin \varpi \text{t}$$

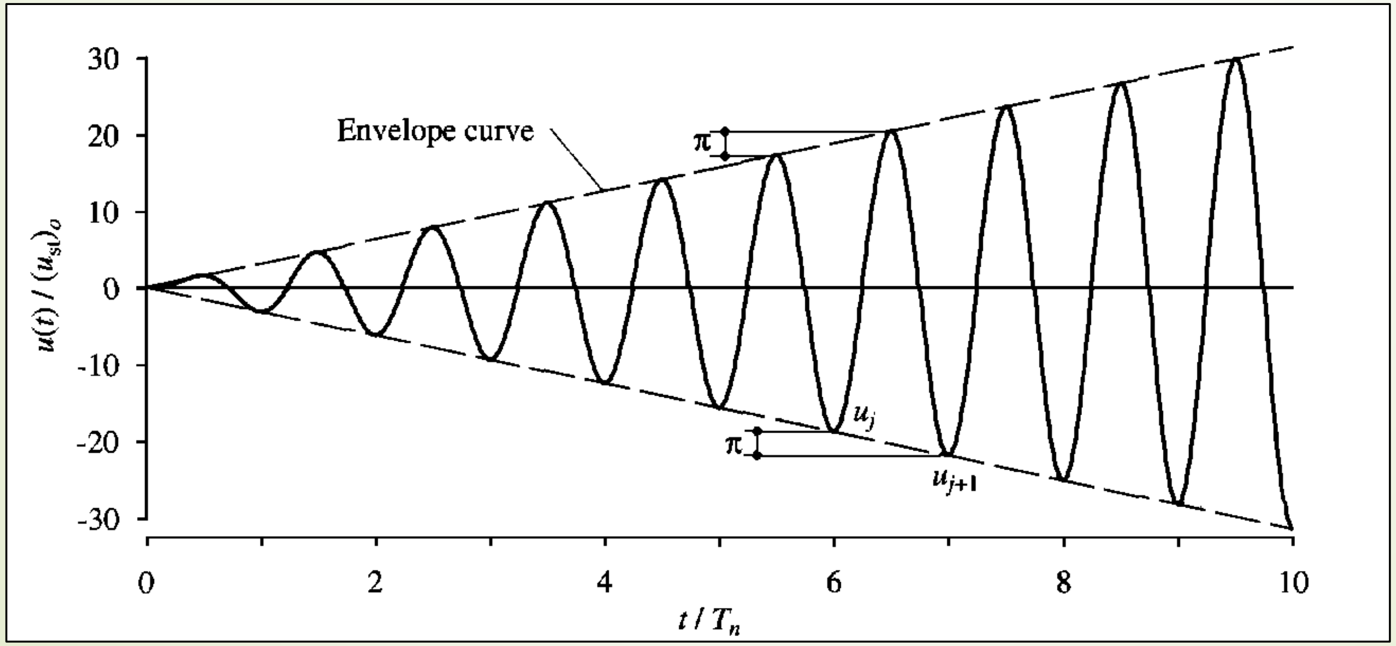
Transient Response

Steady State Response

In all practical cases, damping forces will always be present in the system and will cause the transient response to vanish eventually.

