Formation of Partial diff. Equation

If there are more arbitrary constants than the number of independent variables, by elimination of constants will give to partial diff. Equation of higher order than the first.

* find P.d.E. of $z = ax + by + a^2 + b^2$ — (1)

$$p = \frac{\partial z}{\partial x} = a \quad \text{— (2)}$$

$$q = \frac{\partial z}{\partial y} = b \quad \text{— (3)}$$

By Eq^n (1) $z = px + qy + p^2 + q^2$

*

$$z = ax + (1-a)y + b$$

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = (1-a)$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = a + (1 - a) = 1$$

$$p + q = 1$$

*

$$az + b = a^2 x + y$$

$$a\frac{\partial z}{\partial x} = a^2 \Rightarrow \frac{\partial z}{\partial x} = a$$

$$a\frac{\partial z}{\partial y} = 1 \Rightarrow \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 1$$

$$z = (x-a)^2 + (y-b)^2$$
$$p = 2(x-a)$$
$$q = 2(y-b)$$
$$p^2 + q^2 = 4z$$

H.W

1. $z = a(x+y) + b$
2. $z = ax + by + ab$

$$(x-h)^2 + (y-k)^2 + z^2 = r^2$$
$$2(x-h) + 2p = 0$$
$$2(y-k) + 2q = 0$$

$$ax^2 + by^2 + cz^2 = 1$$

$$2ax + 2cz\frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$2by + 2cz\frac{\partial z}{\partial y} = 0 \Rightarrow b$$

D. w.r.to y.

$$\frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial x} + z\frac{\partial^2 z}{\partial x \partial y} = 0$$

H.W

$$z = ax^2 + bxy + cy^2$$

* $\phi(x+y+z, x^2+y^2-z^2) = 0$

$u = x+y+z$

$v = x^2+y^2-z^2$

$\phi(u,v) = 0$

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0$$

$$\frac{\partial \phi}{\partial u}\left[1 + P\right] + \frac{\partial \phi}{\partial v}\left[2x + P\cdot(-2z)\right] = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(1 + q\right) + \frac{\partial \phi}{\partial v}\left(2y + (-2z)\cdot q\right) = 0$$

$$\frac{1+P}{1+q} = \frac{2x-2Pz}{2y-2qz} = \frac{x-Pz}{y-qz}$$

* $z = F(x^2-y^2)$

$$\frac{\partial z}{\partial x} = F'(x^2-y^2)\cdot 2x$$

$$\frac{\partial z}{\partial y} = F'(x^2-y^2)\cdot(-2y)$$

$z = f\left(\frac{xy}{z}\right)$

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \times \left[\frac{y}{z} \ldots\right.$$

* $z = f\left(\frac{xy}{z}\right)$ $\quad yP + xq = 0$

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\cdot\frac{y}{z}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\cdot\frac{x}{z}$$

$xP - yq = 0$

$$\frac{\partial z}{\partial x}\left[1 - \right.$$

$F(xy+z^2, x+y+z)=0$

$F(u,v)=0$ ~~$u(x,y,z)$~~

$u=xy+z^2, \quad v=x+y+z$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\frac{\partial F}{\partial u}\left(y+2z\cdot p\right)+\frac{\partial F}{\partial v}\left(1+\cancel{\frac{\partial z}{\partial z}}\cdot q\right)=0$$

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)=0$$

$$\frac{\partial F}{\partial u}\left(x+2zq\right)+\frac{\partial F}{\partial v}\left(1+q\right)=0$$

$$\frac{y+2pz}{x+2zp}=\frac{1+q}{1+q}$$

$$\Rightarrow \frac{y+2pz}{x+2zq}=1$$

$$\boxed{x-y=2pz-2zq}$$

$F(x^2+y^2+z^2, z^2-2xy)=0$

$u=x^2+y^2+z^2 \qquad v=z^2-2xy$

$$\frac{\partial F}{\partial u}\left(2x+2z\cdot p\right)+\frac{\partial F}{\partial v}\left(-2y+2z\cdot p\right)=0$$

$$\frac{\partial F}{\partial u}(2y + 2z \cdot q) + \frac{\partial F}{\partial v}(-2x + 2z \cdot q) = 0$$

$$\frac{2x+2zp}{2y+2zq} = \frac{-2y+2zp}{-2x+2zq}$$

$$\frac{x+zp}{y+zq} = \frac{zp-y}{zq-x}$$

* Find the d.e. of all planes through the origin.

