Linear differential Equation with Constant Coeffi

Differential Operator D

Inverse Operator D^{-1}

$$f(D)y = Q$$

$$y = \text{Complementary} + \left\{\begin{array}{l}\text{Particular}\\ \text{Integr}\end{array}\right.$$

$$y = C.F. + P.I$$

Case I $\quad y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots$

Case II $\quad y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + \dots$

Case III $\quad \alpha \pm i\beta, \quad y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$$\left.\begin{array}{l} y = c_1 e^{\alpha x} \cos(\beta x + c_2) \\ \text{or} \quad y = c_1 e^{\alpha x} \sin(\beta x + c_2) \end{array}\right\}$$

Case IV $\quad \alpha \pm i\beta$

$$y = (c_1 + c_2 x) e^{(\alpha + i\beta)x} + (c_3 + c_4 x) e^{(\alpha - i\beta)x}$$

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx\right]$$

Ex. $\quad (D^4 + 2D^3 + 3D^2 + 2D + 1)y = 0$

$$(m^2 + m + 1)^2 = 0$$

$$m = \frac{-1 \pm i\sqrt{3}}{2}, \ \frac{-1 \pm i\sqrt{3}}{2}$$

$$y = e^{-1/2 x}\left[(c_1 + c_2 x)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4 x)\sin\frac{\sqrt{3}}{2}x\right]$$

P.I. — (General) Method

$$\frac{1}{D-a} Q = e^{ax}\int e^{-ax} Q\,dx$$

Ex. $\quad (D^2 + a^2)y = \sec ax$

Sol. $\quad m = \pm ia$

$$C.F. = c_1 \cos ax + c_2 \sin ax$$

$$P.I. = \frac{1}{f(D)} \sec ax = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D + ia)(D - ia)} \sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

Scanned by CamScanner

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\,dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int(\cos ax - i\sin ax)\sec ax\,dx - e^{-iax}\int(\cos ax + i\sin ax)\sec ax\,dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left\{\int dx - i\int \tan ax\,dx\right\} - e^{-iax}\left\{\int dx + i\int \tan ax\,dx\right\}\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\log(\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log(\cos ax\right\}\right]$$

$$= \frac{1}{2ia}\left[(\cos ax + i\sin ax)\left\{x + (i/a)\log(\cos ax\right\} - (\cos ax - i\sin ax)\left\{x - (i/a)\log ax\right\}\right]$$

$$= \frac{1}{2ia}\left[\frac{2i}{a}\cos ax\log\cos ax + 2ix\sin ax\right]$$

$$= \frac{1}{a^2}\cos ax\log(\cos ax) + \frac{x}{a}\sin ax$$

$\frac{\sin ax}{(D^2+a^2)^n} = \frac{(-1)^n}{(2a)^n}$

G.S. $y = C.F. + P.I.$

i.e. $y = c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax) + \frac{x}{a}\sin ax$

Short method for finding P.I.

1. $e^{ax} \Rightarrow \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}$, $\frac{e^{ax}}{(D-a)^r} = \frac{x^r}{r!}e^{ax}$

2. $\sin ax, \cos ax \Rightarrow \frac{\sin ax}{f(D^2)} = \frac{\sin ax}{f(-a^2)}$, $\frac{\sin ax}{(D^2+a^2)^n} = \frac{(-1)^n}{(2a)^n}\frac{x^n}{n!}$ $\sin\left(ax + \frac{n\pi}{2}\right)$

3. $\sinh ax \Rightarrow \frac{\sinh ax}{f(D)} = \frac{\sinh ax}{f(a^2)} = \frac{e^{ax} - e^{-ax}}{2f(D)}$

Scanned by CamScanner

Ex. $(D-1)^2(D^2+1)^2 y = \sin^2\frac{x}{2} + e^x$

Sol. Auxiliary Eq. $(m-1)^2(m^2+1)^2 = 0$

$m = 1, 1, \pm i, \pm i$

$C.F. = (c_1 + c_2 x)e^x + e^{0x}[(c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x]$

$= (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x$ — (1)

$P.I. = \frac{1}{(D-1)^2(D^2+1)^2}\, \sin^2 x/2 + e^x$

$= \frac{e^x}{(D-1)^2(D^2+1)^2} + \frac{1-\cos x}{2(D-1)^2(D^2+1)^2}$

$= \frac{e^x}{(D-1)^2(1+1)^2} + \frac{e^{0x}}{2(D-1)^2(D^2+1)^2} - \frac{\cos x}{2(D-1)^2(D^2+1)^2}$

