First order ODE -

Differential equation. A differential equation is an equation containing derivatives of [illegible]

Some examples. -

1.

$$m\frac{d^2x}{dt^2} = F\left(x, t, \frac{dx}{dt}\right)$$

The Newton's law for the position $x(t)$ of a particle acted upon by a force F which is a function of x, t and dx/dt is represented by a d. E.

Newton's Second law of motion.

2

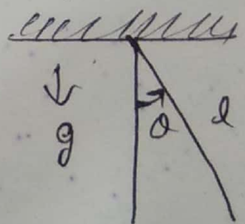
$$m\frac{d^2x}{dt^2} = F(t).$$

Where $x(t)$ is the mass's displacement measured from the origin.

3.

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

This equation governs the angular motion $\theta(t)$ of a pendulum of length l, under the effect of gravity. here g is the acceleration of gravity and t is the time.



Ordinary differential Equations

A d. E. Containing a single independent variable and the derivative

1.9 Solution of differential equations of the first order and first degree

An ordinary differential equation of first order and first degree is of the form $f\left(x, y, \frac{dy}{dx}\right) = 0$ or $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$ or $M(x, y)\,dx + N(x, y)dy = 0$ or $\frac{dy}{dx} = f(x, y)$.

The general solution of such equations will contain only **one arbitrary constant**.

All differential equations of the first order and first degree cannot be solved in every case. Only those which belong to or can be reduced to one of the following types can be solved by the standard methods. These types are :

(1) Equations in which variables are separable.

(2) Homogeneous equations.

(3) Linear equations.

(4) Exact eqauations.

1.10 Variable separable

If a differential equation of the first order and first degree can be put in the form.

$$f_1(x)\,dx + f_2(y)\,dy = 0$$

then it is called **variable separable** equation.

Integrating, we get

$$\int f_1(x)\,dx + \int f_2(y)\,dy = C$$ as its general solution,

C being an arbitrary constant.

Further, equation of the form

$\frac{dy}{dx} = f(ax + by + c)$ can be reduced to a form in which variables are separable by putting $ax + by + c = v$.

EXAMPLE 1 Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

SOLUTION
$$\frac{dy}{dx} = (e^x + x^2)\, e^{-y}$$
or
$$e^y\, dy = (e^x + x^2)\, dx$$

Integrating, we get
$$e^y = e^x + \frac{x^3}{3} + C \text{ or } 3\,(e^y - e^x) = x^3 + A$$
Which is the general solution of the given equation.

EXAMPLE 2 Solve $x \frac{dy}{dx} + \cot y = 0$ given $y = \frac{\pi}{4}$ when $x = \sqrt{2}$.

SOLUTION $x\, dy + \cot y\, dx = 0$
$$\Rightarrow \tan y\, dy = -\frac{dx}{x}$$
Integrating, we get
$$\log \sec y = -\log x + \log c$$
$$\Rightarrow \log \sec y + \log x = \log c$$
$$\Rightarrow \log \frac{x}{\cos y} = \log c \qquad \therefore\ x = c \cos y \quad ...(1)$$

When $x = \sqrt{2},\ y = \frac{\pi}{4}$ then we get
$$\sqrt{2} = c \cos \frac{\pi}{4} \Rightarrow \sqrt{2} = \frac{c}{\sqrt{2}} \qquad \therefore\ c = 2$$
Substituting the value of c in (1), we have

$x = 2 \cos y$ is the particular solution of the given equation.

EXAMPLE 3 Solve $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

SOLUTION
$$y - ay^2 = (x + a)\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{y\,(1 - ay)} = \frac{dx}{x + a}$$
Integrating we get
$$\int\left(\frac{1}{y} + \frac{a}{1 - ay}\right) dy = \int \frac{dx}{x + a} + \log c$$
$$\Rightarrow \quad \log y - a \cdot \frac{1}{a} \log(1 - ay) = \log(x + a) + \log c$$

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Prob^m $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$ (3)

$$\sqrt{(1+x^2)(1+y^2)} + xy\frac{dy}{dx} = 0$$

$$\frac{\sqrt{1+x^2}}{x}dx + \frac{y}{\sqrt{1+y^2}}dy = 0$$

$$\frac{1+x^2}{x\sqrt{1+x^2}}dx + \frac{y}{\sqrt{1+y^2}}dy = 0$$

$$\int\frac{1}{x\sqrt{1+x^2}}dx + \int\frac{x}{\sqrt{1+x^2}}dx + \int\frac{y}{\sqrt{1+y^2}}dy = 0$$

