Computer Graphics

- Computer graphics deals with generation of two dimensional or three dimensional model of an object using graphical software or an application software or both.
- User can generate a model either with the help of programming or by using user friendly software but recently trend is of user friendly program rather than programming because this is very time consuming and not highly effective

Application :

- Science, engineering, medicine, business advertising, education, art etc.
- Major role is simulation & animation.

CAD Softwares

- Graphics Software
- Application Software
  - CNC machine
  - Automated process planning
  - FEA softwares

Line generating Algorithm :

- DDA (Digital Differential Analyser) Algorithms
- Bresenham's Algorithm

DDA Algorithms :

- DDA algorithms is used for generating the straight line betn two points.
- DDA Algorithm works on principle that x and y are simultaneously incremented by a small step proportion

$$y = mx + c$$

$$\frac{dy}{dx} = m = \frac{\Delta y}{\Delta x} = \frac{y_n - y_1}{x_n - x_1}$$

$$\Delta y = \left[\frac{y_n - y_1}{x_n - x_1}\right] \Delta x \quad \text{——— (1)}$$

$$\& \quad \Delta x = \left[\frac{x_n - x_1}{y_n - y_1}\right] \Delta y \quad \text{——— (2)}$$

$$y_{K+1} = y_K + \Delta y$$

$$\text{or} \quad y_{K+1} = y_K + \left[\frac{y_n - y_1}{x_n - x_1}\right] \Delta x$$

Similarly,

$$x_{K+1} = x_K + \Delta x$$

$$= x_K + \left[\frac{x_n - x_1}{y_n - y_1}\right] \Delta y$$

- In DDA algorithm whichever is larger and whichever is chosen as 1 raster unit other increment is eq. ① & ② is calculated.
- Following are two cases

<u>Case-I:</u> $|\Delta x| \geq |\Delta y|$

<u>Case-II:</u> $|\Delta y| \geq |\Delta x|$

- <u>Case I:</u> $|\Delta x| \geq |\Delta y|$

$$|x_n - x_1| \geq |y_n - y_1|$$

So $|\Delta x| = 1$

$$\Delta x = \pm 1$$

$\Delta x = 1$ if $x_n > x_1$

$\Delta x = -1$ if $x_n < x_1$

Now, $\Delta y = \left|\dfrac{y_n - y_1}{x_n - x_1}\right| \Delta x$

- starting with first point (x_1, y_1)

$$x_{k+1} = x_k + \Delta x$$

$$y_{k+1} = y_k + \Delta y$$

<u>Case-II</u>

$$|y_n - y_1| \geq |x_n - x_1|$$

$$|\Delta y| = 1$$

$$\Delta y = \pm 1$$

$\Delta y = 1$ ~~then~~ if $y_n > y_1$

$\Delta y = -1$ if $y_n < y_1$

Now $\Delta x = \left[\dfrac{x_n - x_1}{y_n - y_1}\right] \Delta y$

Now starting with first points corresponding points can be calculated

$$y_{k+1} = y_k + \Delta y$$

$$x_{k+1} = x_k + \Delta x$$

Advantages :

- DDA is a faster method than direct use of the equation $y = mx + c$

Limitations :

- Due to round off gets accumulated in additions of increments. Hence calculated pixels drift away from actual line.

Ex Generate straight line connecting two points (1,2) and (8,6) using DDA algorithm.

$x_1 = 1$ $y_1 = 2$

$x_n = 8$ $y_n = 6$

$x_n - x_1 = 7$ & $y_n - y_1 = 4$

In this case $(x_n - x_1) > (y_n - y_1)$

So $\Delta x = 1$

$$\Delta y = \frac{y_n - y_1}{x_n - x_1} = \frac{4}{7} = 0.571$$

Date :

Pixel positions

- **Point 1:**

$x_1 = x_1' + 0.5 = 1 + 0.5 = 1.5$

$y_1 = y_1' + 0.5 = 2 + 0.5 = 2.5$

$(x_1, y_1) = (1, 2)$

- **Point 2:**

$x_2 = x_1 + 1 = 1.5 + 1 = 2.5$

$y_2 = x_2 + 0.571 = 3.071$

$(x_2, y_2) = (2, 3)$

- **Point 3:**

$x_3 = x_2 + 1 = 2.5 + 1 = 3.5$

$y_3 = y_2 + 0.571 = 3.642$

$(x_3, y_3) = (3, 3)$

- **Point 4:**

$x_4 = x_3 + 1 = 4.5$

$y_4 = y_3 + 0.571 = 4.213$

$(x_4, y_4) = (4, 4)$

- **Point 5:**

$x_5 = x_4 + 1 = 5.5$

$y_5 = y_4 + 0.571 = 4.784$

$(x_5, y_5) = (5, 4)$

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- Paint 6:

$x_6 = x_5 + 1 = 6.5$

$y_6 = y_5 + 0.571 = 5.355$

$(x_6, y_6) = (6, 5)$

- Paint 7:

$x_7 = x_6 + 1 = 7.5$

$y_7 = y_6 + 0.571 = 5.926.$

$(x_7, y_7) = (7, 5)$

- Paint 8:

$x_8 = x_7 + 1 = 8.5$

$y_8 = y_7 + 0.571 = 6.497$

$(x_8, y_8) = (8, 6)$

Ex (Practice)

$(x_1, y_1) = (1, 0)$
$(x_n, y_n) = (10, 3)$

$x_n - x_1 = 9.$
$y_n - y_1 = 3$

Here $(x_n - x_1) > (y_n - y_1)$

So, $\Delta x = 1$

$\Delta y = \frac{3}{9} = 0.333$

Pixel points

- Point 1 :

$x_1 = x_1' + 0.5 = 1 + 0.5 = 1.5$

$y_1 = y_1' + 0.5 = 0 + 0.5 = 0.5$

$(x_1, y_1) = (1, 0)$

- Point 2 :

$x_2 = x_1 + 1 = 2.5$

$y_2 = y_1 + 0.333 = 0.5 + 0.333 = 0.833$

$(x_2, y_2) = (2, 0)$

- Point 3 :

$x_3 = 2.5 + 1 = 3.5$

$y_3 = y_2 + 0.333 = 1.166$

$(x_3, y_3) = (3, 1)$

- Point 4 :

$x_4 = x_3 + 1 = 4.5$

$y_4 = y_3 + 0.333 = 1.499$

$(x_4, y_4) = (4, 1)$

- Point 5 :

$x_5 = x_4 + 1 = 5.5$

$y_5 = y_4 + 0.333 = 1.832$

$(x_5, y_5) = (5, 1)$

- Paint 6:

$x_6 = x_5 + 1 = 6.5$

$y_6 = y_5 + 0.333 = 2.165$

$(x_6, y_6) = (6, 2)$

- Paint 7:

$x_7 = x_6 + 1 = 7.5$

$y_7 = y_6 + 0.333 = 2.498$

$(x_7, y_7) = (7, 2)$

- Paint 8:

$x_8 = x_7 + 1 = 8.5$

$y_8 = y_7 + 0.333 = 2.831$

$(x_8, y_8) = (8, 2)$

- Paint 9:

$x_9 = x_8 + 1 = 9.5$

$y_9 = y_8 + 0.333 = 3.164.$

$(x_9, y_9) = (9, 3)$

- Paint 10:

$x_{10} = x_9 + 1 = 10.5$

$y_{10} = y_9 + 0.333 = 3.497$

$(x_{10}, y_{10}) = (10, 3)$

Bresenham's line Algorithm :

- Bresenham's algorithm is used for making line bet^n two end points
- It increments by one unit in either x or y. Depending upon slope of the line.
- Increment in other variable is taken as either zero or one and is determining by distance bet^n actual line and nearest pixel position
- If $(x_n - x_1) \geq (y_n - y_1)$ then slop is less than or equal to 1.

  Δx as unit value.

  Δy as zero or one. depending upon the value.
- If $(x_n - x_1) < (y_n - y_1)$ then slop is greater than 1.

  Δy as unit value

  & Δx as zero or one depending upon the value.

Bresenham's line Algorithm :

- Two end points of line (x_1, y_1) & (x_n, y_n)
- Calculate $X_c = x_n - x_1$

  $Y_c = Y_n - Y_1$

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- Calulate decision parameter
$$P_k = 2Y_c - X_c \qquad \cancel{P_{k+1} = 2Y_c - 2X_c}$$

- At each value of x_k along the line
① If $P_K < 0$ then $\longrightarrow (x_k + 1, y_k)$ and
$$P_{K+1} = P_K + 2Y_c$$

② If $P_K > 0 \longrightarrow (x_k + 1, y_k + 1)$ and
$$P_{K+1} = P_K + 2Y_c - 2X_c$$

- Repeat this step X_c time.

Ex Generate a straight line connecting two points (21, 11) and (26, 15) by bresenham's algorithm.