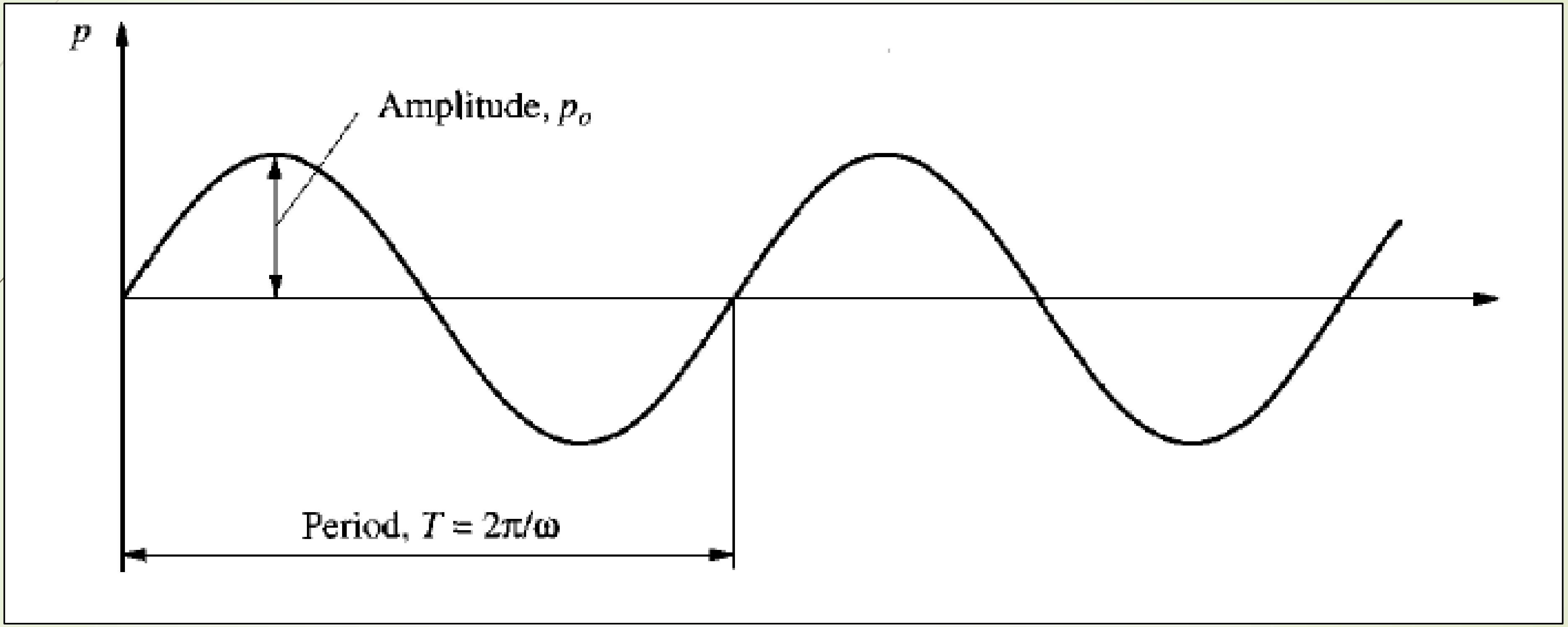
Forced Undamped Vibration of SDOF System

- When forcing frequency is equal to natural frequency (i.e. $\omega = \varpi$) or r = 1.0, the amplitude of motion becomes infinitely large.
- A system acted upon by an external frequency coinciding with the natural frequency is said to be at **RESONANCE**.



