Unit - IIIrd - lattices

→ lattice theory is a study of a sets of objects.

→ It is ~~or~~ result of boolean alzebra. and its provides. frame work to ~~f~~ combining ~~suted~~ study of classes or ~~sort~~ ordered set in mathematices.

Cross product (Cartesian product of Sets) -

Given two Non empty sets P and Q, Cartesian product (P × Q) is the set of all ordered pairs of elements from P and Q. i.e.

$$P \times Q = \{(p, q) / p \in P, q \in Q\}$$

- if either P or Q is null set then ~~to~~ P × Q will also been. empty set.

$$P \times Q = \emptyset \text{ (null set)}$$

e.g.

$P = \{a, b, c\}$.

$Q = \{1, 2, 3\}$.

$P \times Q = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1)(c,2), (c,3)\}$

e.g. -

$P = \{\ \}$

$Q = \{1, 2\}$.

$P \times Q = \{\ \}$.

Relation –

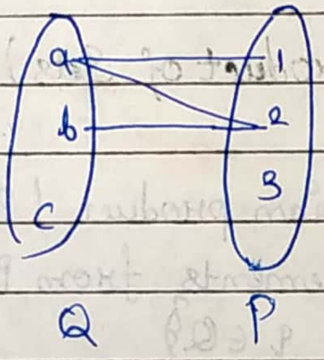
A Relation 'R' from a non empty set 'A' to non empty set 'B' is a subset of Cartesian product $(A \times B)$

e.g. $P \times Q = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$.

$R_1 = \{(a,1), (a,2), (b,2)\}$

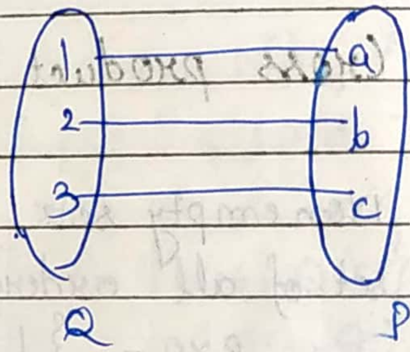
$R_2 = \{(a,1), (b,2), (c,3)\}$

$R_1 =$



$R_2^{-1} =$

$R_2 \Rightarrow$



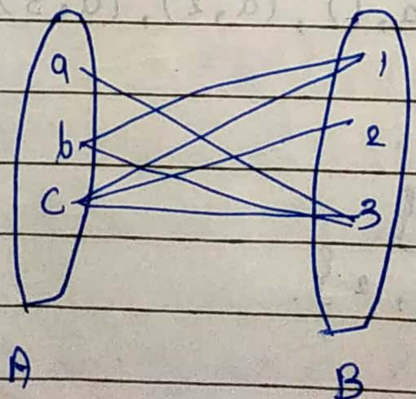
$= R_2$

Complement of relation –

$$\overline{R_1} = (A \times B) - R_1$$

$\overline{R_1} = \{(a,3), (b,1), (b,3), (c,1) (c,2), (c,3)\}$

$\overline{R_1} =$

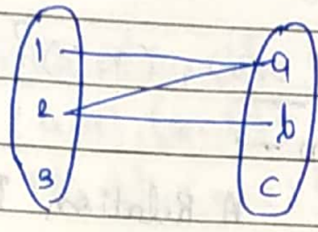


Inverse of Relation –

Inverse of Relation is denoted by.

$R^{-1} = \{(b, a) / (a, b) \in R\}$.

$R_1^{-1} = \{(1, a), (2, a) (2, b)\}$

$R_1^{-1} =$ 

Note – In given above example,
Cardinality of R_1 = Cardinality of R^{-1}.

Types of Relations –

1.) Reflexive relation –

A relation R on a set 'A' is said to be Reflexive if $\forall a \in A, (a, a) \in R$.

e.g. $A = \{a, b, c\}$.
$A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b) (c,c)\}$

Qus find the Relation are Reflexive or Not

× 1.) $R = \{\emptyset\}$. (irreflexive) Not R.

✓ 2.) $R = A \times A$. Biggest Reflexive.

✓ 3.) $R = \{(a, a), (b, b), (c, c)\}$. Smallest "

× 4.) $R = \{(a, b), (b, a), (a, a), (b, b)\}$. Not R.