$= \frac{1}{4}\cdot\frac{x^2}{2!}e^x + \frac{1}{2\times1\times1} - \frac{\cos x}{2(D^2-2D+1)(D^2+1)^2}$

$= \frac{x^2}{8}e^x + \frac{1}{2} - \frac{\cos x}{2(-1-2D+1)(D^2+1)^2}$

$= \frac{x^2e^x}{8} + \frac{1}{2} \cancel{+} + \frac{1}{4}\,\frac{\sin x}{(D^2+1)^2}$

$= \frac{x^2e^x}{8} + \frac{1}{2} + \frac{1}{4}\,\frac{(-1)^2}{(2)^2}\,\frac{x^2}{2!}\sin\left(x + \frac{2\pi}{2}\right)$

$= \frac{x^2e^x}{8} + \frac{1}{2} \cancel{+} - \frac{1}{32}x^2\sin x$

G.S.

$y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x$

$+ \frac{x^2e^x}{8} + \frac{1}{2} - \frac{1}{32}x^2\sin x$ Ans

$\frac{1}{D^4}\left[1 + \frac{1}{D}\right.$

$\frac{1}{D^4}\left[1 - \right.$

$\frac{1}{D^4}\sin$

Ex. $(D^3 + 3D^2 + 2D)y = x^2$

Sol. $m^3 + 3m^2 + 2m = 0$

$m(m^2 + 3m + 2) = 0$

$m = 0, -1, -2$

$C.F. = c_1 e^{0x} + c_2 e^{-x} + c_3 e^{-2x}$

$= c_1 + c_2 e^{-x} + c_3 e^{-2x}$ Ans

Scanned by CamScanner

$$P.I. = \frac{1}{D(D+1)(D+2)} x^2$$

$$= \frac{1}{2D} (1+D)^{-1} \left(1+\frac{D}{2}\right)^{-1} x^2$$

$$1 - \frac{D}{2} + \frac{D^2}{4}$$
$$- D + \frac{D^2}{2} - \frac{D^3}{4}$$
$$+ D^2 - \frac{D^3}{2} + \frac{D^4}{4}$$

$$= \frac{1}{2D} (1 - D + D^2 - \cdots) \left\{1 - \left(\frac{D}{2}\right) + \left(\frac{D^2}{4}\right) - \cdots\right\} x^2$$

$$= \frac{1}{2D} \left(1 - \frac{3}{2} D + \frac{7}{4} D^2 + \cdots\right\} x^2$$

$$= \frac{1}{2D} \left(x^2 - 3x + \frac{7}{2}\right) = \frac{1}{2}\left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{2}x\right)$$

$$= \frac{1}{12}(2x^3 - 9x^2 + 21x)$$

G.S.

$$y = C.F. + P.I.$$

$$y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{12}(2x^3 - 9x^2 + 21x)$$ Ans

Ex. $(D^3 - 7D - 6) y = e^{2x}(1+x)$

Sol. Auxiliary Eq. $m^3 - 7m - 6 = 0$

$$(m+1)(m^2 - m - 6) = 0$$

$$(m+1)(m-3)(m+2) = 0$$

$$m = -1, -2, 3$$

$$C.F. = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

$$P.I. = \frac{1}{(D+1)(D-3)(D+2)} \left\{ e^{2x}(1+x) \right\}$$

$$= e^{2x} \cdot \frac{1}{(D+2+1)(D+2-3)(D+2+2)} (1+x)$$

$$= e^{2x} \cdot \frac{1}{(D+3)(D-1)(D+4)} (1+x)$$

$$= -\frac{1}{12} e^{2x} \cdot (1-D)^{-1} \left(1+\frac{D}{4}\right)^{-1} \left(1+\frac{D}{3}\right)^{-1} (1+x)$$

$$= -\frac{1}{12} e^{2x} \cdot (1 + D + \cdots)\left(1 - \frac{D}{4} - \cdots\right)\left(1 - \frac{D}{3} + \cdots\right)(1+x)$$

$$= -\frac{1}{12} e^{2x} \cdot \left(1 + \frac{5}{12} D\right)(1+x)$$

$$= -\frac{1}{12} e^{2x} \cdot \left(1 + x + \frac{5}{12}\right) = -\frac{1}{144}(17 + 12x) e^{2x}$$