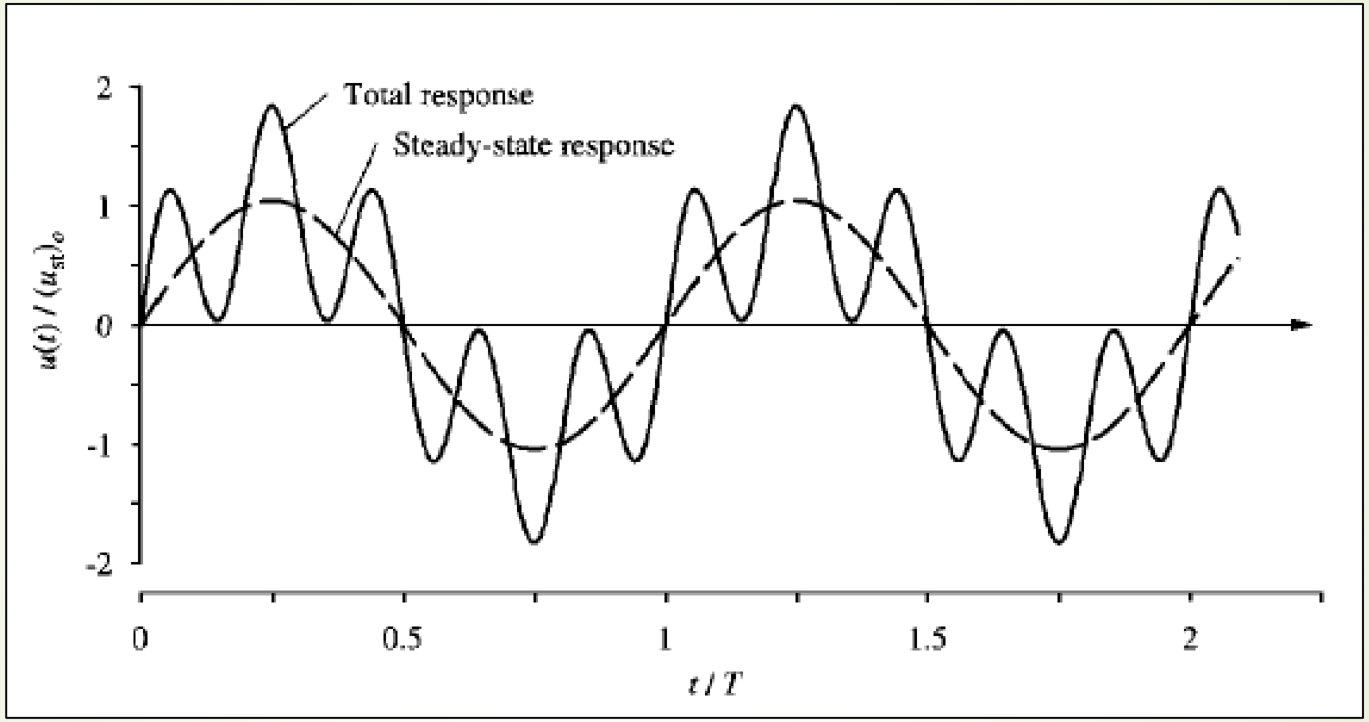
Response of undamped system to sinusoidal force of frequency $\varpi = \omega$; $u(0) = \dot{u}(0) = 0$

Forced Undamped Vibration of SDOF System



Harmonic force with $\varpi = \omega$ (ω is not natural frequency of system)

Forced Undamped Vibration of SDOF System



Response of undamped system to harmonic force with

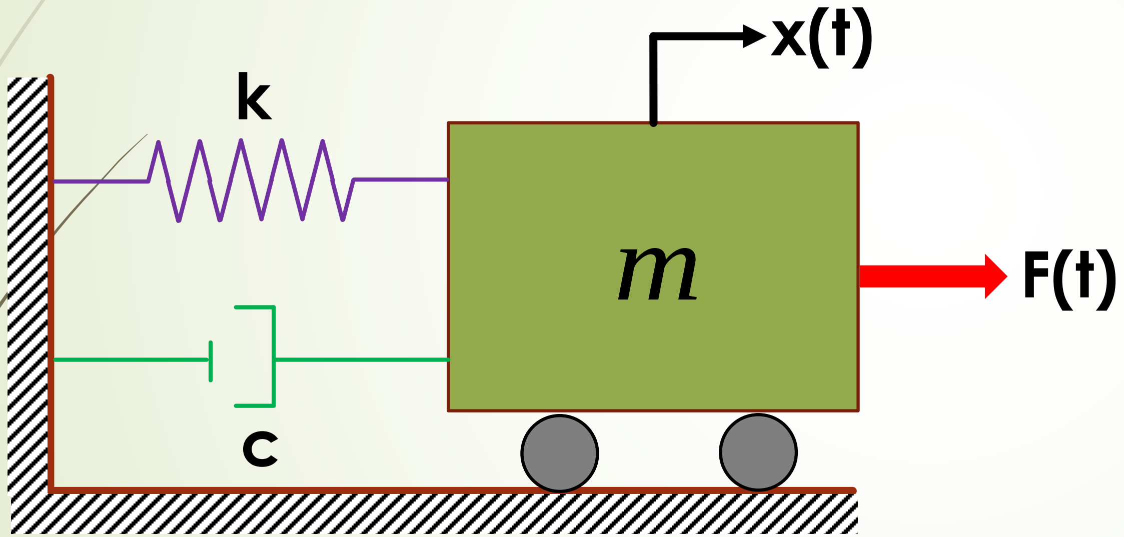
$$\varpi / \omega = 0.2,\ u(0) = 0 \text{ and } \dot{u}(0) = \omega_n P_0 / k$$