$x = 1/t$

$dx = -\frac{1}{t^2}dt$

$$\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{1+\frac{1}{t^2}}} + \int\frac{x}{\sqrt{1+x^2}}dx + \int\frac{y}{\sqrt{1+y^2}}dy = 0$$

$$-\int\frac{dt}{\sqrt{t^2+1}} + \int\frac{x}{\sqrt{1+x^2}}dx + \int\frac{y}{\sqrt{1+y^2}}dy = 0$$

$$-\log\{t+\sqrt{t^2+1}\} + \sqrt{1+x^2} + \sqrt{1+y^2} = C$$

$$-\log\left\{\frac{1+\sqrt{1+x^2}}{x}\right\} + \sqrt{1+x^2} + \sqrt{1+y^2} = C$$

Homogeneous differential Equation -

$$(x^3+3xy^2)dx + (y^3+3x^2y)dy = 0$$

$$\frac{dy}{dx} = -\frac{x^3+3xy^2}{y^3+3x^2y} = -\frac{1+3(y/x)^2}{(y/x)^3+3(y/x)}$$

$y/x = v$ i.e. $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\frac{1+3v^2}{v^3+3v}$$

$$x\frac{dv}{dx} = -\frac{1+3v^2}{v^3+3v} - v = -\frac{v^4+6v^2+1}{v^3+3v}$$

$$\frac{dx}{x} = -\frac{v^3+3v}{v^4+6v^2+1}$$

$$4\frac{dx}{x} = -\frac{4v^3+12v}{v^4+6v^2+1}$$

$$\log x^4 = -\log[v^4+6v^2+1] + \log C$$

$$x^4(v^4+6v^2+1) = C$$

$$y^4 + 6y^2 + x^4 = C$$

Prob

$$(2x + e^y)dx + xe^y dy = 0$$

$M = 2x + e^y$ $\quad\quad$ $N = xe^y$

$\frac{\partial M}{\partial y} = e^y$ $\quad\quad$ $\frac{\partial N}{\partial x} = e^y$

$$\int M dx + \int N(\text{Terms free from } x)\, dy = c$$

$$\int 2x + e^y \, dx + \int 0 \, dy = c$$

$$x^2 + xe^y = c$$

Prob^m

$$((x+1)e^x - e^y)\, dx - xe^y dy = 0$$

$y(1) = 0$.

Sol^n

$M = (x+1)e^x - e^y$

$N = -xe^y$

$\frac{\partial M}{\partial y} = -e^y$ $\quad\quad$ $\frac{\partial N}{\partial x} = -e^y$

$$\int ((x+1)e^x - e^y)dx + \int 0\, dy = c$$

$$(x+1)e^x - e^x - xe^y = c$$

$$\boxed{x(e^x - e^y) = c}$$

$x = 1,\ y = 0$

$$e - 1 = c$$

$$x(e^x - e^y) = e - 1$$

Exact d.E. Defⁿ

If M and N are functions of x and y. the Equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x,y)$ of x and y such that

$$d[f(x,y)] = Mdx + Ndy$$

i.e $$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

~~Note~~ Ex d.E. $y^2dx + 2xy\,dy = 0$.

is an exact d.E. for there exists a function xy^2 such that

$$d(y^2x) = y^2dx + 2xydy = 0$$
$$d(y^2x) = 0$$
$$xy^2 = c$$

The Necessary Condition for a d.E. of f.O f.d to be Exact.

$$d(f(x,y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial x} = M \quad (2), \qquad \frac{\partial f}{\partial y} = N \quad (3)$$

Equating Coefficients of dx and dy in (3).

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Diff. partially w. r. to y

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Diff. partially w. r. to x

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ — (4)

This if (1) is exact M and N Satisfy Condition (4)

Probm

$$(\tan y + x)\,dx + (x\sec^2 y - 3y)\,dy = 0$$

Solm

$$\frac{\partial M}{\partial y} = \sec^2 y \qquad \frac{\partial N}{\partial x} = \sec^2 y$$

$$\int (\tan y + x)\,dx + \int (-3y)\,dy = c$$

$$x\tan y + \frac{x^2}{2} - \frac{3y^2}{2} = c$$

Probm

$$x\,dx + y\,dy = \frac{a(x\,dy - y\,dx)}{x^2+y^2}$$

Solm

$$x\,dx + y\,dy = \frac{ax\,dy}{x^2+y^2} - \frac{ay\,dx}{x^2+y^2}$$

$$\left(x + \frac{ay}{x^2+y^2}\right)dx + \left(y - \frac{ax}{x^2+y^2}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{a(x^2+y^2) - 2ay(2y)}{(x^2+y^2)^2} = \frac{a(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = 0 + \frac{(x^2+y^2)a - ax(2x)}{(x^2+y^2)^2} = \frac{-a(y^2-x^2)}{(x^2+y^2)^2}$$