$\rightsquigarrow$ $x_1 = 21$ & $y_1 = 11$
$x_n = 26$ & $y_n = 15$

- Slope :

$x_n - x_1 = 5 = X_c$
$y_n - y_1 = 4 = Y_c$
$(x_n - x_1) > (y_n - y_1)$

Hence, Slope is less than 1.
$$\Delta x = 1$$

- Estimating X_c and Y_c

$X_c = x_n - x_1 = 5$

$Y_c = y_n - y_1 = 4$

$2Y_c = 8$

$2Y_c - 2X_c = 2 \times 4 - 2 \times 5 = -2$

Points

- **Point 1**: Set pixel $(x_1, y_1) = (21, 11)$

$$P_1 = 2Y_c - X_c = 8 - 5 = 3$$

- **Point 2**:

$$x_2 = x_1 + 1 = 22$$

$$P_1 > 0$$

$$y_2 = y_1 + 1 = 12$$

$$(x_2, y_2) = (22, 12)$$

$$P_2 = P_1 + 2Y_c - 2X_c = 3 - 2 = 1$$

$$P_2 > 0$$

- **Point 3**:

$$x_3 = x_2 + 1 = 23$$

$$P_2 > 0$$

$$y_3 = y_2 + 1 = 13$$

$$P_3 = P_2 + 2Y_c - 2X_c = 1 - 2 = -1$$

$$\boxed{(x_3, y_3) = (23, 13)}$$

- **Point 4**:

$$x_4 = x_3 + 1 = 24.$$

$$P_3 < 1$$

$$y_4 = y_3 + 0 = 13$$

$$\boxed{(x_4, y_4) = (24, 13)}$$

$$P_4 = P_3 + 2Y_c = -1 + 8 = 7$$

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Paint 5 :

$x_5 = x_4 + 1 = 25$

$P_4 > 0$

$y_5 = y_4 + 1 = 14$

$\boxed{(x_5, y_5) = (25, 14)}$

$P_5 = P_4 + 2y_c - 2x_c = 7 - 2 = 5$

Paint 6 :

$x_6 = x_5 + 1 = 26.$

$P_5 > 0$

$y_6 = y_5 + 1 = 15$

$\boxed{(x_6, y_6) = (26, 15)}$

Ex Det^n raster scan location selected by bresenham's algorithm for generating a line from (1, 0) to (10, 3).

$\rightsquigarrow$ $x_1 = 1$ & $y_1 = 0$

$x_n = 10$ & $y_n = 3$

$x_n - x_1 = 9$

$y_n - y_1 = 3$

$(x_n - x_1) > (y_n - y_1)$

so $\Delta x = 1$

- Estimation of x_c & y_c :

$$x_c = 9.$$
$$y_c = 3$$
$$2y_c = 6.$$
$$2y_c - 2x_c = 2\times3 - 2\times9 = -12$$

→ Points :

- Point 1 :

$$(x_1, y_1) = (1, 0)$$
$$P_1 = 2y_c - x_c = \overset{6.}{\cancel{12}} - 9 = -3$$

- Point 2 :

$$x_2 = x_1 + 1 = 2$$
$$P_1 < 0.$$
$$y_2 = y_1 = 0.$$
$$(x_2, y_2) = (2, 0)$$
$$P_2 = P_1 + 2y_c = -3 + 6 = 3$$

- Point 3 :

$$x_3 = x_2 + 1 = 3.$$
$$P_2 > 0.$$
$$y_3 = y_2 + 1 = 1$$
$$(x_3, y_3) = (3, 1)$$
$$P_3 = P_1 + 2y_c - 2x_c = 3 - 12 = -9$$

- Point 4 :

$$x_4 = x_3 + 1 = 4.$$
$$P_3 < 0$$
$$y_4 = y_3 = 1$$
$$(x_4, y_4) = (4, 1)$$

$$P_4 = P_3 + 2Y_c = -9 + 6 = -3$$

- **Paint 5:**

$$x_5 = x_4 + 1 = 5$$
$$P_4 < -3$$
$$y_5 = y_4 = 1$$
$$P_5 = P_4 + 2Y_c = -3 + 6 = 3$$
$$\boxed{(x_5, y_5) = (5, 1)}$$

- **Paint 6:**

$$x_6 = x_5 + 1 = 6.$$
$$P_5 > 0.$$
$$y_6 = y_5 + 1 = 2.$$
$$\boxed{(x_6, y_6) = (6, 2)}$$
$$P_6 = P_5 + 2Y_c - 2X_c = 3 - 12 = -9$$

- **Paint 7:**

$$x_7 = x_6 + 1 = 7$$
$$P_6 < 0.$$
$$y_7 = y_6 = 2.$$
$$\boxed{(x_7, y_7) = (7, 2)}$$
$$P_7 = P_6 + 2Y_c = -9 + 6 = -3$$

- **Paint 8:**

$$x_8 = x_7 + 1 = 8$$
$$P_7 \leq 0.$$
$$y_8 = y_7 = 2$$
$$\boxed{(x_8, y_8) = (8, 2)}$$
$$P_8 = P_7 + 2Y_c = -3 + 6 = 3$$