Forced Damped Vibration of SDOF System

Forced Damped Vibration of SDOF System

$m\ddot{x} =$ Inertia force, $c\dot{x} =$ Damping force and

$kx =$ Restoring force



Free Body Diagram

Forced Damped Vibration of SDOF System

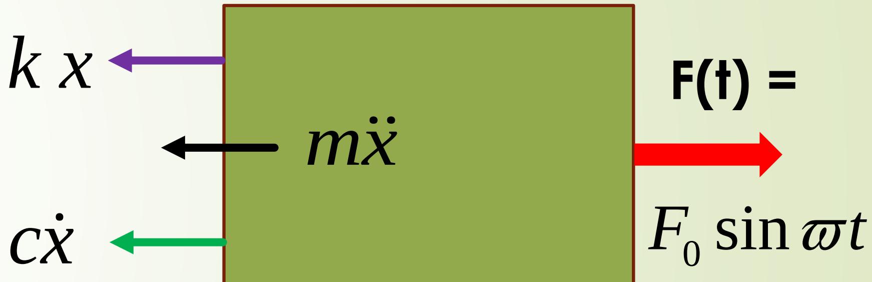
Applying D'Alembert's principle,

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ ---------- (i)}$$

For forced undamped vibrating system,

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = F_0 \sin \varpi t \text{ ---------- (A)}$$

- **Equation (A) is higher order non-homogeneous differential equation.**
- **Solution of such equation consists of two parts namely: Complimentary Function and Particular Integral.**



Free Body Diagram

$\dot{x}$

Forced Damped Vibration of SDOF System

$$x(t) = x_c(t) + x_p(t)$$

Where, $x_c(t) =$ Complimentary solution, satisfying homogeneous differential equation and

$x_p(t) =$ particular solution, satisfying non-homogeneous part of equation

To determine $x_c(t)$:

$m\ddot{x} + c\dot{x} + kx = 0$ -------(Equation represnts free -damped vibration)

Solution of this equation is, $x_c(t) = e^{-\xi\omega t}\left[A\cos\omega_d t + B\sin\omega_d t\right]$

$\dot{x}$

Forced Damped Vibration of SDOF System

To determine $x_p(t)$:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \varpi t$$

Dividing all the terms by m,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin \varpi t$$

Let, $\frac{c}{m} = a,\ \frac{\mathrm{k}}{\mathrm{m}} = b,\ \frac{F_0}{m} = d\ \ \&\ \ \mathrm{D} = \frac{\mathrm{d}}{\mathrm{dt}} \Rightarrow \mathrm{D}^2 = \frac{\mathrm{d}^2}{\mathrm{dt}^2}$

$$\Rightarrow \left(\mathrm{D}^2 + a\mathrm{D} + \mathrm{b}\right)x = d\sin \varpi t$$