Solⁿ

$$\int \left(x + \frac{ay}{x^2+y^2}\right) dx + \int y\, dy = c$$

$$\frac{x^2}{2} + a \cdot \tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c.$$

Probᵐ

$$(\underbrace{y^2 e^{xy^2} + 4x^3}_{M})dx + (\underbrace{2xye^{xy^2} - 3y^2}_{N})dy = 0$$

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2} \times 2xy$$

$$\frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xye^{xy^2} \times y^2 - 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M\, dx + \int N\,(\text{Terms free from } x)\, dy = c$$

$$\int (y^2 e^{xy^2} + 4x^3)\, dx + \int (-3y^2)\, dy = c$$

$$\boxed{\frac{y^2 e^{xy^2}}{y^2} + x^4 - y^3 = c}$$

$$\boxed{y^2 e^{xy^2} + x^4 y^2 - y^5 = c}$$

Probᵐ

Integrating factors

Rule 1. If $Mx + Ny \neq 0$ and the equation is Homogeneous then $\frac{1}{Mx+Ny}$ is an integrating factor.

Rule 2. If $Mx - Ny \neq 0$ and the equation can be written in the form.

$f_1(xy)\, y\,dx + f_2(xy)\, x\,dy = 0$ then $\frac{1}{Mx-Ny} = I.F.$

Rule 3. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say $f(x)$, then $e^{\int f(x)dx} = I.F.$

Rule 4. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$, then $e^{\int f(y)dy} = I.F.$

Prob^m 1. $x^2y\,dx - (x^3+y^3)dy = 0$

$M = x^2y \qquad \frac{\partial M}{\partial y} = x^2$

$N = -(x^3+y^3)$

$\frac{\partial N}{\partial x} = -3x^2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Not Exact. Homogeneous.

$$I.F. = \frac{1}{Mx+Ny} = \frac{1}{x^3y - (x^3y+y^4)} = \frac{-1}{y^4}$$

Multiplying the equation by I.F.

$$-\frac{1}{y^4}\left(x^2y\,dx\right) - \left(-\frac{1}{y^4}\right)(x^3+y^3)dy = 0$$

$$-\frac{x^2}{y^3}dx + \frac{x^3+y^3}{y^4}dy = 0$$

$$\int\left(-\frac{x^2}{y^3}\right)dx + \int \frac{1}{y}dy = c$$

$$-\frac{x^3}{3y^3} + \log y = c$$

Probm

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \qquad \frac{\partial N}{\partial x} = -(3x^2 - 6xy)$$

Not Exact.

- Homogeneous.

then $\quad I.F. = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

$$Mx+Ny = (x^2y - 2xy^2)x + [-x^3 + 3x^2y)y$$

$$Mx+Ny = \cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2 = x^2y^2$$

$$I.F. = \frac{1}{x^2y^2}$$

$$\frac{x^2y - 2xy^2}{x^2y^2}dx - \frac{(x^3 - 3x^2y)}{x^2y^2}dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

$$\int M dx + \int N (\text{without } x)\, dy = c$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

Probm $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin yx - \cos xy)\,x\,dy = 0$

Solm $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

$M = xy^2\sin xy + y\cos xy$

$N = x^2y\sin yx - x\cos xy$

$\frac{\partial M}{\partial y} = 2xy\sin xy + xy^2\cos xy \cdot x + \cos xy - y\sin xy \times x$

$= xy\sin xy + x^2y^2\cos xy + \cos xy$

$\frac{\partial N}{\partial x} = 2xy\sin yx + x^2y\cos xy \cdot y - \cos xy + x\sin xy \cdot y$

$= xy\sin xy + x^2y^2\cos xy - \cos xy$

$Mx - Ny = (xy\sin xy + \cos xy)\,xy - (xy\sin yx + \cos xy)\cdot xy$

$Mx - Ny = 2xy\cos xy$

$I.F. = \frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$

$\frac{(xy\sin xy + \cos xy)\,y\,dx}{2xy\cos xy} + \frac{(xy\sin yx - \cos xy)\,x\,dy}{2xy\cos xy} = 0$

$\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0$

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$\int M\,dx + \int N(\text{without } x)\,dy = c$

$\int \left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \int \left(\frac{-1}{2y}\right)dy = c$

$-\frac{y}{2}\frac{\log\cos xy}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

$\log\sec xy + \log x - \log y = c$

$\frac{x\sec xy}{y} = c$

$\boxed{x\sec xy = cy}$

H.W.