✓ 5.) $R = \{(a, a), (b, b), (c, c), (a, b)\}$. R

× 6.) $R = \{(a, b), (b, c), (a, c)\}$. Not R ~~Not~~ irreflexive.

Note: All diagonal elements should be in Relation then and then it is Reflexive Relation.

② Irreflexive Relation –

A Relation 'R' on a Set A is said to be irreflexive if $\forall a \in A$, $(a, a) \notin R$.

(Not diagonal element)

e.g.

$R = \{(a,b), (b,c), (a,c)\}$

$R = \phi$

$R = \{(a,b), (b,b), (a,c)\}$

③ Symmetric Relation –

A Relation R on a Set 'A' is said to be symmetric if $\forall a, b \in A$, $(a, b) \in R$ then $(b, a) \in R$.

e.g.

$A = \{a, b, c\}$

$A \times A =$

	a	b	c
a	(a, a)	(a, b)	(a, c)
b	(b, a)	(b, b)	(b, c)
c	(c, a)	(c, b)	(c, c)

$\{a, b, c\}$

R

Note - diagonal elements of A×A have existance of diagonal elements have symmetric relation.

Some example of Symmetric Relation –

$R = \{(a,b), (b,a)\}$

$R = \{(b,c), (c,b), (a,a), (c,c)\}$

$R = \{(a,a), (b,b), (c,c)\}$

$R = \{\ \} = \phi$

$R = A \times A$

$\times R = \{(a,b), (b,c), (a,c)$

$\times R = \{(a,b), (b,a), (a,c)\}$

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④ Anti Symmetric Relation -

A Relation R on a Set 'A' is said to be Anti symmetric if $\forall a, b \in A$, $(a,b) \in R$, $(b,a) \in R$ then $a = b$.

find the Sets are Anti symmetric or not?

✓ $R = \{(a,b), (b,c) (a,c)\}$. Anti symmetric
✓ $R = \{(a,b), (a,a), (b,b)\}$. Anti symmetric
✓ $R = \{(a,a), (b,b), (c,c)\}$. "
✓ $R = \emptyset$. "
✗ $R = A \times A$. Not "
✗ $R = \{(a,b), (b,a), (b,c), (c,c)\}$. Not "
✗ $R = \{(a,b), (b,c), (a,c) (c,a), (a,a), (c,c)\}$ Not A.S.

Note: diagonal elements are Allowed

⑤ Asymmetric Relation -

A Relation R in a set A is said to be Asymmetric if $\forall a, b \in A$, $(a,b) \in R$, $(b,a) \notin R$
$A = \{a, b, c\}$.

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✓ $R = \{(a,b), (b,c), (c,a)\}$ → Asymmetric
✗ $R = \{(a,b), (b,c), (c,b)\}$. Not "
✗ $R = \{(a,a) (b,b), (c,c)\}$. Not "
✗ $R = \{(a,b), (b,a), (a,a) (b,b)\}$. Not "
✓ $R = \emptyset$ "
✗ $R = A \times A$. Not "
✓ $R = \{(a,b), (a,c), (b,c)\}$. "

Note: diagonal elements are not allowed

⑥ Transitive Relation :-

A Relation "R" on a set A is said to be Transitive if all ~~∀ {a, b} ∈ A~~ $\forall\, a, b \in A$, $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$.

$A = \{a, b, c\}$.

1) $R = \{(a,b), (b,a), (a,a), (b,b)\}$. ✓ a – b – a   b – a – b

2) $R = \{(a,a), (b,b), (c,c)\}$. ✓

3) $R = \{(a,b)\}$. ✓

4) $R = \{(a,b), (a,c)\}$. ✓

~~$R = \{(a,b), (a,c)\}$~~

5) $R = \{(a,b), (c,b)\}$ ✓

6) $R = \{(a,b), (b,c), (a,c), (a,a)\}$ ✓ a – b – c

7) $R = \{(c,a), (b,c)\}$. b – c – a ✗ ~~b – c – a~~

8) $R = \{(a,b), (b,a), (c,c)\}$. ✗ a – b – a

9) $R = \{(b,a), (a,b), (a,c)\}$. ✗ b – a – b

10) $R = \emptyset$ ✓

(7) Equivalance Relation -

A Relation R on set A is said to be Equivalent if R is.

(a) Reflexive.

(b) Symmetric.

(c) Transitive.

e.g. - $A = \{1, 2, 3\}$.

