Unit-2

Newton Forward And Backward Interpolation

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called **extrapolation**.

Forward Differences : The differences y1 – y0, y2 – y1, y3 – y2, ……, yn – yn–1 when denoted by dy0, dy1, dy2, ……, dyn–1 are respectively, called the first forward differences. Thus the first forward differences are :

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
$(= x_0 + h)$		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
$(= x_0 + 2h)$		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
$= (x_0 + 3h)$		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
$= (x_0 + 4h)$		Δy_4				
x_5	y_5					
$= (x_0 + 5h)$						

NEWTON'S GREGORY FORWARD INTERPOLATION FORMULA :

This formula is particularly useful for interpolating the values of f(x) near the beginning of the set of values given. h is called the interval of difference and **u = (x – a) / h**, Here a is first term.

$\theta°$	45°	50°	55°	60°
$\sin\theta$	0.7071	0.7660	0.8192	0.8660

$x°$	Differences			
	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^2 y$	$10^4 \Delta^3 y$
45°	7071			
		589		
50°	7660		– 57	
		532		– 7
55°	8192		– 64	
		468		
60°	8660			

Backward Differences : The differences y1 – y0, y2 – y1, ……, yn – yn–1 when denoted by dy1, dy2, ……, dyn, respectively, are called first backward difference. Thus the first backward differences are :

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
$(= x_0 + h)$		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
$(= x_0 + 2h)$		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
$(= x_0 + 3h)$		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
$(= x_0 + 4h)$		∇y_5				
x_5	y_5					
$(= x_0 + 5h)$						

NEWTON'S GREGORY BACKWARD INTERPOLATION FORMULA :

This formula is useful when the value of f(x) is required near the end of the table. h is called the interval of difference and **u = (x – an) / h**, Here an is last term.

Example :

Input : Population in 1925

Year (x):	*1891*	*1901*	*1911*	*1921*	*1931*
Population (y): (in thousands)	*46*	*66*	*81*	*93*	*101*

Output :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		– 5		
		15		2	
1911	81		– 3		– 3
		12		– 1	
1921	93		– 4		
		8			
1931	101				

Value in 1925 is 96.8368

Stirling Interploation

Stirling Approximation or Stirling Interpolation Formula is an interpolation technique, which is used to obtain the value of a function at an intermediate point within the range of a discrete set of known data points .

Stirling Formula is obtained by taking the average or mean of the Gauss Forward and Gauss Backward Formula . Both the Gauss Forward and Backward formula are formulas for obtaining the value of the function near the middle of the tabulated set .

How to find

Stirling Approximation involves the use of forward difference table, which can be prepared from the given set of x and f(x) or y as given below –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

This table is prepared with the help of x and its corresponding f(x) or y . Then, each of the next column values is computed by calculating the difference between its preceeding and succeeding values in the previous column, like

$\Delta y0= y1 - y0, \Delta y1 =y2 - y1,$

$\Delta^2 y0 = \Delta y1 - \Delta y0$, and so on.

Now, the Gauss Forward Formula for obtaining f(x) or y at **a** is:
where,p=a-x0/h ,
a is the point where we have to determine f(x), x is the selected value from the given

Δy

x which is closer to **a** (generally, a value from the middle of the table is selected), and h is the difference between any two consecutive x. Now, y becomes the value corresponding to x and values before x have negative subscript and those after have positive subscript, as shown in the table below –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
⋮	⋮				
x_{-2}	y_{-2}				
		Δy_{-2}			
			$\Delta^2 y_{-2}$		
x_{-1}	y_{-1}				
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0				
			$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
x_1	y_1			$\Delta^3 y_{-1}$	
		Δy_0	$\Delta^2 y_0$		
x_2	y_2				
		Δy_1			
⋮	⋮				

Stirling's Formula gives a good approximation for n! in terms of elementary functions. Before stating the formula, we introduce the following notation:

if f(n) is a function and g(n) is a function, then we write f(n) ∼ g(n) ↔ limn→∞ f(n) g(n) = 1. The statement f(n) ∼ g(n) is read f(n) is asymptotic to g(n) as n → ∞.

For example, one verifies that n 2 ∼ (n + 1)2 and √ 1 + n ∼ √ n.