Forced Damped Vibration of SDOF System

$$\Rightarrow PI = \frac{d \sin \varpi t}{\left(D^2 + aD + b\right)}$$

Putting, $D^2 = -(\varpi)^2$

$$PI = \frac{d \sin \varpi t}{\left(-\varpi^2 + aD + b\right)} = \frac{d \sin \varpi t}{\left(b - \varpi^2\right) + aD}$$

$$= \frac{d \sin \varpi t}{\left(b - \varpi^2\right) + aD} * \frac{\left(b - \varpi^2\right) - aD}{\left(b - \varpi^2\right) - aD} = \frac{d \sin \varpi t \left(b - \varpi^2\right) - aDd \sin \varpi t}{\left(b - \varpi^2\right)^2 - \left(aD\right)^2}$$

$\dot{x}$

Forced Damped Vibration of SDOF System

$$\Rightarrow PI = \frac{d\sin\varpi t\left(b-\varpi^2\right)-ad\varpi\cos\varpi t}{\left(b-\varpi^2\right)^2+\left(a\mathrm{D}\right)^2}$$

$$=\frac{d\left[\sin\varpi t\left(b-\varpi^2\right)-a\varpi\cos\varpi t\right]}{\left(b-\varpi^2\right)^2-a^2\mathrm{D}^2}$$

$$=\frac{d\left[\sin\varpi t\left(b-\varpi^2\right)-a\varpi\cos\varpi t\right]}{\left(b-\varpi^2\right)^2-a^2\left(-\varpi^2\right)}=\frac{d\left[\sin\varpi t\left(b-\varpi^2\right)-a\varpi\cos\varpi t\right]}{\left(b-\varpi^2\right)^2+a^2\left(\varpi^2\right)}$$

Forced Damped Vibration of SDOF System

$$\Rightarrow Let,\left(b-\varpi^2\right)=R\cos\phi \text{ and } a\varpi = R\sin\phi$$

$$\Rightarrow\left[\left(b-\varpi^2\right)^2+\left(a\varpi\right)^2\right]=R^2$$

$$PI=\frac{d\left[\sin\varpi t\left(R\cos\phi\right)-\left(R\sin\phi\right)\cos\varpi t\right]}{\left(b-\varpi^2\right)^2+a^2\left(\varpi^2\right)}$$

$$PI=\frac{dR\left[\sin\varpi t\left(\cos\phi\right)-\left(\sin\phi\right)\cos\varpi t\right]}{\left(b-\varpi^2\right)^2+a^2\left(\varpi^2\right)}=\frac{d\left[\sin\left(\varpi t-\phi\right)\right]}{\sqrt{\left(b-\varpi^2\right)^2+a^2\left(\varpi^2\right)}}$$

Forced Damped Vibration of SDOF System

$$PI = \frac{\frac{F_0}{m}}{\sqrt{\left(\frac{k}{m} - \varpi^2\right)^2 + \left(\frac{c}{m}\varpi\right)^2}} \sin(\varpi t - \phi)$$

$$PI = \frac{F_0}{\sqrt{\left(k - m\varpi^2\right)^2 + \left(c\varpi\right)^2}} \sin(\varpi t - \phi)$$

$\dot{x}$

Forced Damped Vibration of SDOF System

Complete solution is given by

$$x(t) = x_c(t) + x_p(t)$$

$$= e^{-\xi\omega t}\left[\mathrm{A}\cos\omega_d t + \mathrm{B}\sin\omega_d t\right]$$