✗ (1) $R = \emptyset$

(2) $R_2 = \{(1,1), (2,2) (3,3)\}$. Yes Equivalance

(3) $R_3 = \{(1,1), (2,2), (3,3), (2,1)\}$. Not

✗ ✗ ✗ (4) $R_4 = \{(1,1), (1,3), (2,1), (3,1)\}$. Not

(5) $R_5 = \{(1,1), (2,2), (3,3), (1,2) (1,3) (2,1) (3,1)\}$. Yes

✗ ✗ (6) $R_6 = \{(1,1), (1,2), (2,1), (3,1), (3,2), (3,3)\}$.

(7) $R_7 = A \times A$. Yes

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Qus (IV) $R = \{(a,b) / a-b$ is divisble by $3\}$.

Reflexive – $(a,a) \in R$

$1-1=0 \Rightarrow 0/3 = 0$

$\sqrt{2}-\sqrt{2}=0 \Rightarrow 0/3 = 0$.

$2/3 - 2/3 \Rightarrow 0/3 = 0$.

R is Reflexive.

Symmetric – $(a,b) \in R$ $(b,a) \in R$ then $(b,a) \in R$

(a,b) $(4,1) = 4-1 = 3 \Rightarrow 3/3 = 1$.

(b,a) $(1,4) = 1-4 = -3 \Rightarrow -3/3 = -1$.

(a,b) $(7,4) = 7-4 = 3. \Rightarrow 3/3 = 1$

(b,a) $(4,7) = 4-7 = -3 \Rightarrow -3/3 = -1$.

R is symmetric

Transitive – $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$

(a,b), $(b,c) = (a,c)$

$(7,1)$, $(1,4) = (7,4)$

$6/3 = 2$ $\quad -3/3 = -1$ $\quad 3/3 = 1$

R is Transitive.

So from condition the given set is shows the equivalance relation.

★ Partial order Relation:-

A Relation 'R' on a set A is said to be ~~eqv~~ partial order relation if R is (a.) Reflexive (b.) Anti-symmetric (c.) Transitive.

1.) ✗ $R = \emptyset$
2) — $R = \{(1,1), (2,2), (3,3)\}$.
3) ✗ $R = \{(1,1), (2,2), (3,3) (1,2) (2,1)\}$.
4) — $R = \{(1,1) (2,2) (3,3) (1,3) (2,3)\}$.
5) ✗ $R = \{(1,1), (1,2), (2,3), (1,3)\}$.
6) — $R = \{(1,1), (1,3) (2,2), (3,3)\}$.
7) ✗ $R = \{A \times A$.

1.) $R = \emptyset$, ~~foll~~ null is ~~it~~ satify all condition. except Reflexive so it is not Reflexive and also not Partial order.

Ex which of the following Relation. are not Poset (Partial order set)

1.) $R_1 = \{(a, b) / a, b \in Z, a < b\}$.

Reflexive – $(a, a) \in R$.
$(2, 2) \rightarrow 2 < 2$ ✗
$(-2, -2) \rightarrow -2 < -2$. ✗.

Here This Relation 'R_1' is not Reflexive. So R_1 is not Partial order set relation.

Imp

Hasse - diagram -

let X be a finite set and $\langle X, \leq \rangle$ is a poset. The Hasse Diagram of a poset is a diagiamatical representation of a poset in the plane.

1) Every element of X is marked by small $(\cdot)$ dott in the plane.

2) If for $a, b \in X$, $a < b$ the dott $(\cdot)$ of 'b' is place at a level heigher than the dott $(\cdot)$ of 'a'.

3.) If $a < b$ and there is not point 'c' in 'x' such that $a < c < b$ than the dott $(\cdot)$ representing a is Joined by a line segment or b covers a then there is a line segment Joining a dott $(\cdot)$ of a and dott $(\cdot)$ of b.

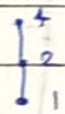
Note-

$$S_n = \{x \mid x \text{ divides } n\}.$$

$= \{$" The set of positive ~~devisors~~ divisors of n; $\forall n \in N$"$\}$

e.g. 1.) if $n = 4$ then.

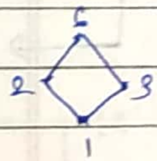
$S_4 = \{1, 2, 4\}$. Hasse-diagram =



1 D 2 D 4 ·

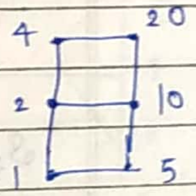
D: Devides.