Sol:
$$ax+by+cz=0$$

$$a + c\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} = 0$$

$$b + c\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial^2 z}{\partial y^2} = 0$$

*
$$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1$$

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1$$

$$ax + ay + bz = 1$$

$$a + b\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} = 0$$

$$a + b\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial^2 z}{\partial y^2} = 0$$

$(x-a)^2+(y-b)^2+z^2=c^2$

* a, b are Arbitrary Constant

* $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x}+\log y\right)\left(-\frac{1}{x^2}\right)$

$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x}+\log y\right)\times\frac{1}{y}$

~~Lo~~ Lagrange's method of solving the quasi-linear partial diff. equation of order one namely

$Pp + Qq = R$, known as Lagrange equation,

~~Sol^n of equations~~ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Lagrange's auxillary equations

* $\frac{y^2 z}{x} p + xzq = y^2$

The Lagrange auxiliary equations

$\frac{dx}{y^2 z/x} = \frac{dy}{xz} = \frac{dz}{y^2}$ ~~dz~~

$\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}} \Rightarrow x^2 dx = y^2 dy$

$\frac{x^3}{3} = \frac{y^3}{3} + C$

First and third fractions

$$\frac{dx}{y^2 z/x} = \frac{dz}{y^2}$$

$$x\,dx = z\,dz$$

$$x^2 - z^2 = c_2$$

$$\phi\left(x^3 - y^3,\ x^2 - z^2\right) = 0$$

ϕ is arbitrary function

* $a(p+q) = z$

The Lagrange's Equation

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{z}$$

$$\frac{dx}{a} = \frac{dy}{a} \Rightarrow x - y = c_1$$

$$\frac{dx}{a} = \frac{dz}{z}$$

$$\frac{x}{a} = \log z + c_2$$

$$x - a\log z = c_2$$

$p + q = \sin x$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sin x}$$

$$\sin x\,dx = dz$$

$$-\cos x - z = c_1$$

$$x - y = c_2$$

* $yzp + 2xq = xy$

$$\frac{dx}{yz} = \frac{dy}{2x} = \frac{dz}{xy}$$

$$\frac{y}{2}dy = dz$$

$$\frac{y^2}{4} - z = c_1$$

$$\frac{dx}{\cancel{y}z} = \frac{dz}{xy}$$

$$x^2 - z^2 = C_2$$

* $yzp + 2xq = xy$

$$\frac{dx}{yz} = \frac{dy}{2x} = \frac{dz}{xy}$$

First and Third

$$\frac{dx}{\cancel{y}z} = \frac{dz}{xy} \qquad x^2 - z^2 = C_1$$

$$y\,dy = 2\,dz$$

$$\frac{y^2}{2} - 2z = C_2$$

* $p\tan x + q\tan y = \tan z$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\log \sin x - \log \sin y = C_1$$

$$\frac{\sin x}{\sin y} = C_1$$

$$\log \frac{\sin y}{\sin z} = C_2$$

* $zp = -x$

$$zp + 0q = -x$$

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \qquad \bigg| \quad y = C_2$$

$$x^2 + z^2 = C_1$$

11.4 Solutions of Partial Differential Equations by the Method of Direct Integration

This method is applicable to those problems, where direct integration is possible. The solutions of which depend only on the definition of the partial differentiation.

