Solution of ordinary differential equation

* Taylor's series method
* Euler's method
* Modified Euler's method
* Runge-Kutta method
* Picard's method of successive approximation.

* Taylor's series method

⇒ Suppose we wish to solve the differential equation

$$\frac{dy}{dx} = f(x,y) \text{ with the initial condition } y(x_0) = y_0 \quad \text{—①}$$

If $y(x)$ is the exact solution of (1), then the taylor series for $y(x)$ around $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots \quad ②$$

Now at $x = x_0 + h$, equation ② becomes

$$y(x_0+h) = y_0 + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots \quad ③$$

As $y'(x) = f(x,y)$ and y is a function of x, we get

$$y''(x) = \frac{d}{dx}[y'(x)] = \frac{d}{dx}[f(x,y)]$$

$$= \frac{\delta}{\delta x} f(x,y) + \frac{\delta}{\delta y} f(x,y) \frac{dy}{dx}$$

$$= f_x(x,y) + f_y(x,y) \times f(x,y)$$

Now ③ can be written as

$$y_1 = y(x_0+h) = y_0 + h f(x_0,y_0) + \frac{h^2}{2!}\left[f_x(x_0,y_0) + f_y(x_0,y_0) \times f(x_0,y_0)\right] + \cdots$$

similarly taking (x_1,y_1) as the starting point, we get

$$y_2 = y_1 + h f(x_1,y_1) + \frac{h^2}{2!}\left[f_x(x_1,y_1) + f_y(x_1,y_1) \times f(x_1,y_1)\right] + \cdots$$

In general

we have

$$y_{n+1} = y_n + h f(x_n, y_n) + \frac{h^2}{2!}\left[f_x(x_n, y_n) + f_y(x_n, y_n) \times f(x_n, y_n)\right] + \cdots$$

This formula is known as the taylor series formula. It can be used repetitively to obtain $y(x)$ for successive values of x. This method can easily be extended to simultaneous and higher order differential equation.

* Euler's method :

⇒ Euler method is a technique of developing a piecewise linear approximation to the soln of an ordinary differential equation.

⇒ consider the first order ordinary differential equation

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0.$$

We can expand a funcn $y(x)$ about a point $x = x_0$ using taylor's series expansion as

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x) \quad \text{--- (1)}$$

Considering the first two term of (1), we have

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0). \quad \text{--- (2)}$$

Now $y(x_0) = y_0$ and $y'(x_0) = f(x_0, y_0)$

Hence (2) can be written as

$$y(x) = y_0 + (x - x_0)\, f(x_0, y_0)$$

Now the value of $y(x)$ at $x = x_1$ is given by

$$y(x_1) = y_0 + (x_1 - x_0)\, f(x_0, y_0)$$

Taking $h = x_1 - x_0$ we obtain

$$y_1 = y_0 + h f(x_0, y_0)$$

similarly for $x = x_2$ we obtain

$$y_2 = y_1 + h f(x_1, y_1)$$

In general, we obtain

$$y_{n+1} = y_n + h f(x_n, y_n)$$

⇒ This formula is known as the Euler's method and can be used recursively to evaluate $y_1, y_2, \ldots$ starting from initial condition $y_0 = y(x_0)$, Note that this formula does not involve derivatives.

Modified Euler's method:

This is also known as Improved Euler's or Heun's method.

$$y'(x) = \frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

The solution is

$$y(x_1) = y(x_0) + \frac{h}{2}\left[f(x_0, y(x_0)) + f(x_1, y_1)\right]$$

or $$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, x_1)\right]$$

where $y_1 = y_0 + h f(x_0, y_0)$

simly. $$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

Repeat the same procedure untill we get the desired result.

Runge - Kutta method of order four.

Given the IVP

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ at equidistant pt.

$k_1 = h f(x_n, y_n)$

$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$k_4 = h f(x_n + h, y_n + k_3)$

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

In particular If $n = 0$ then

$k_1 = h f(x_0, y_0)$, $k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$, $k_4 = h f(x_0 + h, y_0 + k_3)$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$