2) $S_6 = \{1, 2, 3, 6\}$.



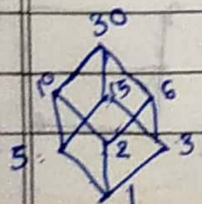
1 D 2 D 6, 1 D 3 D 6 $\} \to$ 1 D (2 ~~D~~ 3) D 6.

3) $S_{20} = \{1, 2, 4, 5, 10, 20\}$.



1 D 2 D 4 D 20
1 D 2 D 10 D 20
1 D 5 D 10 · D 20

4) $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.



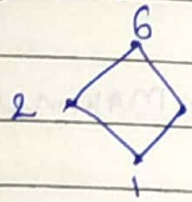
1 D 2 D 6 D 30 | 1 D 5 D 15 D 30
1 D 2 D 10 D 30 | ~~1 D 3 D 15 D 30~~
1 D 5 D 10 D 30 | 1 D 3 D 6 D 30

2 ~~D~~ 3
3 ~~D~~ 5
5 ~~D~~ 6

The least member –

let $\langle X, \leq \rangle$ be a poset and $a \in X$. If for every $b \in X$, $a \leq b$. then a is called a least member of the poset X.

e.g. In the poset $\langle S_6, D \rangle$.

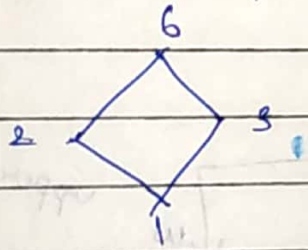


1 is least member of S_6.

The Greatest Member –

let $\langle X, \leq \rangle$ be a poset and $b \in X$. If for every $b \in X$, $a \leq b$ then b is called a Greatest member of the poset X.

In the poset $\langle S_6, D \rangle$.

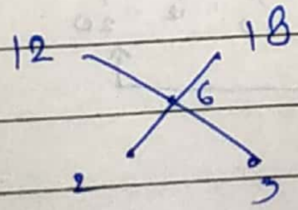


6 is greatest member of S_6.

Minimal Member –

let $\langle x, \leq \rangle$ be a poset and $b \in X$. If ~~for every~~ $a \in X$, ~~b~~ $a \leq b$. ~~Then b is called~~ there is a no $a \in X$. Such that $a \leq b$ then b is called minimal member of X.

$S = \{2, 3, 6, 12, 18\}$.

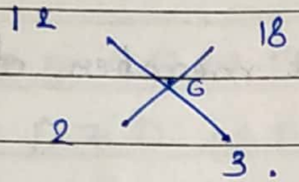


2, 3 are minimal member for given poset.

Maximal Member –

let $<x, \leq>$ be a poset and $a \in X$. if no. there is a no. $b \in X$. such that $a \leq b$. then a is called maximal member of X.

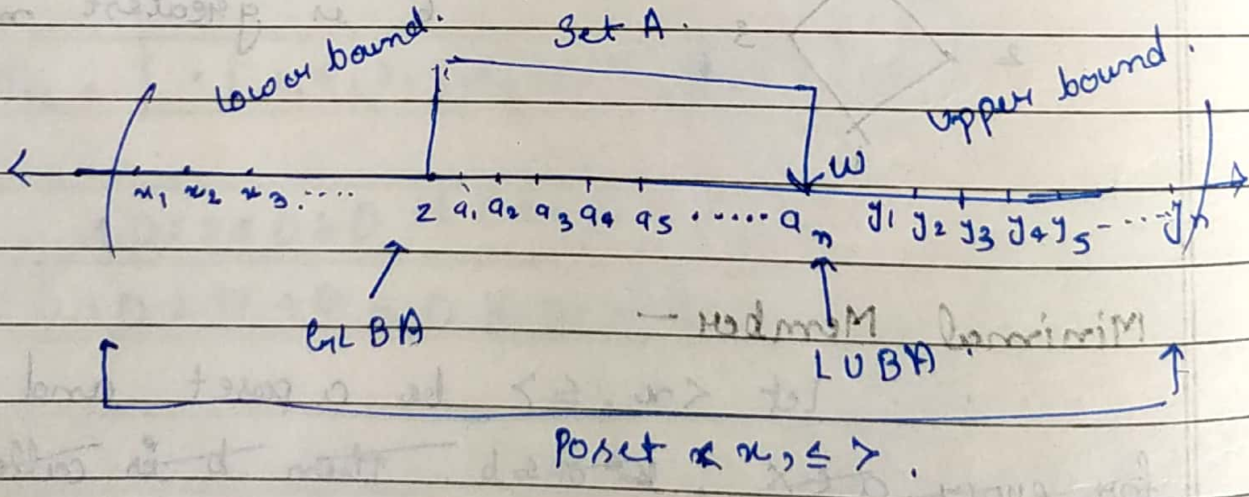
$S = \{2, 3, 6, 12, 18\}$.