- Paint 9:

$x_9 = x_8 + 1 = 9$

$P_8 > 0.$

$y_9 = y_8 + 1 = 3.$

$\boxed{(x_9, y_9) = (9, 3)}$

$P_9 = P_8 + 2y_c - 2x_c = 3 - 12 = -9$

- Paint 10:

$x_{10} = x_9 + 1 = 10$

$P_9 < 0.$

$y_{10} = y_9 = 3$

$\boxed{(x_{10}, y_{10}) = (10, 3)}$

Practice ex

(1, 1)

(8, 5)

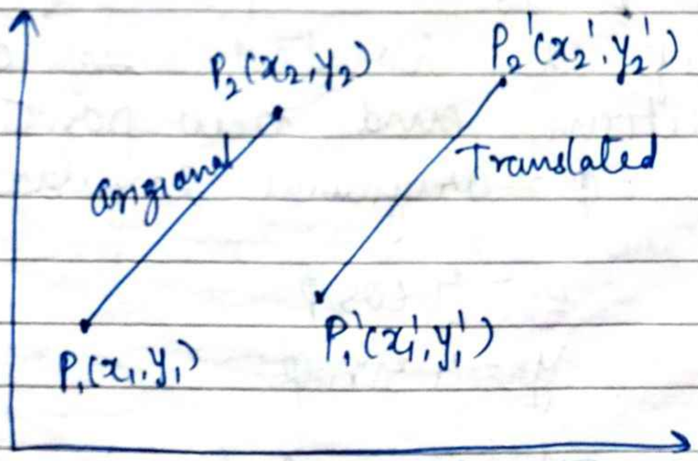
Bresenham's Algo.

Two dimensional geometric transformation

- It changes orientation, size, shape of the objects.
- The geometric transformation play a central role in construction of model CAD/CAM software commands like : translate, rotate, zoom, mirror, array etc.
- This also used in animation.
- Basic transformations are
  1. Translation
  2. Rotation
  3. Scaling
  4. Reflection
  5. Shear

Translation :

It involves moving graphic element from one location to another.



- Consider a point $P(x, y)$ is translated by distance t_x in x-position & t_y in y position. new point $P'(x', y')$

$$x' = x + t_x$$

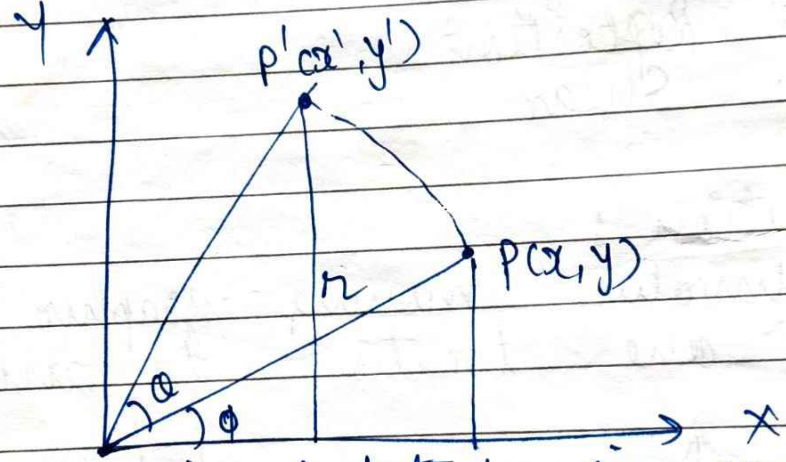
$$y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\underbrace{[P']}_{\text{New position}} = \underbrace{[P]}_{\text{Old position}} + \underbrace{[T]}_{\text{Translation Matrix}}$$

→ <u>Rotation</u> : In rotation the graphics element is rotated about origin (Z-axis) by angle θ

- For positive angle, rotation is counter clockwise direction.



$P(x, y)$ is rotated in counter clockwise direction and new position $P'(x', y')$

ϕ = origional angular position

$x = r\cos\phi$

$y = r\sin\phi$

$x' = r\cos(\phi + \theta)$

$= r\cos\phi\cos\theta - r\sin\phi\sin\theta$

$= x\cos\theta - y\sin\theta$

$y' = r\sin(\theta + \phi)$

$= r\sin\phi\cos\theta + r\cos\phi\sin\theta$

$= x\sin\theta + y\cos\theta$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [R]\,[P]$$

$\hookrightarrow$ Rotation Matrix.

→ Scaling : It alters the size of the graphics element. It is used to enlarge or reduce the size of element.