Scanned by CamScanner

$y = C.F. + P.I.$

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{144}(17+12x)e^{2x}$

Ex. $(D^2 - 2D + 5) y = e^{2x} \sin x$

Sol. Auxiliary Eq. $m^2 - 2m + 5 = 0$

$m = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$C.F. = e^{x}(c_1 \cos 2x + c_2 \sin 2x)$

$P.I. = \frac{e^{2x} \sin x}{D^2 - 2D + 5} = e^{2x} \cdot \frac{1}{(D+2)^2 - 2(D+2) + 5} \sin x$

$= e^{2x} \cdot \frac{1}{D^2 + 2D + 5} \sin x = e^{2x} \cdot \frac{1}{-1 + 2D + 5} \sin x$

$= e^{2x} \cdot \frac{\sin x}{2D + 4} = \frac{e^{2x}}{2} \cdot \frac{\sin x}{D+2} = \frac{e^{2x}}{2} \cdot \frac{(D-2)\sin x}{D^2 - 4}$

$= \frac{e^{2x}}{2} \cdot \frac{(D-2)\sin x}{-1-4} = -\frac{1}{10} e^{2x} [\cos x - 2\sin x]$

General Sol.

$y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{10} e^{2x} [\cos x - 2\sin x]$

$$\boxed{\frac{1}{f(D)} xV = \left\{ x - \frac{1}{f(D)} f'(D) \right\} \frac{1}{f(D)} V}$$

Where V, is the function of x.

Ex. $(D^2 - 2D + 1) y = x \sin x$

Sol. Auxiliary Eq. $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$C.F. = (c_1 + c_2 x) e^{x}$ — (1)

$P.I. = \frac{1}{D^2 - 2D + 1} (x \sin x)$

$= \left\{ x - \frac{1}{D^2 - 2D + 1} (2D - 2) \right\} \frac{1}{D^2 - 2D + 1} \sin x$

$\left\{ x - \frac{2}{D-1} \right\} \frac{\sin x}{-1 - 2D + 1}$

$$= \left\{ x - \frac{2}{D-1} \right\} \frac{\sin x}{-2D} = \frac{x\cos x}{2} - \frac{\cos x}{D-1}$$

$$= \frac{1}{2} x\cos x - \frac{(D+1)}{-1-1} \cos x \qquad D^2 \to -1$$

$$= \frac{1}{2} x\cos x + \frac{1}{2}\sin x + \frac{\cos x}{2}$$

$$= \frac{1}{2}(x\cos x + \cos x - \sin x)$$

$$y = (c_1 + c_2 x)e^x + \frac{1}{2}(x\cos x + \cos x - \sin x) \text{ Ans}$$

Ex. 2 $(D^4 + 2D^2 + 1)y = x^2 \cos x$

Sol. Auxiliary Eq.

$$m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0$$

$$m = \pm i, \pm i$$

$$\text{C.F.} = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$$

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x \qquad \boxed{e^{ix} = \text{Consideration}}$$

$$= \frac{1}{(D^2+1)^2} (x^2 e^{ix}) \text{ Real part}$$

$$= e^{ix} \frac{1}{[(D+i)^2 + 1]^2} x^2 \text{ Real part}$$

$$= e^{ix} \frac{1}{(D^2 + 2iD)^2} x^2 \text{ Real Part} \qquad Dx = 1,\ \frac{1}{D^2}x = \frac{x^2}{2} = \frac{x^3}{6}$$

$$= e^{ix} \frac{1}{2iD}\left[1 + \frac{D}{2i}\right]^{-2} x^2 \text{ Real Part}$$

$$= e^{ix} \frac{1}{-4D^2}\left[1 - \frac{D}{i} + \frac{3D^2}{4i^2} + \cdots\right) x^2 \text{ Real part}$$

$$= -\frac{1}{4} e^{ix} \left[\frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right] \text{ Real part}$$

$$= -\frac{1}{4} e^{ix} \frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right) \text{ Real part}$$

$$= -\frac{1}{4} e^{ix} \left(\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3x^2}{4}\right) \text{ Real part}$$

Scanned by CamScanner

$= \left[-\frac{1}{4}(\cos x + i\sin x)\left(\frac{x^4}{12} + i\frac{x^3}{3} - \frac{3}{4}x^2\right)\right]$ Real part

$= -\frac{1}{4}\left[\left(\frac{x^4}{12} - \frac{3}{4}x^2\right)\cos x - \frac{x^3}{3}\sin x\right]$