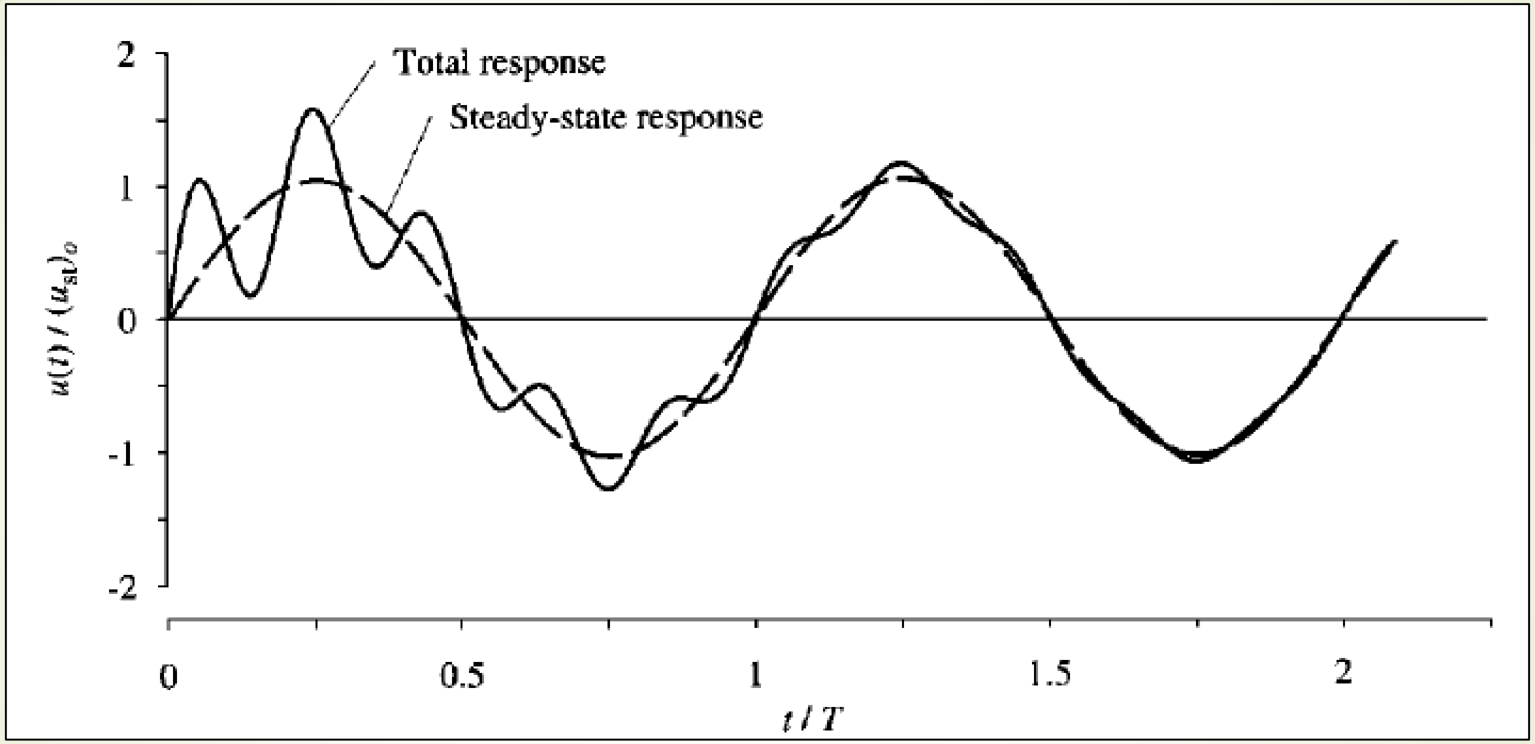
Transient Response

$$+\frac{F_0}{\sqrt{\left(k - m\varpi^2\right)^2 + \left(c\varpi\right)^2}}\sin\left(\varpi t - \phi\right)$$

Steady State Response

$\dot{x}$

Forced Damped Vibration of SDOF System



Response of damped system to harmonic force with

$$\varpi / \omega = 0.2,\ \xi = 0.05,\ u(0) = 0 \text{ and } \dot{u}(0) = \omega_n P_0 / k$$

$\dot{x}$

Forced Damped Vibration of SDOF System

- **Complimentary solution will become negligible with time, as the term e$^{\infty}$ = 0.**
- **Steady state response of the system is given by,**

$$x = \frac{F_0}{\sqrt{\left(k - m\varpi^2\right)^2 + \left(c\varpi\right)^2}} \sin\left(\varpi t - \phi\right)$$