- If S_x and S_y is scaling factor in x and y direction

$$x' = S_x \cdot x$$
$$y' = S_y \cdot y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [S]\,[P]$$

$\hookrightarrow$ scaling matrix

- If S_x and S_y are less than 1 then size of element is reduced and greter than 1 then size enlarge
- If $S_x = S_y \rightarrow$ Uniform Scaling

  $S_x \neq S_y \rightarrow$ Differential scaling
- By differential scaling circle → ellipse or ellipse → circle.

Reflection

- It is the Transformation that produces a mirror image of the graphics element about the axis or line

Types

- Reflection about X-axis
- Reflection about Y-axis
- Reflection about origin
- Reflection about $y = x$
- Reflection about $y = -x$

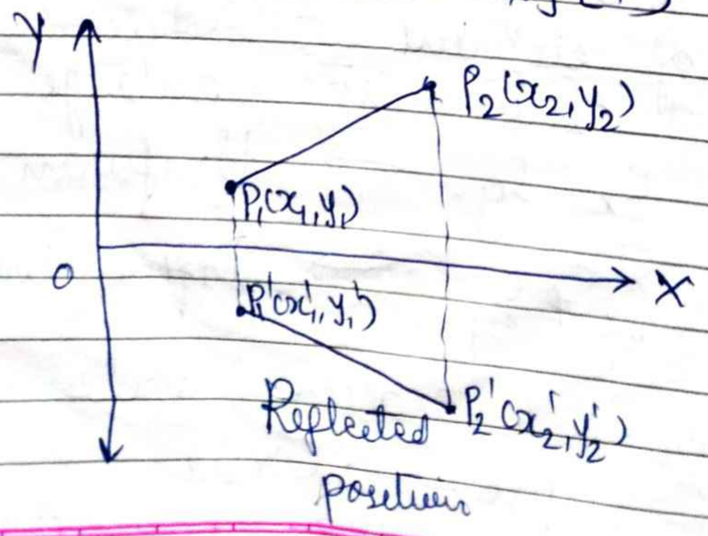
Reflection about X-axis :

- Consider a point $P(x, y)$ is reflected about X-axis to new position $P'(x', y')$

$$x' = x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [M_x][P]$$



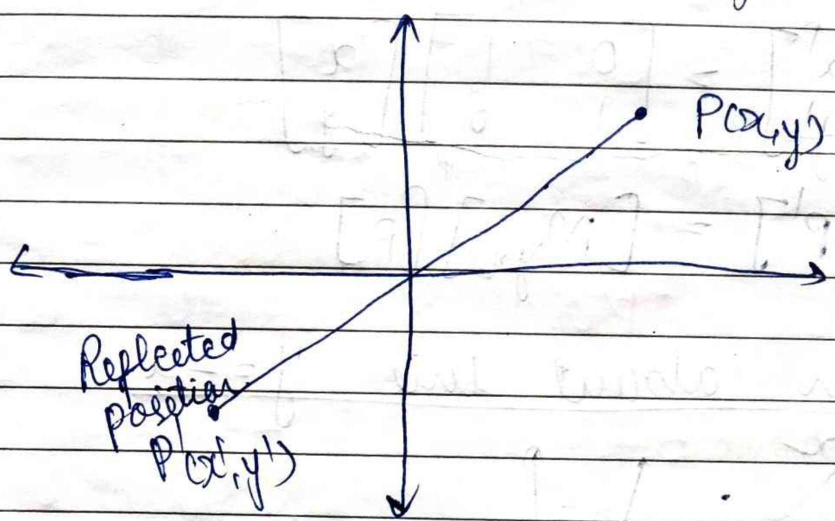
Reflected about Y-axis

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [M_y][P]$$

Reflected about Origin :

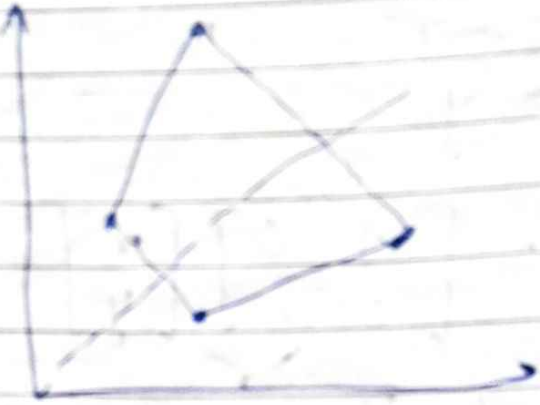


$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [M_o][P]$$

Reflection about $y = x$ line



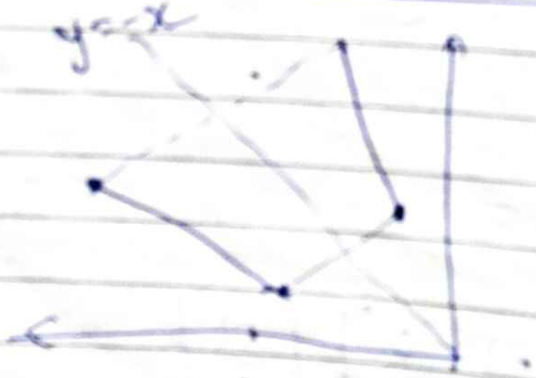
$$x' = y$$
$$y' = x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [M_{yx}][P]$$