$= -\frac{1}{48}\left[(x^4 - 9x^2)\cos x - 4x^3\sin x\right]$

$y = C.F. + P.I.$

$y = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x - \frac{1}{48}\left[(x^4 - 9x^2)\cos x - 4x^3\sin x\right]$

Ex. $(D^2 - 2D + 1)y = xe^x\sin x$

Sol: $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1.$

$C.F. = (C_1 + C_2x)e^x$

$P.I. = \frac{1}{(D-1)^2} xe^x\sin x = e^x \frac{1}{(D+1-1)^2} x\sin x$

$= e^x \cdot \frac{1}{D^2} x\sin x = e^x \cdot \left[x - \frac{1}{D^2} 2D\right] \frac{1}{D^2}\sin x$

$= -e^x\left[x - \frac{2}{D}\right]\frac{\cos x}{D} = -e^x\left[x - \frac{2}{D}\right]\sin x$

$= -e^x x\sin x + 2e^x\cos x$

$= -e^x(x\sin x + 2\cos x)$

$y = C.F. + P.I.$

$y = (C_1 + C_2x)e^x - e^x(x\sin x + 2\cos x)$ Ans

Ex.

$(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x$

Sol. Auxiliary Eq. $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2.$

$C.F. = (C_1 + C_2x)e^{2x}$

$P.I. = \frac{1}{(D-2)^2} 8x^2e^{2x}\sin 2x$

Scanned by CamScanner

$$= 8e^{2x} \cdot \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 8e^{2x} \cdot \frac{1}{D^2} x^2 \sin 2x = 8e^{2x}\left[\frac{1}{D^2}(x^2 e^{i2x})\right] \text{Imaginary part}$$

$$= 8e^{2x}\left[e^{i2x} \cdot \frac{1}{(D+2i)^2} x^2\right] \text{Imaginary part}$$

$$= 8e^{2x}\left[e^{i2x} \cdot \frac{1}{(2i)^2}\left(1+\frac{D}{2i}\right)^{-2} x^2\right] \text{Imaginary part}$$

$$= 8e^{2x}\left[\left\{-\frac{1}{4} e^{i2x}\left(1 - \frac{D}{i} + \frac{3}{4i^2} D^2 + \cdots\right) x^2\right\}\right] \text{Imaginary P.}$$

$$= 8e^{2x}\left[-\frac{1}{4} e^{i2x}\left\{x^2 - \frac{2x}{i} - \frac{3}{2}\right\}\right] \text{Imaginary Part}$$

$$= -\frac{8e^{2x}}{4}\left[e^{i2x}\left\{x^2 - \frac{2x}{i} - \frac{3}{2}\right\}\right] \text{Imaginary Part}$$

$$= -2e^{2x}\left\{(\cos 2x + i\sin 2x)\left(x^2 - \frac{2x}{i} - \frac{3}{2}\right)\right\} \text{Imaginary part.}$$

$$= -2e^{2x}\left\{+2x\cos 2x + x^2 \sin 2x - \frac{3}{2}\sin 2x\right\}$$

$$= -2e^{2x}\left[2x\cos 2x + \left(x^2 - \frac{3}{2}\right)\sin 2x\right]$$

$$y = (c_1 + c_2 x)e^{2x} - 2e^{2x}\left[2x\cos 2x + \left(x^2 - \frac{3}{2}\right)\sin 2x\right]$$

Scanned by CamScanner

Deflection of Beams

With usual notations, for a bending of a beam :

The internal bending moment = EI $\frac{d^2y}{dx^2}$.

Where, E = Young's modulus of elasticity of the material of the beam.

I = Moment of inertia of the cross–section of the beam about the neutral axis.

If the external bending moment is M, then from the equilibrium condition, e have

$$M = EI \frac{d^2y}{dx^2} \quad ...(1)$$

Scanned by CamScanner

Which is the basic differential equation of the elastic curve.