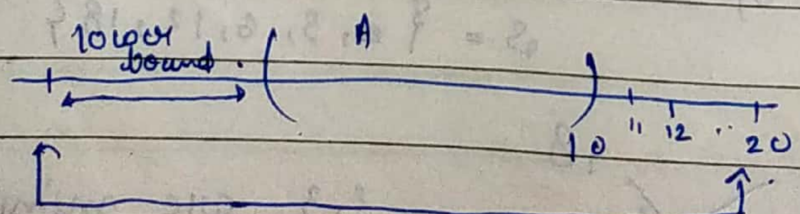


12, 18 are maximal member

lower bound –

let $<x, \leq>$ be a post. be a poset and $A \subset X$, if there exists $x \in X$, such that $x \leq a$ $\forall a \in A$ then x is called a lower bound of the set A.





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ASSIGNMENT – 3

Q1. Show that $(Z, \leq)$ where $x \leq y \Rightarrow x \leq y$ for every $x, y \in Z$ is a poset. Also verify if it is a totally ordered set.

→ (i) Reflexive: $\forall a \in A$; $(a, a) \in R$
for ex.: $(2,2), (3,3), \ldots$
Here, $x \leq y$ is satisfied so it is reflexive.

(ii) Anti Symmetric: $\forall a, b \in A$; $(a, b) \in R$ and $(b, a) \in R$, then $a = b$. ex.: $(2,2)$
Here, $(a,b) \in R$; $(b,a) \in R$ and $a = b$. So, it is Anti-Symmetric.

(iii) Transitive: $\forall a, b, c \in R$; $(a,b) \in R$; $(b,c) \in R$; $(a,c) \in R$, ex.: $(2,4), (4,6), (2,6), \ldots$
Here, $(a,b) \in R$; $(b,c) \in R$ and $(a,c) \in R$. So, it is Transitive.

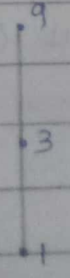
→ Here, it satisfies these relation's. Hence, this set is a POSET.

→ For every $x \leq y$ condition, the set satisfy the compatability condition. So, we can say that it is a totally Ordered set.