check

Reflection about line $y = -x$



$$x' = -y$$
$$y' = -x$$

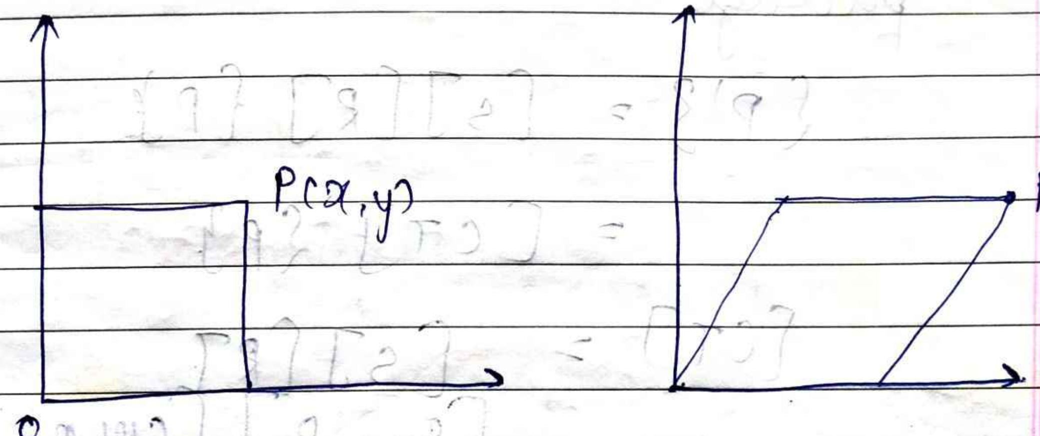
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[P'] = [M_{y-x}][P]$$

Shear

- A shear transformation distorts shape of the graphics element such that internal layers that had been slide over each other.
  - X-direction shear
  - Y-direction shear

X-direction shear



$x' = x + Sh_x \cdot y$

$y' = y$

Sh_x = Shear parameter

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} 1 & Sh_x \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

→ x-direction shear matrix

Y-direction shear

$x' = x$

$y' = y + Sh_y \cdot x$

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ Sh_y & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

→ Y-direction shear Matrix.

Combined / Composite / Concatenated Transform

- It's a combination of two or more transformation such as : translation, rotation, scaling or reflection
- Combination of transformations into single composite transformation improves performance of graphical package.

$$\{P'\} = [S][R]\{P\}$$

$$= [CT]\{P\}$$

$$[CT] = [S][R]$$

$$= \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} S_x \cos\theta & -S_x \sin\theta \\ S_y \sin\theta & S_y \cos\theta \end{bmatrix}$$

Ex P (2,3) and Q (7,8) is to be rotated about origin by 30° clockwise direction

$$\theta = -30^\circ$$

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-30) & -\sin(-30) \\ \sin(-30) & \cos(-30) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix}$$

$$[R] = \begin{bmatrix} 0.8660 & 0.5 \\ -0.5 & 0.8660 \end{bmatrix}$$

$$\{P'\} = [R]\{P\}$$

$$= \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3.232 \\ 1.598 \end{bmatrix}$$

$$[Q'] = [R][Q]$$

$$= \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$[Q'] = \begin{bmatrix} 10.062 \\ 3.428 \end{bmatrix}$$

Ex Det^n new co-ordinates of triangle A(0,0), B(3,2), C(2,3) when rotated at 45° clockwise direction.

→ $\theta = -45°$

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-45) & -\sin(-45) & 0 \\ \sin(-45) & \cos(-45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A'] = [R][A]$$

$$= \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[B'] = [B][R]$$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.536 \\ -0.7071 \\ 1 \end{bmatrix}$$

$$[C'] = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.526 \\ 0.7071 \\ 1 \end{bmatrix}$$

New co-ordinates of triangle is
$A'(0, 0)$, $B'(3.536, -0.7071)$ and
$C'(3.526, 0.7071)$