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3} = \text{shear force along the cross-section of the beam} \quad ...(2)$$

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = \text{Intensity of loading (i.e. load per unit length)} \quad ...(3)$$

Boundary Conditions

The general solution of the differential equation (1) will contain two arbitrary constants which in any particular problem are to be determined from the boundary (or end) conditions given below :

(i) End freely supported (Fig. 4.9)

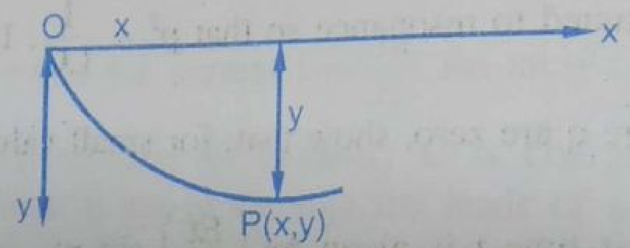


Fig. 4.9

At the freely supported end O, there is no deflection of the beam so that $y = 0$. Also there is no bending moment so that $\frac{d^2y}{dx^2} = 0$.

(ii) End fixed horizontally (Fig. 4.10)

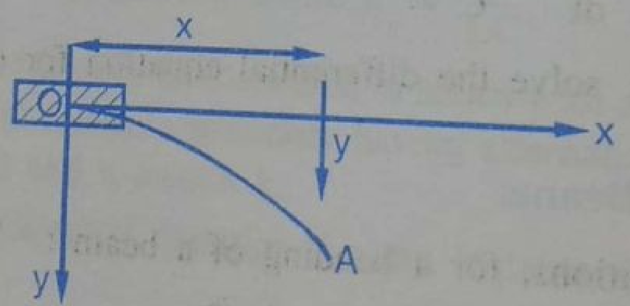


Fig. 4.10

At the fixed end, the deflection and the slope of the beam both are zero.

$$\therefore \quad y = 0 \text{ and } \frac{dy}{dx} = 0.$$

(iii) End perfectly free (Fig. 4.10)

At the free end A (refer Fig. 4.10), there is no bending moment and no shear force, so that

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = 0$$

Scanned by CamScanner

Struts and Columns

A member of a structure or a machine when subjected to end thrusts only is called a **strut** and a vertical strut is called a **column**.

There are four possible ways of the end fixation of a strut.

(a) Both ends fixed, called a *built in strut.*

(b) One end fixed and the other freely supported, *hinged* or *pin–jointed.*

(c) One end fixed and other end free, called a *cantilever.*

(d) Both ends freely supported or *pin–jointed.*

EXAMPLE 1 The deflection of a strut of length l with one end $(x = 0)$ built–in and the other supported and subjected to end thrust P, satisfies the equation

$$\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{p}(l - x)$$

Given that $y = 0, \frac{dy}{dx} = 0$ when $x = 0$ and $y = 0$ when $x = l$.

Prove that the deflection curve is

$$y = \frac{R}{P}\left(\frac{\sin ax}{a} - l\cos ax + l - x\right), \text{ where } al = \tan al$$

SOLUTION The given equation in symbolic form is

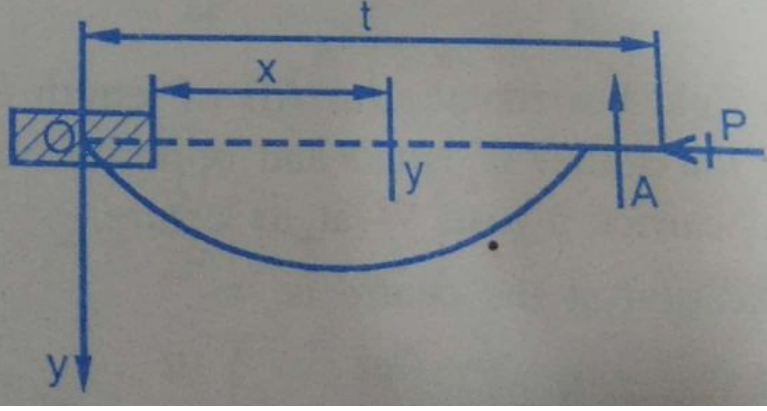
$$(D^2 + a^2)\,y = \frac{a^2R}{P}(l - x), \text{ where } D = \frac{d}{dx} \qquad \text{...(1)}$$

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm\, ia$

C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{P. I.} = \frac{a^2R}{P} \cdot \frac{1}{D^2 + a^2}(l - x)$$

$$= \frac{a^2R}{P} \cdot \frac{1}{a^2\left(1 + \frac{D^2}{a^2}\right)}(l - x)$$



Scanned by CamScanner

$$\text{P. I.} = \frac{R}{P}\left(1 + \frac{D^2}{a^2}\right)^{-1} (l - x)$$

$$= \frac{R}{P}\left(1 - \frac{D^2}{a^2} + \ldots\right)(l - x)$$

$$= \frac{R}{P}(l - x)$$

$\therefore$ The general solution of (1) is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{R}{P}(l - x) \quad \text{...(2)}$$