1 $(x^2y^2+2)y\,dx + (2-x^2y^2)x\,dy = 0$,

2. $x(x-y)\frac{dy}{dx} = y(x+y)$

3 $(1+xy)y\,dx + (1-xy)x\,dy = 0$

4. $y(xy+2x^2y^2)dx + x(xy-2x^2y^2)dy = 0$

Prob^m

$(x^2+y^2+1)dx - 2xy\,dy = 0$

Sol^n

$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y + 2y = 4y$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{2xy}\times 4y = \frac{-2}{x} = f(x)$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$$

$$\text{I.F.} = \frac{1}{x^2}$$

$$\frac{x^2+y^2+1}{x^2}dx - \frac{2xy}{x^2}dy = 0$$

$$\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

~~$\frac{\partial M}{\partial y} = \frac{2y}{x^2}$~~ Exact.

$$\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx + \int 0\,dy = c$$

$$x - \frac{y^2}{x} - \frac{1}{x} = c \Rightarrow x^2 - y^2 - 1 = cx$$

Probm

Soln

$$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$$

$$(2xy^3 - 1)x^2$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 6x^2y^3 + 4x = 2x(3xy^3+2)$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy(3xy^3+2)} \times 2x(3xy^3+2)$$

$$= \frac{2}{y} = f(y)$$

$$\text{I.F.} = e^{\int \frac{2}{y}dy} = e^{2\log y} = y^2$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{x}\sec y + \frac{1}{x} y \sec y \tan y - \tan^2 y - 1 + 1 - \frac{\sec y}{x}$$

$$= \tan y \left[\frac{y \sec y}{x} - \tan y \right]$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{\left(\frac{y}{x}\sec y - \tan y\right)} \times \tan y \left(\tan y - \frac{y \sec y}{x}\right)$$

$$= -\tan y = f(y)$$

$$I.F. = e^{-\int \tan y \, dy} = e^{-\log \sec y} = \cos y$$

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)dx - \cos y\left(x - \sec y \log x\right)dy = 0$$

$$\left(\frac{y}{x}\cos y \sec y - \cos y \tan y\right)dx - \left(x \cos y - \cos y \sec y \log x\right)dy = 0$$

$$\int \frac{y}{x} \cos x \sec y$$

$$\int\left(\frac{y}{x} - \sin y\right)dx = c$$

$$\boxed{y \log x + \cos y = c}$$

Linear Differential Equations

A first-order differential equation is said to be linear if it can be written

$$y' + P(x)y = Q(x).$$

Where $P(x)$ and $Q(x)$ are functions of x or constants.

$$I.F. = e^{\int P dx}.$$

Solution $y(I.F.) = \int Q(x) \cdot I.F. \, dx + C$

Similarly $\dfrac{dx}{dy} + Px = Q$

Where P and Q are functions of y.

Prob^m $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1} y = \dfrac{1}{(x^2+1)^3}$

$$P = \frac{4x}{x^2+1} \qquad Q = \frac{1}{(x^2+1)^3}$$

$$I.F. = e^{\int \frac{4x}{x^2+1} dx} = e^{2\log(x^2+1)} = ~~\frac{1}{(x^2+1)^2}~~ (x^2+1)^2$$