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Ex Triangle ΔPQR, $P(20,20)$, $Q(50,20)$ $R(20,140)$, $S_x = 2$ and $S_y = 0.5$

$$[S] = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$[P'] = [S][P]$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$$

$$[Q'] = [S][Q]$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 50 \\ 20 \end{bmatrix} = \begin{bmatrix} 100 \\ 10 \end{bmatrix}$$

$$[R'] = [S][R]$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 20 \\ 140 \end{bmatrix} = \begin{bmatrix} 40 \\ 70 \end{bmatrix} \#$$

Ex ΔPQR : $P(50,20)$, $Q(110,20)$, $R(80,60)$
Detn reflection of ΔPQR w.r.t.
(1) X-axis (2) $y = x$

About x-axis:

$$[M_x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[P'] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 50 \\ 20 \end{bmatrix} = \begin{bmatrix} 50 \\ -20 \end{bmatrix}$$

$$[Q'] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 110 \\ 20 \end{bmatrix} = \begin{bmatrix} 110 \\ -20 \end{bmatrix}$$

$$[R'] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \end{bmatrix} = \begin{bmatrix} 80 \\ -60 \end{bmatrix}$$

New co-ordinates of ΔPQR are
$P'(50,-20)$, $Q'(110,-20)$, $R'(80,-60)$

→ About $y = x$

$$[M_{yx}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[P'] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 50 \\ 20 \end{bmatrix} = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$$

$$[Q'] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 110 \\ 20 \end{bmatrix} = \begin{bmatrix} 20 \\ 110 \end{bmatrix}$$

$$[R'] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \end{bmatrix} = \begin{bmatrix} 60 \\ 80 \end{bmatrix}$$

Hence new co-ordinates
$P'(20, 50)$, $Q'(20, 110)$, $R'(60, 80)$

Two dimensional geometric transform using homogeneous co-ordinates:

- For rotation, scaling, shearing reflection matrix multiplication needed while in translation matrix addition is required
- So translation is also required to represent in terms of matrix multiplication which can be done by homogeneous co-ordinates
- In homogeneous co-ordinates, a point in n-dimensional space is represented by $(n+1)$ co-ordinates
- A two dimensional point P with cartesian co-ordinates (x, y) has the homogeneous co-ordinates (x_h, y_h, h) where h = non-zero scalar factor.

$x_h = h \cdot x$
$y_h = h \cdot y$

Common value of h is 1. Therefore if P(x, y) then homogeneous coordinate ~~$(x_h, y_h, 1)$~~

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection:

$$[M_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and corresponding

Shear

x-direction
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y direction
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Two dimensional inverse transformation

- Inverse translation

$$[T_h]^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[P] = [T_h]^{-1} [P']$$

- Inverse rotation

It is done by replacing θ by $(-\theta)$

$$[R]^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Inverse Scaling

$$[S]^{-1} = \begin{bmatrix} 1/S_x & 0 & 0 \\ 0 & 1/S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex ΔABC with vertices $A(30,20)$, $B(90,20)$ $C(30,80)$ is to be scaled 0.5 about point $X(50,40)$. Determine:
1. Composite transformation Matrix
2. Co-ordinates of scaled triangle

→ $S_x = S_y = 0.5$

- Translation:

$$[T_4] = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -50 \\ 0 & 1 & -40 \\ 0 & 0 & 1 \end{bmatrix}$$

- Scaling:

$$[S] = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Inverse translation

Translate the triangle to original posi

$$[T_4]^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 40 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Composite transformation matrix

$$[CT] = [T_u]^{-1}[S][T_u]$$

$$= \begin{bmatrix} 1 & 0 & 50 \\ 0 & 1 & 40 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -50 \\ 0 & 1 & -40 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & 50 \\ 0 & 0.5 & 40 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -50 \\ 0 & 1 & -40 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[CT] = \begin{bmatrix} 0.5 & 0 & 25 \\ 0 & 0.5 & 20 \\ 0 & 0 & 1 \end{bmatrix}$$

→ New co-ordinates

$$[A'] = [CT][A]$$

$$= \begin{bmatrix} 0.5 & 0 & 25 \\ 0 & 0.5 & 20 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ 30 \\ 1 \end{bmatrix}$$

$$[B'] = [CT][B]$$

$$= \begin{bmatrix} 0.5 & 0 & 25 \\ 0 & 0.5 & 20 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} 70 \\ 30 \\ 1 \end{bmatrix}$$

$$[C'] = [CT][C]$$

$$= \begin{bmatrix} 0.5 & 0 & 25 \\ 0 & 0.5 & 20 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 \\ 20 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 40 \\ 60 \\ 1 \end{bmatrix}$$

Ex Practice ΔABC, $A(50,40)$, $B(100,60)$
$C(70,80)$
$S_x = 0.5$ and $S_y = 0.7$ about point A
1. Composite transformation matrix
2. Coordinate of vertices for a scaled triangle.

