Lagrangian & Hamiltonian Dynamics

Consider a system of N particles in 3D space where the position vector is $r_i(t)$ (where $i = 1, 2, \cdots N$).

As per 2nd law of Newtonian dynamics,

$$\frac{d\vec{p}_i}{dt} = \dot{\vec{p}}_i = \underbrace{\vec{F}_i^{\,ext}}_{\text{ext - force}} + \underbrace{\vec{F}_i^{\,cnt}}_{\text{constraint force.}}$$

Due to constraints, the motion of a particle will be restricted, as degrees of freedom will be less.

Constraints are of two types $\rightarrow$ Holonomic / Non-holonomic

Holonomic constraints :— this can be expressed in terms of eqn.

$$f_k(r_1, \cdots r_n) = 0$$

Non-holonomic constraints →

this cannot be expressed in terms of equations,

$$f_k(r_1, \ldots r_N) \leq 0$$

D'Alembert's principle :-

In mathematical statement,

$$\sum_{i=1}^{N} \left(\dot{\vec{p}}_i - \vec{F}_i^{\,ext} \right) \cdot \delta \vec{r}_i = 0$$

where $\delta \vec{r}_i$ is the virtual displacement. The virtual work done would be zero.

Here, we can define,

generalised co-ordinates q_j and

generalised force

$$Q_j = \sum_i \vec{F}_i^{\,ext} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

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Starting from D'Alembert's principle, we can derive Lagrange's eqn. of motion as,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

where $\underset{\text{Lagrangian}}{L} = \underset{\text{K.E}}{T} - \underset{\text{P.E}}{V}$

Generalised momenta :—

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

where, $p_j = p_j(q, \dot{q}, t)$ in general.

Also, $L = L(q, \dot{q}, t)$

Exp :— Simple harmonic motion

$$L(x,\dot{x}) = \underbrace{\tfrac{1}{2}m\dot{x}^2}_{T} - \underbrace{\tfrac{1}{2}kx^2}_{V}$$

Now, $\dfrac{\partial L}{\partial x} = -kx$

$\dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$

Thus, the eqn. of motion becomes,

$$\boxed{m\ddot{x} + kx = 0}$$

Hamiltonian Dynamics :—

The Hamiltonian function can be defined as, $H(q,p,t)$ through Legendre transform,

$$H(q,p,t) = \dot{q}p - L(q,\dot{q},t)$$

or, $$\boxed{H = \sum_j \dot{q}_j p_j - L}$$

Hamilton's eqn. of motion →

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \text{1st order eqn. of motion}$$

$$H = H(p, q, t)$$

$$\frac{dH}{dt} = \sum \frac{\partial H}{\partial p_i} \dot{p}_i + \sum \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial t}$$

$$= \cancel{\sum -\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i}} + \cancel{\sum \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i}} + \frac{\partial H}{\partial t}$$

$$\Rightarrow \boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

If H does not explicitly depend on time

⇒ H is a constant of motion