Amplitude of steady state response will be,

$$\frac{F_0}{\sqrt{\left(k - m\varpi^2\right)^2 + \left(c\varpi\right)^2}} = \frac{F_0 / k}{\sqrt{\left(1 - \frac{m}{k}\varpi^2\right)^2 + \left(\frac{c}{k}\varpi\right)^2}}$$

Forced Damped Vibration of SDOF System

$$\text{Steady State Amplitude} = \frac{F_0/k}{\sqrt{\left(1-\frac{\varpi^2}{\omega^2}\right)^2 + \left(\frac{c}{k}\varpi\right)^2}}$$

Using, $\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}$ and simplifying we get,

$$\text{Steady State Amplitude} = \frac{F_0/k}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}$$

The numerator F_0/k is static deflection of spring with stiffness k under the force F_0, and can be denoted by δ_{st}.

$\dot{x}$

Forced Damped Vibration of SDOF System

Magnification Factor or Deformation Response Factor (R_d)

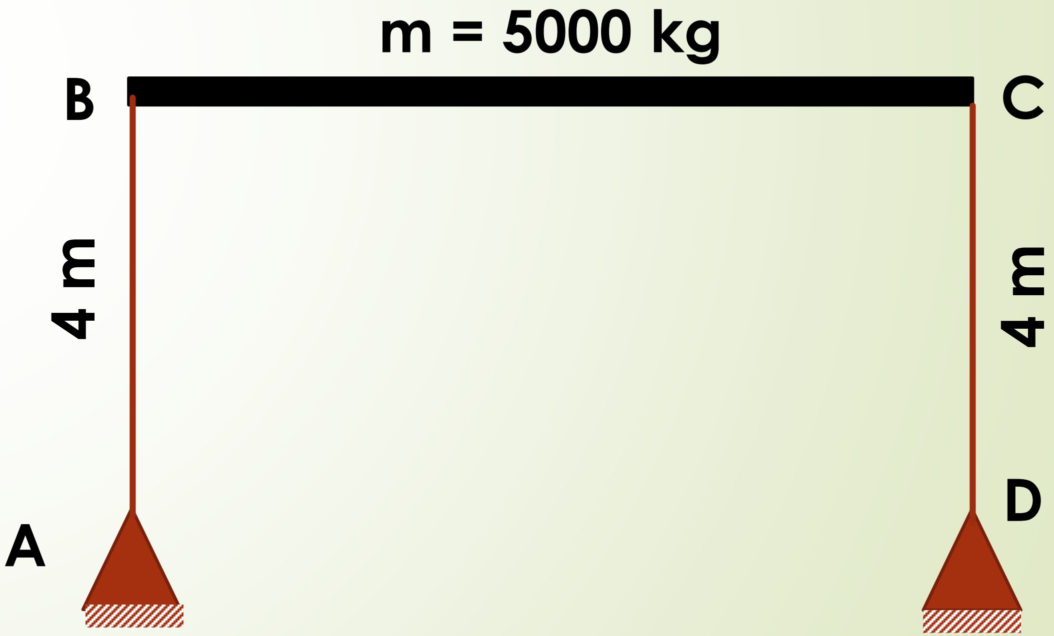
The ratio of Amplitude of steady state response to to static deflection under action of force F_0 is known as Magnification Factor (R_d) or M.F.

$$R_d = \frac{\dfrac{F_0 / k}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}}{F_0 / k}$$

$$X_{\max} = R_d * \delta_{st}$$

$$\text{Magnification Factor} = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}$$

EX-8 A rigid frame shown in figure supports a rotating machine which exerts a force at the girder level of 50000 sin11t N. Assuming 4% of critical damping what would be the steady state amplitude of vibration? Take, I for columns = 1500 x10^{-7} m^4 and **E = 21 x 10^{10} N/m^2.**



EX-9 Determine the magnification factor for a forced vibration produced by a oscillator fixed at the middle of the beam at a speed of 600 rpm. The concentrated load at the middle of the beam is 5000 N and produces a static deflection of the beam is 0.025 cm. Neglect the weight of beam and assuming the damping co-efficient equal to 20 Ns/mm.