Differentiating (2) w.r.t.x

$$\frac{dy}{dx} = -c_1 a \sin ax + c_2 a \cos ax - \frac{R}{P} \quad \text{...(3)}$$

Since the end O is built–in (Fig. 4.11)

$$y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0$$

$\therefore$ From (2), $0 = c_1 + \frac{Rl}{P} \Rightarrow c_1 = -\frac{Rl}{P}$

and From (3), $0 = ac_2 - \frac{R}{P} \Rightarrow c_2 = \frac{R}{aP}$

Substituting the values c_1 and c_2 in (2), we have

$$y = \frac{R}{P}\left(\frac{\sin ax}{a} - l\cos ax + l - x\right) \quad \text{...(4)}$$

Which is the required equation of the deflection curve.

Also, at the end A, $y = 0$ when $x = l$

$\therefore$ From (4) $0 = \frac{R}{P}\left(\frac{\sin al}{a} - l\cos al\right)$

or $\frac{\sin al}{a} = l\cos al \qquad \therefore\ al = \tan al$

Scanned by CamScanner

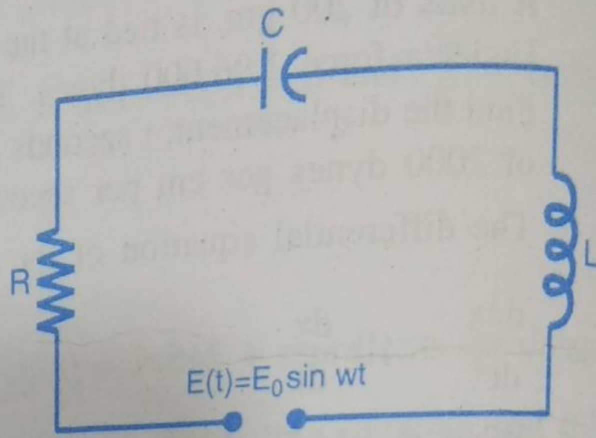
.2 Modelling : Electrical Circuit System

We devoted last section to the study of a mechanical system that is of great practical interest. We shall now consider a similar important *electrical system*, which may be regarded as a basic building block in electrical networks.

In the present section we shall demonstrate an impressive *unifying power* of mathematics by taking *entirely different physical systems which may correspond to the same mathematical model* – in the present case, to the same differential equation–so that they can be treated and solved by the same methods. The practical importance of such an *analogy between mechanical and electrical systems* is *almost obvious*. The analogy may be used for constructing an "electrical model" of a given mechanical system.

Setting up the Model

Consider the LRC–circuit in Fig. 4.8, in which a resistor of resistance R [ohms], an inductor of inductance L [henrys], and a capacitor of capacitance C [farads] are connected in series to a source of electromotive force E(t) [volts], where t is time.



$E(t) = E_o \sin \omega t$

Fig. 4.8

The equation for the current i (t) [amperes] in the LRC–circuit is obtained by considering the three voltage drops.

$E_L = L \dfrac{di}{dt}$ across the inductor,

$E_R = Ri$ across the resistor (Ohm's law), and

$E_C = \dfrac{1}{C} \int i(t)\, dt$ or $\dfrac{q}{C}$ across the Capacitor.

By **Kirchhoff's voltage law**, the analog of **Newton's second law** for mechanical systems, the sum of the voltage drops equals the electromotive force E(t). For a sinusoidal $E(t) = E_o \sin \omega t$ (E_o constant), this law gives

$$L \frac{di}{dt} + Ri = \frac{1}{C} \int i\, dt = E(t) = E_o \sin \omega t \qquad ...(1)$$

To form the second order differential equation, we differentiate (1) with respect to t, obtaining

$$L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = E_o\, \omega \cos \omega t \qquad ...(2)$$

Also if we put $\dfrac{dq}{dt} = i$ i.e. $\dfrac{d^2q}{dt^2} = \dfrac{di}{dt}$ and $q = \int i\, dt$

Scanned by CamScanner

We obtain from (1) the differential equation for the charge q on the capacitor

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0 \sin \omega t \qquad ...(3)$$

Equation (2) is of the same form as (11'), sec. 4.1 (d). Hence our LRC–circuit is the electrical analog of the mechanical system in Sec. 4.1 (d). The corresponding analogy of electrical and mechanical quantities is shown in Table 4.1.