$$y \cdot \frac{1}{(x^2+1)^2} = \int \frac{1}{(x^2+1)^3} (x^2+1)^2 \, dx + C$$

$$= \int \frac{dx}{x^2+1} + C = \tan^{-1} x + C$$

Prob^m

$$\frac{dy}{dx} + y = x$$

$P = 1 \qquad Q = x$

$I.F. = e^{\int dx} = e^{x}$

Sol^n $\quad y.e^{x} = \int x.e^{x}dx + C$

$$ye^{x} = xe^{x} - e^{x} + C$$

$$y = x - 1 + Ce^{-x}$$

Prob^m

$$\frac{dy}{dx} + (\sin x)y = e^{\cos x}$$

$P = \sin x$

$Q = e^{\cos x}$

$I.F. = e^{\int \sin x\, dx} = e^{-\cos x}$

Sol^n $\quad y.e^{-\cos x} = \int e^{\cos x}.e^{-\cos x}dx + C$

$$ye^{-\cos x} = x + C$$

Prob

$y' + y\tan x = \sin 2x, \quad y(0) = 1$

$P = \tan x \qquad Q = \sin 2x$

$I.F. = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

$y.\sec x = \int \sin 2x.\sec x\, dx + C = \int \frac{2\sin x\cos x}{\cos x}dx$

$y \cdot \sec x = \int 2 \sin x \, dx + c$

$y \sec x = -2 \cos x + c$

$y = c \cos x - 2 \cos^2 x$

$y(0) = 1$

$1 = c - 2 \Rightarrow c = 3$

$$y = 3 \cos x - 2 \cos^2 x$$

Bernoulli Equation

$y' + Py = Qy^n$

$\frac{y'}{y^n} + \frac{Py}{y^n} = Q$

$$\frac{y'}{y^n} + \frac{P}{y^{n-1}} = Q$$

$$\frac{1}{y^{n-1}} = t$$

$$-(n-1)\, y^{-(n-1)-1}\, dy = dt$$

$$-(n-1)\, y^{-n}\, dy = dt$$

$$\frac{dy}{y^n} = \frac{dt}{-(n-1)}$$

P→ 6

$\frac{dy}{dx} + \frac{1}{x} y = x^2 y^6$

$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{y^5}\cdot\frac{1}{x} = x^2$

Now $\frac{1}{y^5}$ $y^{-5} = t$

$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$

$y^{-6}\frac{dy}{dx} = \frac{-1}{5}\frac{dt}{dx}$

$-\frac{1}{5}\frac{dt}{dx} + \frac{t}{x} = x^2$

$\frac{dt}{dx} - \frac{5t}{x} = -5x^2$ linear int.

$P = -\frac{5}{x}$, $Q = -5x^2$

$I.F. = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = e^{\log x^{-5}}$
$= \frac{1}{x^5}$

Solⁿ $t\cdot\frac{1}{x^5} = \int (-5x^2)\cdot\frac{1}{x^5}\,dx + C$

$\frac{t}{x^5} = -5\int\frac{dx}{x^3} + C$ $\quad \frac{x^{-3+1}}{-2}$

$\frac{y^{-5}}{x^5} = \frac{+5}{+2}\cdot\frac{1}{x^2} + C$

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Mathematical Modellings

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March 14, 2020

Constructing Mathematical Models

Real wold problems

Mathematical world problems Models

Constructing Mathematical Models

Real wold problems

Mathematical world problems Models

Observed behaviour or Phenomenon

Mathematical operations and rules Mathematical conclusions

Steps are usually involved

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Determine the variables and their relationships

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By using concepts and equations create a mathemtical model

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Determine the variables and their relationships

By using concepts and equations create a mathemtical model

Solve the model by mathematical techniques

Steps are usually involved

Determine the variables and their relationships

By using concepts and equations create a mathemtical model

Solve the model by mathematical techniques

Compute the conclusions and predictions with the real world problem

Motion of a particle falling under gravity

A particle falls from rest in a medium in which the resistence is kv^2 per unit mass. Find the velocity of the body and the distance it has fallen in t seconds.

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Solution

- let x be its distance from the starting point after time t

Motion of a particle falling under gravity

A particle falls from rest in a medium in which the resistence is kv^2 per unit mass. Find the velocity of the body and the distance it has fallen in t seconds.

Solution

- let x be its distance from the starting point after time t
- If v is its velocity at this point

Motion of a particle falling under gravity

A particle falls from rest in a medium in which the resistence is kv^2 per unit mass. Find the velocity of the body and the distance it has fallen in t seconds.

Solution

- let x be its distance from the starting point after time t
- If v is its velocity at this point
- The resistance on the particle is mkv^2 in the vertically upward direction.

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General solution

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- **Kirchhoff's current law**- At any point of a circuit, the sum of the inflowing current is equal to the sum of the outflowing currents.

RL-Circuit

Show that the current in a circuit containing a resistance R, an inductance L in series, with constant e.m.f.E. at time t is given by $i = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$.

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General Solution

Equation(1) $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$

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The charge Q on the plate of a condenser C charged through a resistance R by a steady voltage V satisfies the differential equation.
$R\frac{dQ}{dt} + \frac{Q}{C} = V$, if Q=0 at t=0, show that $i = \frac{V}{R}e^{-\frac{t}{RC}}$

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