Forced Damped Vibration of SDOF System

Transmissibility (T_r)

The ratio of the transmitted force (F_{TO}) to the applied force (F_O) is known as Transmissibility (T_r).

$$\text{Transmissibility, } T_r = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

$$\text{or } = \frac{\text{Maximum mass displacement}}{\text{Maximum support displacement}}$$

$$F_0 = Y_0 k\sqrt{1+(2\xi r)^2}$$

EX-10 A steel frame shown in figure is subjected to sinusoidal ground motion, x = 0.2sin15t cm, assuming damping ratio to be 0.1, determine:

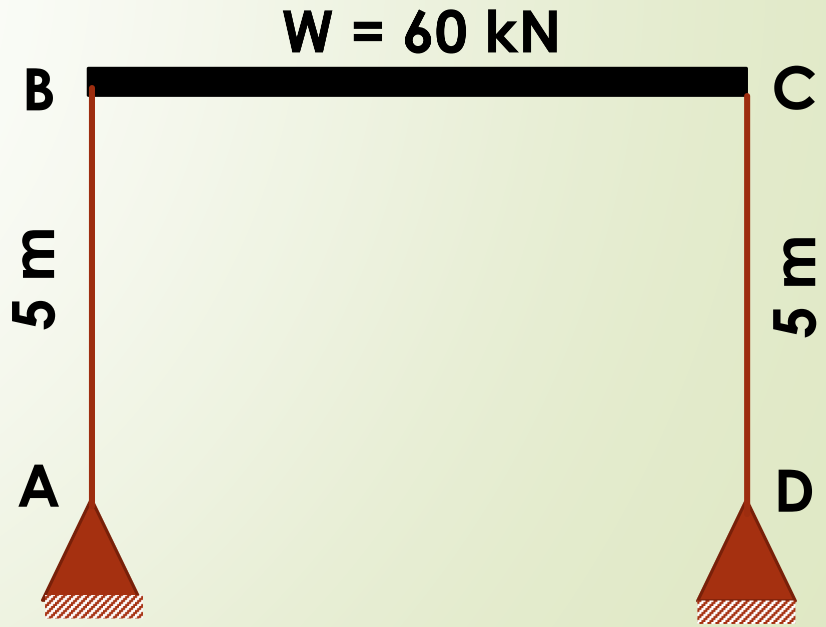
(a) Transmissibility of motion

(b) Maximum SF in column

(c) Maximum bending stress in column

Take, $E = 2 \times 10^5$ N/mm^2,

$Z = 1404.2$ cm^3 and $I = 28083.5$ cm^4



EX-11 A two bay single storey RCC plane frame which is supporting lumped mass of 20 tonne on three columns namely AB, CD & EF as shown in figure. $L_{AB} = 0.5L_{CD} = 0.25L_{EF} = 2m.$

Calculate:

(a) Natural frequency of damped vibration

(b) Bending Moment and Shear Force at support for the RCC frame after 5 cycles if the floor is displaced horizontally by 300 mm and suddenly released.

Assume rigid diaphragm action and damping to be 8%.

Take, M25 grade concrete and size of column to be 600 mm x 600 mm.

Types of Damping

- External Viscous Damping
- Body-friction damping
- Internal Viscous Damping
- Hysteresis Damping
- Radiation Damping

Damping Ratio for Various Building Materials

Material	Damping Ratio (ξ)
Concrete	5%
Steel	<2%
Wood	12%
Clay	8-12%
Brick	5-7%