Table 4.1

Analogy of Electrical and Mechanical Quantities in (2), This section, and (11), Sec. 4.1 (d)

Mechanical System		Electrical System
Mass m	$\leftrightarrow$	Inductance L
Damping constant λ	$\leftrightarrow$	Resistance R
Spring modulus (stiffness) k	$\leftrightarrow$	Reciprocal $\frac{1}{C}$ of capacitance
Displacement x(t)	$\leftrightarrow$	Current i(t) or charge q(t)
Driving force Q cos nt	$\leftrightarrow$	Derivative $E_0\omega \cos \omega t$ of electromotive force

EXAMPLE 1 Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is $\frac{30}{\pi\sqrt{LC}}$ cycles/minute.

SOLUTION Let q be the charge in the condenser plate of capacity C and i the current at any time t. The voltage drops across L and C are $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$ and $\frac{q}{C}$ respectively. Since there is no applied electromotive force in the circuit, we have by Kirchhoff's voltage law,

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

or $$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

or $$(D^2 + \omega^2)\, q = 0, \text{ Where } \frac{1}{LC} = \omega^2 \text{ and } D = \frac{d}{dt}.$$

G.S. = C.F. $q = c_1 \cos \omega t + c_2 \sin \omega t$

It represents oscillatory current with the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

Scanned by CamScanner

$$\therefore \quad \text{Frequency} = \frac{1}{T} \text{ cycles/second}$$

$$= \frac{60}{2\pi\sqrt{LC}} \text{ cycles/minute}$$

$$= \frac{30}{\pi\sqrt{LC}} \text{ cycles/minute}$$

EXAMPLE 2 An uncharged condenser of capacity C is charged by applying an electromotive force $E \sin \frac{t}{\sqrt{LC}}$, through the leads of self inductance L and of negligible resistance. Prove that the charge at any time t is

$$q = \frac{EC}{2}\left(\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right)$$

SOLUTION Let q be the charge on the condenser of capacity C at any time t. The differential equation for the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\frac{t}{\sqrt{LC}} \quad \text{...(1)}$$

or $\left(LD^2 + \frac{1}{C}\right) q = E\sin\frac{t}{\sqrt{LC}}$, where $D = \frac{d}{dt}$

Its A.E. is $LD^2 + \frac{1}{C} = 0$

or $D^2 = -\frac{1}{LC} \qquad \therefore \quad D = \pm\frac{i}{\sqrt{LC}}$

$$\text{C.F.} = c_1\cos\frac{t}{\sqrt{LC}} + c_2\sin\frac{t}{\sqrt{LC}}$$

$$\text{P. I.} = \frac{1}{LD^2 + \frac{1}{C}} E\sin\frac{t}{\sqrt{LC}}$$

$$= \frac{E}{L}\cdot\frac{1}{\left[D^2 + \left(\frac{1}{\sqrt{LC}}\right)^2\right]}\sin\frac{t}{\sqrt{LC}}$$

$$= -\frac{E}{L}\cdot\frac{t}{2\sqrt{\frac{1}{LC}}}\cos\frac{t}{\sqrt{LC}} \qquad \left[\because \frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax\right.$$

$$= -\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$$

$\therefore$ The general solution of (1) is

$$q = c_1 \cos \frac{t}{\sqrt{LC}} + c_2 \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}} \quad ...(2)$$

Initially, when $t = 0$, $q = 0$ $\quad\therefore$ from (2), $c_1 = 0$

Differentiating (2) w.r.t. t, we have

$$\frac{dq}{dt} = -\frac{c_1}{\sqrt{LC}} \sin \frac{t}{\sqrt{LC}} + \frac{c_2}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}}$$

$$- \frac{E}{2}\sqrt{\frac{C}{L}} \left(\cos \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \sin \frac{t}{\sqrt{LC}} \right) \quad ...(3)$$

Initially, when $t = 0$, $\frac{dq}{dt} = i = 0 \therefore$ from (3) $\quad c_2 = \frac{EC}{2}$.

Substituting the values of c_1 and c_2 in (2), the charge q on the condenser at any time t is given by

$$q = \frac{EC}{2} \sin \frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}} \cos \frac{t}{\sqrt{LC}}$$

$$\text{or } q = \frac{EC}{2}\left(\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} \right)$$

Scanned by CamScanner