EXAMPLE 1 Solve $\frac{\partial^2 z}{\partial x \partial y} = x^3 + y^3$

SOLUTION $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^3 + y^3$

Integrating with respect to x, keeping y as a constant,

$$\frac{\partial z}{\partial y} = \frac{x^4}{4} + xy^3 + \phi(y)$$

Now, integrating with respect of y keeping x as a constant

$$z = \frac{x^4}{4} y + x \frac{y^4}{4} + \int \phi (y)\, dy + g(x)$$

Writing $\int \phi (y)\, dy = f(y)$, we have

$$z = \frac{x^4 y}{4} + \frac{xy^4}{4} + f(y) + g(x)$$

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$\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$, $\frac{\partial z}{\partial y} = -2\cos y$ when $x = 0$

$\cancel{x=0}$ $z = 0$, $y = n\pi$

Sol:

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \cos y$

In. w. r. to x

$\frac{\partial z}{\partial y} = -\cos x \cos y + \phi(y)$

∴ I w. r. to y

$z = -\cos x . \sin y + \int \phi(y) dy + f(x)$

$z = -\cos x \sin y + f(x) + g(y)$

$y = n\pi$, $z = 0$

$0 = 0 + f(x) + g(y)$

$\frac{\partial z}{\partial y} = -\cos x \cos y + g'(y)$

$\frac{\partial z}{\partial y} = -2\cos y$, $x = 0$

$-2\cos y = -\cos y + g'(y)$

$\int g'(y) dy = -\int \cos y \, dy$

I:

$g(y) = -\cos y + A$

$f(x) = \cos y - A$

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* $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$

$\frac{\partial z}{\partial y} = -2\cos y$

When $x=0$ and $z=0$ when y is a multiple of π.

Solⁿ $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \cos y$

on integration w.r. to x

$\frac{\partial z}{\partial y} = -\cos x \cos y + \phi(y)$

on int, w.r.to y

$z = -\cos x \sin y + \int \phi(y)\,dy + f(x)$

$z = -\cos x \sin y + g(y) + f(x)$ | $z = -\cos x \sin y$

$$\frac{\partial z}{\partial y} = -\cos x \cos y + g'(y) + 0$$

$$-2\cos y = -\cos y + g'(y)$$

$$g'(y) = -\cos y$$

$$g(y) = -\sin y + A$$

Separation of variables

Ex.1.

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{K}\frac{\partial z}{\partial t} \quad — (1)$$

Where $z = z(x,t)$, with the conditions that $z=0$ when $x=0$ and $x=l$ for all values of t.

Soln. Boundary conditions.

$$z(0,t) = z(l,t) = 0 \text{ for all } t.$$

$$z(x,t) = X(x)\,T(t) \quad — (2)$$

By (1) $\quad T\frac{d^2X}{dx^2} = \frac{1}{K} X \frac{dT}{dt}$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{KT}\frac{dT}{dt}$$

The left hand side of equation (1) is independent of t and the R.H.S is in.d of x, each side is equal to the same constant

$$\frac{d^2X}{dx^2} = \mu X \text{ and } \frac{dT}{dt} = \mu K T$$

Either $\mu = 0$ or $\mu > 0$ or $\mu < 0$

Case I $\mu = 0$ $\quad \frac{d^2X}{dx^2} = 0$ and $\frac{dT}{dt} = 0$

on Integration

$X(x) = Ax + B$ and $T(t) = C$

By boundary conditions $A = 0, B = 0$

$X(x) = 0$ i.e. $z(x,t) = 0$

Case II Let $\mu = \lambda^2, \lambda \neq 0$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} - K\lambda^2 T = 0$$

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ and $T(t) = Ce^{K\lambda^2 t}$

by boundary conditions $A = 0, B = 0$

$X(x) = 0$ and $\therefore z(x,t) = 0$

Case III Let $\mu = -\lambda^2, \lambda \neq 0$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + K\lambda^2 T = 0$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$X(x) = A\cos\lambda x + B\sin\lambda x$

by boundary conditions.