Here is Stirling's Formula: Stirling's Formula n! ∼nn e −n √ 2πn. The following graph shows a plot of the function h(n) = n!/nn e −n √ 2πn, confirming Stirling's Formula: h(n) → 1 as n → ∞. It turns out that h(n) is decreasing so n n e −n √ 2πn always underestimates n! by a small amount.

10 20 30 40 50 60 70 1.03 1.04 1.02 1.01 Figure 1 :

Stirling's Formula The proof of Stirling's Formula is beyond the scope of this course. Instead of proving the formula, we rather give a proof of a weaker statement: we show that for every positive integer n, n n e −n < n! < (n + 1)n+1e −n . (1) This does not prove Stirling's Formula, but it gives motivation for the n n e −n term in the formula. The proof of the √ 2πn part of the formula is more difficult. 1 First Proof — To prove (1), we just have to show (by taking logarithms): n log n − n <log(n!) < (n + 1) log(n + 1) − n. Since n! = n · (n − 1) · (n − 2)· · · 2 · 1, log(n!) = log 1 + log 2 + . . . + log n. The sum on the right can be estimated by integrals: let's show that log 1 + log 2 + . . . + log n < Z n+1 1 log x dx. To see this, note that the integral represents the area under the curve y = log x (the red curve in the left plot below) for 1 ≤ x ≤ n+1, whereas the sum log 1+log 2+. . .+ log n represents adding up the areas of rectangles with height log k for k = 1, 2, . . . , n (see green step function in the left plot below). Now we can work out the integral: Z n+1 1 log x dx = x log x − x + 1in+1 1 = (n + 1) log(n + 1) − n. Therefore log(n!) < (n + 1) log(n + 1) − n. We're going to do the same thing to prove log(n!) >n log n − n: we claim that log 1 + log 2 + . . . + log n > Z n 0 log x dx.

This is shown in the figure on the right, with the red curve representing log x and the rectangles representing log(1) + log(2) + . . . + log(n). 2.0 0.5 x 2 6 1.5 1.0 0.0 4 8 2 x 1.5 0.5 2.0 4 6 0.0 1.0 8 Figure 2 : Approximating log(n!) Therefore log(n!) > R n 0 log xdx = n log n − n, which completes the proof of (1).

Bessel's Interpolation

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called **extrapolation**.

Central differences : The central difference operator d is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, \ldots\ldots, y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}.$$

Similarly, high order central differences are defined as :

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \quad \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2$$

Note – The central differences on the same horizontal line have the same suffix

Central difference table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Bessel's Interpolation formula –

$$
\begin{aligned}
f(u) = &\left\{\frac{f(0)+f(1)}{2}\right\} + \left(u-\frac{1}{2}\right)\Delta f(0) \\
&+ \frac{u(u-1)}{2!}\left\{\frac{\Delta^2 f(-1)+\Delta^2 f(0)}{2}\right\} \\
&+ \frac{(u-1)\left(u-\frac{1}{2}\right)u}{3!}\,\Delta^3 f(-1) \\
&+ \frac{(u+1)\,u\,(u-1)(u-2)}{4!}\left\{\frac{\Delta^4 f(-2)+\Delta^4 f(-1)}{2}\right\} + \ldots\ldots
\end{aligned}
$$

It is very useful when **u = 1/2**. It gives a better estimate when **1/4 < u < 3/4** Here f(0) is the origin point usually taken to be mid point, since bessel's is used to interpolate near the centre. h is called the interval of difference and u = (x – f(0)) / h, Here f(0) is term at the origin chosen.

Examples –

Input : Value at 27.4 ?

x:	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333.

Output :

u	$10^3 f(u)$	$10^3\ \Delta f(u)$	$10^3\ \Delta^2 f(u)$	$10^3\ \Delta^3 f(u)$	$10^3\ \Delta^4 f(u)$	$10^3\ \Delta^5 f(u)$
– 2	4000					
		– 154				
– 1	3847		12			
		– 142		– 3		
0	3704		9		4	
		– 133		1		– 7
1	3571		10		– 3	
		– 123		– 2		
2	3448		8			
		– 115				
3	3333					

Value at 27.4 is 3.64968