Q2. Draw the Hasse Diagram for S_n :-

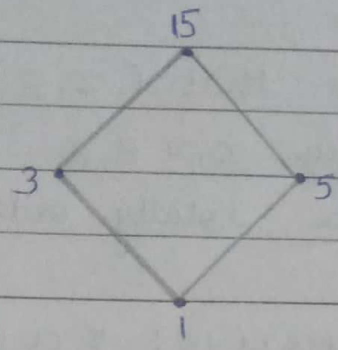
→ a) $n = 9$
$S_n = S_9 = \{1, 3, 9\}$

1D3D9



b) $n = 15$

$S_n = \{1, 3, 5, 15\}$

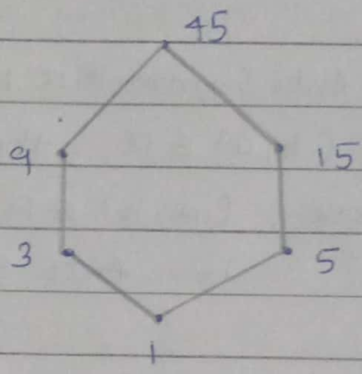


$1 D 3 D 15 D 5$

$1 D 5 D 15$

c) $n = 45$

$S_n = \{1, 3, 5, 9, 15, 45\}$



$1 D 3 D 9 D 45$

$1 D 5 D 15 D 45$

d). $n = 70$

$S_n = S_{70} = \{1, 2, 5, 7, 10, 14, 35, 70\}$

$1 D 2 D 10 D 70$

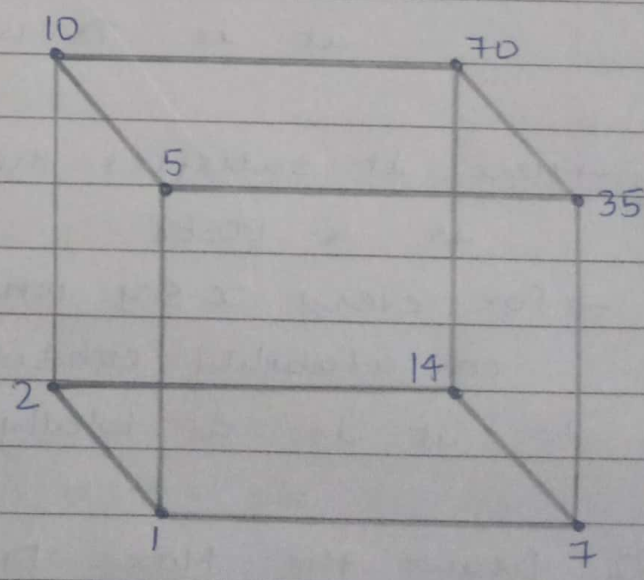
$1 D 5 D 10 D 70$

$1 D 5 D 35 D 70$

$1 D 7 D 14 D 70$

$1 D 7 D 35 D 70$

$1 D 2 D 14 D 70$



Q3. Show that $(N, /)$ is a POSET, show that it is also a lattice.

→ A lattice is a poset (a, K) in which every pair of element $a, b \in L$ has greatest lower bound and atleast one upper bound in L.

→ Let $a, b \in N$

$a \; D \; L \; N(a, b), b, D \; LCM\{a, b\}$

$\therefore$ $LCM\{a, b\}$ is an upper bound of $\{a, b\}$

Again let c be an upper bound of $\{a, b\}$.

$\therefore$ $a D c$, $b D c$

$\therefore$ c is multiple of a and b.

$\therefore$ c is multiple of least of a and b.

$\therefore$ $LCM\{a, b\} \; D \; C$.

→ The $LCM\{a, D\}$ is an upper bound of $\{a, b\}$ and bound of $\{a, b\}$ then $LCM\{a, b\}$ C.

$\therefore$ $LUB\{a, b\} = LCM\{a, b\}$

$a \oplus b = LCM\{a, b\}$

similarly,

$a * b = GCD\{a, b\}$

$\therefore$ $a \oplus b = LCM\{a, b\} \in N$

$\therefore$ $a * b = GCD\{a, b\} \in N$

$\therefore$ (N, D) is a Lattice.

Q4. Define complement of an element be $X = \{a, b, c\}$ and $< P(X), \cup, \cap >$ be a lattice. Find the complement of $\{b, c\}$.

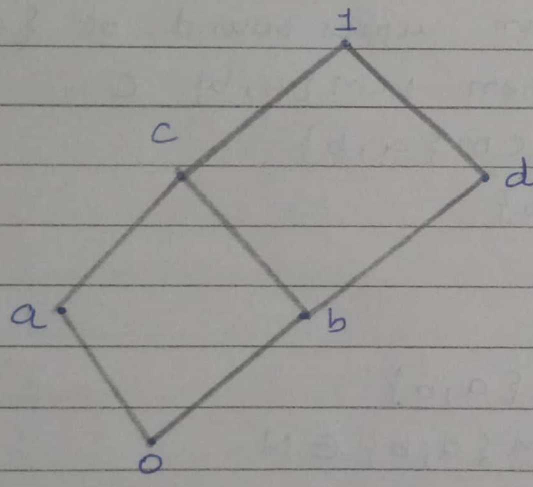
→ Let $< L; *; \oplus; 0 I >$ be a bounded lattice and $a \in b$ if there is an element $b \in L$, such that $a * b = 0$ and $a \oplus b = I$, then element b is called complement of a.

Q5. Define complemented lattice. Give an example of a complemented lattice.

→ Let L be a bounded lattice with lower bound 0 and upper bound 1. Let a be an element of L. an element x in L is called a complement of a if $a \vee x = 1$ and $a \wedge x = 0$.

→ A Lattice L is said to be complemented if L is bounded and every element in L has a complement.

→ Example: Determine the complement of a and c below:



→ The complement of a is d. Since, $a \vee d = 1$ and $a \wedge d = 0$.

→ The complement of c DNE. Since, there DNE any element c such that $c \vee c' = 1$ and $c \wedge c' = 0$.