Ex A rectangle ABCD, $A(1,1)$, $B(2,1)$, $C(2,3)$
$D(1,3)$. It is rotated by 30° clockwise about $P(3,2)$. Detn.
1. Composite transformation matrix.
2. New coordinates of rectangle.

Practice

$$[CT] = \begin{bmatrix} 0.866 & 0.5 & -0.598 \\ -0.5 & 0.866 & 1.768 \\ 0 & 0 & 1 \end{bmatrix}$$

$A(0.768, 2.134)$, $B(1.634, 1.634)$
$C(2.634, 3.366)$

Ex ΔPQR, $P(2,5)$, $Q(6,7)$, $R(2,7)$ is to be reflected about $y = 0.5x + 3$

Detⁿ: ① Concatenated trans. matrix
② Co-ordinates of vertices

→ Translation :

$y = 0.5x + 3$

$x = 0$ then $y = 3$

line passes from point $(0,3)$

$t_x = 0$ & $t_y = -3$

$$[T_4] = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

→ Rotation :

$\theta = -\tan^{-1}(m) = -\tan^{-1}(0.5)$

$\boxed{\theta = -26.565^\circ}$

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-26.565) & -\sin(-26.565) & 0 \\ \sin(-26.565) & \cos(26.565) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ -0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about X-axis :

$$[M_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Rotation :

$$[R]^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(+\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(26.565) & -\sin(26.565) & 0 \\ \sin(26.565) & \cos(26.565) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse translation :

$$[T_n]^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Concatenated Matrix

$$[CT] = [T_n]^{-1}[R]^{-1}[M_x][R][T_n]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ -0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8944 & -0.4472 & 0 \\ 0.4472 & 0.8944 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & -1.3416 \\ -0.4472 & 0.8944 & -2.6832 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8944 & 0.4472 & 0 \\ 0.4472 & -0.8944 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8944 & 0.4472 & -1.3416 \\ -0.4472 & 0.8944 & -2.6832 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.8 & -2.4 \\ 0.8 & -0.6 & 4.8 \\ 0 & 0 & 1 \end{bmatrix}$$

New coordinates
P (2.8, 3.4), Q (6.8, 5.4), R (4.4, 2.2)

Ex Reflect diamond shape polygon whose vertices are A(-2,0), B(0,-1), C(2,0), D(0,1) about line $y = 0.5x + 1$

$$[CT] = \begin{bmatrix} 0.6 & 0.8 & -0.8 \\ 0.8 & -0.6 & 1.6 \\ 0 & 0 & 1 \end{bmatrix}$$

A (-2, 0), B (-1.6, 2.2), C (0.4, 3.2), D (0, 1.6)

Ex Show that in concatenated matrix, final position of the object is dependent upon the sequence of concatenation.

$$[CT_1] = [S]\,[T_r]$$

$$= \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[CT_1] = \begin{bmatrix} S_x & 0 & S_x t_x \\ 0 & S_y & S_y t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Now $[CT_2] = [T_1][S]$

$$= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & t_x \\ 0 & S_y & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$[CT_1] \neq [CT_2]$$

Three-Dimensional geometric transformation

- Translation
- Rotation
- Scaling
- Reflection.

Translation: The translation of a point from $P(x, y, z)$ to new position $P'(x', y', z')$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[P'] = [T][P]$$

→ Rotation :

- Rotation about z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[P'] = [R_z][P]$$

- Rotation about x-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[P'] = [R_x][P]$$

- Rotation about y-axis :

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$[P'] = [R_y][P]$$

Scaling :

$$[P'] = [S][P]$$

$$[S] = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection :

① About XY plane:

$$[R_{xy}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

② About YZ plane :

$$[R_{YZ}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

③ About XZ plane !

$$[R_{XZ}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ex. Construct concatenated matrix based on following conditions.
① Rotation through 120° about z axis
② Translation through 10 & −20 units along x and y-direction
③ Rotation through 30° about x-axis.

→ Rotation of 120° about z-axis

$\theta_z = 120^\circ$

$$[R_z] = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 120 & -\sin 120 & 0 & 0 \\ \sin 120 & \cos 120 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.866 & 0 & 0 \\ 0.866 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Translation through 10 & −20 in X & Y direction

$$[T_r] = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Date: / /

⟶ Rotation through 30° about x-axis

$$\theta_x = 30^\circ$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⟶ Concatenated Matrix

$$[CT] = [R_x][T_h][R_z]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 0.866 & -0.5 & -17.32 \\ 0 & 0.5 & 0.866 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.5 & -0.866 & 0 & 0 \\ 0.866 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.866 & 0 & 10 \\ & & & \\ & & & \end{bmatrix}$$