$A = 0$

$A\cos\lambda l + B\sin\lambda l = 0$

i.e. $B\sin\lambda l = 0$ $\left[\because B \neq 0 \text{ otherwise } X(x) = 0\right]$

$\Rightarrow \sin\lambda l = 0$

$\lambda l = n\pi$

or $\lambda = \frac{n\pi}{l}, n = 1, 2, \ldots$

$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$

$$\frac{dT}{T} = -\frac{n^2\pi^2}{l^2} K\,dt$$

$$\frac{dT}{T} = -C_n^2\,dt$$

$$C_n^2 = \frac{n^2\pi^2 K}{l^2}$$

general solution

$T_n(t) = D_n e^{-C_n^2 t}$

$z_n(x,t) = X_n(x)\,T_n(t)$

$= E_n \sin\left(\frac{n\pi x}{l}\right) e^{-C_n^2 t}$

$E_n = B_n D_n$

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D_{mn} $-D_m D_n F_{mn}$

ex3. Obtain a solution of Laplace's equation in rectangular Cartesian co-ordinates (x, y, z) by the method of separation of variables.

Soln Three dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(x,y,z) = X(x)\, Y(y)\, Z(z)$$

$$\frac{\partial^2 u}{\partial x^2} = YZ\frac{\partial^2 X}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2} = XZ\frac{\partial^2 Y}{\partial y^2}, \quad \frac{\partial^2 u}{\partial z^2} = XY\frac{\partial^2 Z}{\partial z^2}$$

$$YZX'' + XZY'' + XYZ'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z}$$

Case I $\quad X'' = 0, \; Y'' = 0, \; Z'' = 0$

$$X(x) = Ax + B$$
$$Y(y) = Cy + D$$
$$Z(z) = Ez + F$$

$$u(x,y,z) = (Ax + B)(Cy + D)(Ez + F)$$

Case II Let $\frac{X''}{X} = \lambda_1^2, \; \lambda_1 \neq 0$

$$\frac{Y''}{Y} = \lambda_2^2, \; \lambda_2 \neq 0$$

and $\lambda_1^2 + \lambda_2^2 = \lambda^2$

$$X'' - \lambda_1^2 X = 0$$
$$Y'' - \lambda_2^2 Y = 0$$
$$Z'' + \lambda^2 Z = 0$$

$$\therefore X = Ae^{\lambda_1 x} + Be^{-\lambda_1 x}$$
$$Y = Ce^{\lambda_2 y} + De^{-\lambda_2 y}$$
$$Z = E\cos\lambda z + F\sin\lambda z$$

$$u(x,y,z) = (Ae^{\lambda_1 x} + Be^{-\lambda_1 x})(Ce^{\lambda_2 y} + De^{-\lambda_2 y})(E\cos\lambda z + F\sin\lambda z)$$

Case III Let $\frac{X''}{X} = -\lambda_1^2$, $\frac{Y''}{Y} = -\lambda_2^2$

$-(\lambda_1^2 + \lambda_2^2) = -\lambda^2$

$X'' + \lambda_1^2 X = 0$, $Y'' + \lambda_2^2 Y = 0$, $Z'' - \lambda^2 Z = 0$

$\therefore$ $X = A\cos\lambda_1 x + B\sin\lambda_1 x$, $Y = C\cos\lambda_2 y + D\sin\lambda_2 y$

$Z = Ee^{\lambda z} + Fe^{-\lambda z}$

$u(x,y,z) = X(x)\,Y(y)\,Z(z)$

Ex.5. Wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} \text{ is } \phi = f(x+ct) + g(x-ct)$$

or

Discuss D' Alembert's Solution of wave equation

Solⁿ. $u = x+ct \quad , \quad v = x-ct$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial u^2} + 2\frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial t}$$

$$\frac{\partial \phi}{\partial t} = c\left(\frac{\partial \phi}{\partial u} - \frac{\partial \phi}{\partial v}\right)$$

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\left(\frac{\partial^2 \phi}{\partial u^2} - 2\frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

$$\therefore \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial u^2} - 2\frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial v^2}$$

Putting in given equation

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0$$

Integrating $\frac{\partial\phi}{\partial u} = F(u)$
w.r. to v

$$\phi = \int F(u)\,du + g(v)$$
$$= f(u) + g(v)$$
$$\phi = f(x+ct) + g(x-ct)$$